



IMF Working Paper

Stock Markets and the Real Exchange Rate: An Intertemporal Approach

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Abstract

The views expressed in this Working Paper are those of the author(s) and do not necessarily represent those of the IMF or IMF policy. Working Papers describe research in progress by the author(s) and are published to elicit comments and to further debate.

The paper presents an N-country model with stock markets, in which a closed-form solution for the real exchange rate is derived. Risky asset prices and allocation of risky assets among countries are determined endogenously. Such a framework allows an analysis of how fundamental parameters, such as the variance and covariance of the risky assets or demographic variables, affect the real exchange rate. The predictions of the model are contrasted with the Balassa-Samuelson effect. A new transmission channel of the real exchange rate for parameters such as income on net foreign assets, risk aversion, and risk-hedging opportunities is also explored.

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I. INTRODUCTION

Given the extraordinary developments in financial markets in recent years, one would expect the stock market to play a role in the determination of the real exchange rate. Yet there are surprisingly few models in the literature that deal with this issue.² The goal of this paper is to contribute to filling this gap.

We develop a model with stock markets in which a closed-form solution for the real exchange rate is derived. Our model allows for a given number of risky assets, which form an incomplete market. Risky asset prices and allocations of risky assets among countries are determined endogenously. The risk-free rate is exogenous, so our model is an intermediate step toward a full general equilibrium. Our new framework allows analysis of how fundamental parameters of the model, such as the variance and covariance of the risky assets or demographic variables, affect the real exchange rate. We contrast the predictions of the model to the Balassa-Samuelson effect. We also suggest a new transmission channel of the real exchange rate for parameters such as returns on net foreign assets, risk aversion, and risk-hedging opportunities.

Let us present briefly the main results of the paper. We first analyze equilibrium asset prices and portfolio allocation; we then present the determinants of the real exchange rate in the model.

As a beginning, we derive closed-form solutions for equilibrium asset prices and portfolio allocations. In our framework, the dynamics of asset prices are much richer than the mere fluctuation of present discounted value of expected dividends. Changes of asset prices also depend on demographic dynamics, on variations in the variance and covariance of the risky assets' distribution, and on changes in the countries' risk aversion. A closed-form solution for portfolio holdings is also derived. It expresses how risky assets are allocated between countries in a very intuitive way.

This done, we can study the determinants of the real exchange rate in the model. We contrast the predictions of the model to the celebrated Balassa-Samuelson effect. In a Balassa-Samuelson framework, a country whose relative productivity advantage in tradables is larger than its relative productivity advantage in nontradables should have a higher real exchange rate.³ Our model underscores that what matters in an intertemporal framework is not the current productivity in tradables, but the **expectations** of future productivity. In our case, the real exchange rate is determined by the relative scarcity of tradable and nontradable goods.

² To the author's knowledge, no existing *H*-country-model with incomplete markets and endogenous determination of risky asset prices offers a closed-form solution for the real exchange rate.

³ Throughout the paper, a higher real exchange rate means an appreciated real exchange rate.

The model thus suggests a new transmission channel for parameters such as returns on net foreign assets, risk aversion, and risk-hedging opportunities. More precisely, the new effects are the following:

- *Gross income on the net foreign position*: The higher the gross income on the net foreign position of the country, the higher the real exchange rate.
- *Risk structure*: The risky assets quoted on a country's stock exchange have a certain cost in terms of risk. This cost corresponds to the variance of the dividends minus the risk-hedging benefits offered by the asset. Countries with lower structural risk cost should also have a higher real exchange rate.
- *Future financial gains*: Countries with larger expected financial gains should have a higher real exchange rate. The magnitude of these financial gains depends on parameters such as risk aversion and risk-hedging opportunities. More risk-averse countries should have lower exchange rates, as should countries with less attractive risk-hedging opportunities.
- *Discount rate*: A more patient country should have a lower real exchange rate.

We are also able to derive a closed-form solution for the volatility of the real exchange rate. The volatility of the real exchange rate was rarely studied in a general-equilibrium framework with incomplete markets. We find that exchange rate volatility depends on the asymmetries of the countries. This includes both asymmetries in endowments' shocks and in shocks on dividends, and in the risk-hedging opportunities of the two countries.

The remainder of the paper is organized as follows: Section II presents the model and its solution; Section III analyzes the real exchange rate; and Section IV concludes.

II. THE MODEL AND ITS SOLUTION

This section presents the model of the paper. As noted above, it is one of the very few existing models on the role of stock markets in exchange rate determination. The model allows for an arbitrary number of risky assets, which form an incomplete market, as well as a risk-free bond. A closed-form solution for the real exchange rate is derived from the optimal portfolio and consumption/saving choices of a representative agent. There are several countries in the world, and risky asset prices are determined endogenously.

Let's now look at the model in greater detail.

A. The Model

The basic mechanism of the model is a stock-market-augmented version of a consumption-smoothing story. A country's representative agent receives two stochastic endowments at each period: one in tradable goods (T), and one in nontradables (NT). The agent will use all the financial instruments at her disposal to maximize her expected inter-temporal utility.

These financial instruments include an arbitrary number of risky assets (both foreign and domestic), which form an incomplete market, as well as a risk-free bond⁴.

Using the tradable good as numéraire, and writing δ as the rate of time preference, the program of the agent is the following:

$$\text{Max}_{\{c_{T,t}; c_{NT,t}; \omega_{0,t}; \omega_{j,t}\}_{t=0}^{+\infty}} U(C) = E_0 \left\{ \sum_{t=0}^{+\infty} \delta^t [u(c_{T,t}) + v(c_{NT,t})] \right\},$$

under the budget constraints BC_t :

$$c_{T,t} + p_{NT,t} c_{NT,t} + \omega_{0,t} + \sum_{j=1}^J \omega_{j,t} = y_{T,t} + p_{NT,t} y_{NT,t} + R_0 \omega_{0,t-1} + \sum_{j=1}^J R_{j,t} \omega_{j,t-1}$$

(we have the initial conditions: $\omega_{0,-1} = \omega_{j,-1} = 0$).

And a sufficient condition of transversality is the following⁵:

$$\lim_{s \rightarrow +\infty} E_t \left(\frac{1}{1+r} \right)^s \left[\omega_{0,t+s} + \sum_{j=1}^J \omega_{j,t+s} \right] \leq 0.$$

Let us now define more precisely the variables of the model. (Appendix III summarizes the notation).

There are H countries, each with a representative agent (RA). An upper script h will refer to the representative agent of country $h=1, \dots, H$. The tradable good serves as a numéraire. As a consequence, all variables are expressed in units of this consumption good. The price of nontradables in units of the tradable good is written p_{NT} . All the payments from risky assets are made in tradables. The major variables involved in the model are thus the following:

- The representative agent of a country receives two stochastic endowments at each period t : one in traded goods, $y_{T,t}$, and one in nontradables, $y_{NT,t}$. The only other income the agent receives at period t is the revenues from her past financial investments.
- All agents can lend/borrow at the constant international risk-free rate $R_0 = 1+r$. This international interest rate is assumed to be constant over time

⁴ The framework of the model was developed in a different context by Davis and Willen (2000), and Davis, Nalewaik and Willen (2001).

⁵ A discussion of a No-Ponzi- Game condition is presented in Mercereau (2003).

and exogenously given.⁶

- There are J stocks available on the world stock markets. They form an incomplete market. They can be either domestic or foreign stocks. Each risky asset $j=1, \dots, J$ pays a stochastic dividend $d_{j,t}$ at time t , and has a market price

$$P_{j,t}. \text{ The gross rate of return is defined by: } R_{j,t} = \frac{d_{j,t} + P_{j,t}}{P_{j,t-1}}.$$

The $J \times 1$ vector of gross rate of risky asset return at time t is R_t .

We also write the following:

$\omega_t^h = (\omega_{j,t}^h)_{j=1}^J$ the portfolio $J \times 1$ holding of risky assets by the representative agent of country h at time t .

$\omega_{0,t}^h$ denotes her holdings of risk-free assets (all these variables are expressed in units of the tradable good).

Market Clearing Condition

In order to derive the asset prices, we need to write the market-clearing equation for risky

$$\text{assets: } \sum_{h=1}^H N_t n_t^h \omega_t^h = S_t.$$

n_t^h is the share of the population of country h in the world population at time t (consequently,

$$\text{we have: } \sum_{h=1}^H n_t^h = 1).$$

N_t is the world population at time t .

$S_t = (S_{j,t})_{j=1, \dots, J}$ is the vector of the J assets' market valuations. Assets are indeed in positive

supply. There are ϕ_j shares of asset j (let $\Phi = (\phi_j)_{j=1, \dots, J}$). The total market valuation of asset j is thus $S_{j,t} = P_{j,t} \times \phi_j$.

Before solving the program of the consumer and solving it, we need to make a few more assumptions, which will facilitate the derivation of the results. They are the following:

⁶ Since the interest rate is exogenously given, the risk-free bond is of course not in zero net supply in the world economy. We therefore implicitly assume that there are countries other than the H studied here, which will supply the necessary risk-free bonds. In that sense, we have an " H -small-country-model" rather than an " H -country-model", since risky asset prices, but not the interest rate, are determined endogenously. It is therefore an intermediate step toward a full general equilibrium model. To simplify the presentation, we nevertheless refer to our model as a "general equilibrium" model, by opposition to the partial equilibrium model developed in Mercereau (2003) –in which risky asset prices were exogenously given.

- The utility function is separable between tradables and nontradables.
- All agents have the same exponential utility function for tradables and nontradables: $u(c) = v(c) = \frac{-1}{A} \exp(-Ac)$, where A is the coefficient of absolute risk aversion⁷.
- Dividends and endowments both follow an ARIMA process. We also assume that their innovations have a joint normal distribution⁸. ε_t^h is the endowments innovation; η_t is the dividends innovation. The moments of their distribution can be time varying, but they are non-stochastic⁹ and bounded.

$$\underbrace{\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix}}_{(H+J,1)} \sim N \left[0; \underbrace{\begin{pmatrix} \Theta_t & \tilde{\beta}_t \\ \tilde{\beta}_t & \tilde{\Sigma}_t \end{pmatrix}}_{(H+J;H+J)} \right].$$

Finally, throughout the paper primed variables denote the transpose of the corresponding vector. (e.g., Z' is the transpose of vector Z).

The full solution to this problem, and its proof, are given in Appendix II¹⁰. Let us now present the solution for the main variables of the model.

⁷ Similar results can be derived with a quadratic utility function. With a quadratic utility function, though, one could not have a closed-form solution for the portfolio ω_t . The main intuition would not be altered by alternative forms of utility functions. What would be different is the presence of a “wealth effect”. With an exponential utility function, portfolio holdings have the unrealistic characteristic not to depend on wealth. So the main implication of this exponential utility framework is the absence of this wealth effect. A perfectly realistic model should include wealth effects in the analysis. The advantage of using an exponential utility function is that one is able to derive closed-form solutions for all the variables in the model, which one could not do with traditional utility functions.

⁸ The normality assumption is made for convenience only. It makes the model easier to solve with the exponential utility function. It can indeed be relaxed, even within the exponential framework (see Gron, Jorgensen, and Polson (2000)). This assumption is also not needed with a quadratic utility function.

⁹ The exogeneity of the variance-covariance matrix is not fully realistic at higher frequencies. In reality, the covariance between assets from different countries is affected by the real exchange rate, especially at higher frequencies. At these frequencies, the relative price of tradables is a major cause of volatility. Our model does not reflect this fact, because of the assumption that all risky assets are paid in tradable goods, for which the law of one price holds. The model is therefore more relevant in describing lower frequency fluctuations, at which the law of one price holds reasonably well.

¹⁰ For an extensive study of financial macroeconomic models using an exponential utility function, see Davis and Willen (2000), and Davis, Nalewaik and Willen (2001).

B. Solution

We first briefly give the solution for consumptions. We will then discuss in greater detail the equilibrium risky asset prices and portfolio allocations.

Consumptions

The consumption in nontradables is given by the market-clearing condition for nontradables:

$$c_{NT,t} = y_{NT,t}$$

The consumption in tradables is given by:

$$c_{T,t} = \frac{r}{1+r} \left\{ \underbrace{\sum_{i=0}^{+\infty} \frac{1}{(1+r)^i} E_t(y_{T,t+i})}_{(1) \text{ endowment income wealth}} + \underbrace{R_0 \omega_{0,t-1} + R'_t \omega_{t-1}}_{(2) \text{ current financial income}} + \underbrace{\sum_{i=1}^{+\infty} \frac{1}{(1+r)^i} E_t(X'_{t+i} \omega_{t+i-1})}_{(3) \text{ future financial excess gains}} \right. \\ \left. - \underbrace{\frac{1}{r A_t} \text{Ln}[\delta \cdot (1+r)]}_{(4) \text{ consumption tilting}} - \underbrace{\frac{A_t}{2} \sum_{i=1}^{+\infty} \frac{1}{(1+r)^i} \text{var}_t c_{T,t+i}}_{(5) \text{ precautionary savings}} \right\},$$

where X_t is the $J \times 1$ vector of *excess returns* of the risky assets: $X_t = [R_{j,t} - R_0]_{j=1}^J$.

Equilibrium Risky Assets Prices and Portfolio Allocations

In this section, we derive closed-form solutions for equilibrium asset prices and portfolio allocations. In our framework, dynamics of asset prices is much richer than the mere fluctuation of present discounted value of expected dividends. Fluctuations in asset prices also depend on demographic dynamics, on variations in the variance and covariance of the risky assets' distribution, and on changes in the countries' risk aversion. A closed-form solution for portfolio holdings is also derived. It expresses in a very intuitive way how risky assets are allocated between countries.

Asset Prices

Risky assets prices are determined by their market-clearing equations: $\sum_{h=1}^H N_t n_t^h \omega_t^h = S_t$.

We have the following proposition:

Proposition. The equilibrium asset prices are given by:

$$P_t = \sum_{i=1}^{+\infty} \left(\frac{1}{1+r} \right)^i \left[E_t(d_{t+i}) - \frac{r}{1+r} A_{t+i}^{\#} \left(\frac{1}{N_{t+i-1}} \Delta \tilde{\Sigma}_{t+i} \Delta \Phi + \sum_{h=1}^H n_{t+i-1}^h \Psi^h \Delta \tilde{\beta}_{t+i}^h \right) \right],$$

where Δ is the diagonal $J \times J$ matrix with Λ_j ($j=1 \dots J$) on the diagonal. Λ_j is the present discounted value of dividends j implied by a unit dividend innovation¹¹.

In the same way, Ψ^h is the present discounted value of endowment income for country implied by a unit endowment innovation¹².

This can also be written as:

$$P_t = \sum_{i=1}^{+\infty} \left(\frac{1}{1+r} \right)^i \left\{ E_t(d_{t+i}) - \frac{r}{1+r} A_{t+i}^{\#} \left[\frac{1}{N_{t+i-1}} \left(\Lambda_k \Lambda_j \tilde{\Sigma}_{(k,j),t+i} \right)_{(k,j)=(1 \dots J)^2} \cdot \Phi + \sum_{h=1}^H n_{t+i-1}^h \Psi^h \left(\Lambda_j \tilde{\beta}_{j,t+i}^h \right)_{j=(1 \dots J)} \right] \right\}$$

Proof: see Appendix II.

Interpretation. The price of an asset includes two components:

- The traditional present discounted sum of expected future dividends. The more apples a tree produces, the higher its price.
- Minus the present discounted sum of a measure of risk:

$$\left(-\frac{r}{1+r} A_{t+i}^{\#} \left(\frac{1}{N_{t+i-1}} \Delta \tilde{\Sigma}_{t+i} \Delta \Phi + \sum_{h=1}^H n_{t+i-1}^h \Psi^h \Delta \tilde{\beta}_{t+i}^h \right) \right).$$

This measure of risk depends on several parameters. The higher the variance of the dividends, the riskier the asset, and the lower its price. But the more negative the covariance between the dividend process and the endowment of the countries, the more hedging the asset allows, and, as a consequence, the higher its price. This measure of risk is also affected by the average risk aversion of the agents, as well as by the availability of the asset (the scarcer, the more expensive).

¹¹ Formally, let $\lambda_{j,i}$ be the i^{th} coefficient in the moving average representation of the dividend process for asset

j . That is, $E_t(d_{j,t+i}) - E_t(d_{j,t+i}) = \lambda_{j,i} \eta_{j,t}$. Then, we define $\Lambda_j = \sum_{s=0}^{+\infty} \frac{1}{(1+r)^s} \lambda_{j,s}$.

¹² Formally, let ψ_i^h be the i^{th} coefficient in the moving average representation of the endowment process for

country h . That is, $E_t(y_{t+i}^h) - E_t(y_{t+i}^h) = \psi_i^h \varepsilon_t^h$. Then, we define $\Psi^h = \sum_{s=0}^{+\infty} \frac{1}{(1+r)^s} \psi_s^h$.

Demographic changes also have an impact on this term: if, for example, the population of countries with more negative betas increases faster than the rest of the world, then the asset price will increase. In a similar way, if the population of more risk-averse countries grows faster than the rest of the world, then the price of the asset will go down. The world population itself also plays a role: if total population grows while the number of shares stays constant, then the asset becomes scarcer, and its price goes up¹³.

Most of these parameters are time varying. As a consequence, the asset prices vary over time. These fluctuations are structural, and have nothing to do with speculation or irrationality. The expected financial excess gains are also determined by the same parameters (see expression below). To put it differently, the financial expected gains reflect the expected changes in the structural parameters such as demographic changes, changes in the variance and covariance of the dividends processes, changes in risk aversion, etc... In Mercereau (2003), some Granger propositions raised the possibility that the current account might help forecast future expected excess financial gains. This claim has to be put in light of our general equilibrium analysis. What the current account may reflect is the expected mid-and long-run changes in the structural parameters mentioned above. As these changes are likely to take place at low frequencies, this property is not in contradiction with an absence of arbitrage on the financial markets. The empirical question is then whether these predictable fluctuations are large enough to be significantly reflected by current account data. This is a point that we will leave for future research.

Portfolio Holding

We will derive an expression for the portfolio. This will illustrate better how risky assets are allocated between countries.

Proposition. Portfolio holding of country h is given by:

$$\omega_t^h = \underbrace{\frac{1+r}{rA_{t+1}^h} \sum_{t+1}^{-1} E_t X_{t+1}}_{\text{risk premium exploitation}} - \Psi^h \underbrace{\sum_{t+1}^{-1} \beta_{t-1}^h}_{\text{risk hedging}}.$$

This can also be written as:
$$\omega_t^h = \frac{A_{t+1}^h}{A_{t+1}^h} \left(\frac{S_t}{N_t} \right) + \sum_{t+1}^{-1} \left(\frac{A_{t+1}^h}{A_{t+1}^h} \sum_{k=1}^H n_t^k \Psi^k \beta_{t+1}^k - \Psi^h \beta_{t+1}^h \right).$$

In terms of fundamental parameters of the model, this gives:

¹³ This might partially explain the rise of the stock market in recent years: the share of households participating in the stock markets has significantly risen in the United States and in Europe. If it were the case that the number of shares available had not risen in the same proportion, then the asset prices had to go up in order to clear the market. A more detailed study would be needed to assess the empirical relevance of this argument.

$$\omega_t^h = \Pi_t \left[\frac{A_{t+1}^{\#}}{A_{t+1}^h} \frac{\Phi}{N_t} + \Delta^{-1} \left(\tilde{\Sigma}_{t+1} \right)^{-1} \frac{1}{A_{t+1}^h} \left(A_{t+1}^{\#} \sum_{k=1}^H n_t^k \Psi^k \tilde{\beta}_{t+1}^k - A_{t+1}^h \Psi^h \tilde{\beta}_{t+1}^h \right) \right].$$

Proof: see Appendix II.

Interpretation. The first expression for the portfolio is the traditional expression in a partial equilibrium. Portfolio holding includes two terms: a risk-premium exploitation term, which depends on expected excess returns, and a risk-hedging term.

The general equilibrium-expression is more interesting, because it shows how the available risky assets are allocated between countries at equilibrium. This expression includes several components. The first one means that a country owns a share of the total (per capita) stock of available assets. This share corresponds to the ratio of the world average risk aversion to its own risk aversion. More risk-averse countries will thus own a smaller share of the available risky assets. The second term reflects the fact that countries to which the risky assets offer more risk-hedging benefits than to the world average will own more risky assets. These risk-hedging benefits are characterized by the covariance of endowment and dividends' innovations, but also by the coefficient of risk aversion. More risk-averse countries indeed derive more utility from the opportunity to get rid of some risk.

Some Other Expressions

The expressions of some other variables are useful to interpret the equations for the exchange rate. They are listed in Appendix I.

We can now turn to the expression of the real exchange rate.

III. REAL EXCHANGE RATE IN AN INTERTEMPORAL N-COUNTRY MODEL WITH INCOMPLETE MARKETS

In this section, we will study the closed-form solution for the real exchange rate. This closed-form solution allows the study of the determinants of the real exchange rate, which include the stock market.

We contrast the predictions of the model to the Balassa-Samuelson effect. In a Balassa-Samuelson framework, a country whose relative productivity advantage in tradables is larger than its relative productivity advantage in nontradables should have a higher real exchange rate. Our model underscores that what matters in an inter-temporal framework is not the current productivity in tradables, but the **expectations** of future productivity. In the model, the real exchange rate is determined by the relative scarcity of tradable and nontradable goods. The model thus suggest a new transmission channel for parameters such as returns on net foreign assets, risk aversion, and risk-hedging opportunities. More precisely, the new effects are the following:

- Gross income on the net foreign position: the higher the gross income on the net foreign position of the country, the higher the real exchange rate.
- Risk structure: the risky assets quoted on a country's stock exchange have a certain cost in terms of risk. This cost corresponds to the variance of the dividends minus the risk-hedging benefits offered by the asset. Countries with lower structural risk cost should also have higher real exchange rate.
- Future financial gains: countries with larger expected financial gains should have a higher real exchange rate. The magnitude of these financial gains depends on parameters such as risk aversion, and risk-hedging opportunities. More risk-averse countries should have lower exchange rates, as should countries with less attractive risk-hedging opportunities.
- Discount rate: a more patient country should have a lower real exchange rate.

We are also able to derive a closed-form solution for the volatility of the real exchange rate. The volatility of the real exchange rate was rarely studied in a general equilibrium framework with incomplete markets. We find that exchange rate volatility depends on the asymmetries of the countries. This includes asymmetries in endowments' shocks and on shocks on dividends, but also in the risk-hedging opportunities of the two countries.

This section is organized as follows. Part A presents the expression for the real exchange rate; B discusses the determinants of the level of the real exchange rate; Part C focuses on exchange rate volatility.

Let us now study the model in greater detail.

A. Real Exchange Rate: Closed-Form Solution

The real exchange rate is defined as the ratio of the price of the nontradable goods in two countries. The Real Exchange Rate of Home (h) relative to Foreign (f) will, therefore, be given by:

$$RER_t = \frac{p_{NT,t}^h}{p_{NT,t}^f} = \frac{\frac{v^{h'}(c_{NT,t}^h)}{u^{h'}(c_{T,t}^h)}}{\frac{v^{f'}(c_{NT,t}^f)}{u^{f'}(c_{T,t}^f)}}, \text{ with the expression for consumption given above.}$$

In order to keep the analysis simple, we had assumed that the utility function in both tradables and nontradables are exponential, and that the agents have the same coefficient of absolute risk aversion for tradables and for nontradables: $u(c) = v(c) = \exp(-Ac)$.

The expression for the real exchange rate then becomes:

$$RER_t = \exp \left[\left(A_t^f c_{NT,t}^f - A_t^h c_{NT,t}^h \right) + \left(-A_t^f c_{T,t}^f + A_t^h c_{T,t}^h \right) \right].$$

It is useful to give an intuitive interpretation of what drives real exchange rate. The real exchange rate is the ratio of the price of nontradables relative to tradables in the two countries. For example, a rise in the price of nontradables in units of tradables at home corresponds to a real exchange rate appreciation (the price of home nontradables increases relative to the price of foreign nontradables). In the model, the price of nontradables in units of tradables is given by the ratio of marginal utilities for nontradables and tradables, respectively. The level and dynamics of the real exchange rate is, therefore, driven by **the factors, which make nontradables more or less desirable relative to tradables in both countries.**

What determines this relative desirability? The marginal utility of nontradables is the marginal utility of the country's endowment in nontradables. Consumption in tradables, on the other hand, has a more complex expression. As previously discussed, today's consumption depends on the expected present discounted of future endowments, on today's revenues on past financial investment, on the expected present discounted value of financial excess gains, on the country's discount rate, and on the variability of consumption. This has implications for both the level and the volatility of the real exchange rate. Let us first analyze the consequences for the real exchange rate level. We will then study the exchange rate volatility implied by our model.

B. Determinants of the Real Exchange Rate

In order to make the analysis easier, let us write $rer_t = \log(RER_t)$.

$$\text{We have: } rer_t = \left(A_t^f c_{NT,t}^f - A_t^h c_{NT,t}^h \right) + \left(-A_t^f c_{T,t}^f + A_t^h c_{T,t}^h \right).$$

Plugging in the expressions for consumption, we get:

$$rer_t = A_t^f y_{NT,t}^f - A_t^h y_{NT,t}^h$$

$$\begin{aligned}
 & \left. \begin{aligned}
 & \sum_{i=0}^{+\infty} \frac{1}{(1+r)^i} E_t (A_t^h y_{T,t+i}^h - A_t^f y_{T,t+i}^f) + \underbrace{\left[A_t^h (R_0 \omega_{0,t-1}^h + R'_t \omega_{t-1}^h) - A_t^f (R_0 \omega_{0,t-1}^f + R'_t \omega_{t-1}^f) \right]}_{\text{Revenue on past financial investments element}} \\
 & + \frac{r}{1+r} \left\{ \underbrace{\sum_{i=1}^{+\infty} \frac{1}{(1+r)^i} E_t \left[X'_{t+i} (A_t^h \omega_{t+i-1}^h - A_t^f \omega_{t+i-1}^f) \right]}_{\text{Future financial excess gains element}} - \underbrace{\frac{1}{r} Ln \left[(\delta^h - \delta^f) \cdot (1+r) \right]}_{\text{Discount rate element}} \right\} \\
 & - \underbrace{\left(\frac{(A_t^h)^2}{2} \sum_{i=1}^{+\infty} \frac{1}{(1+r)^i} \text{var}_t c_{T,t+i}^h - \frac{(A_t^f)^2}{2} \sum_{i=1}^{+\infty} \frac{1}{(1+r)^i} \text{var}_t c_{T,t+i}^f \right)}_{\text{Precautionary savings element}}
 \end{aligned} \right\}
 \end{aligned}$$

Let us now study each term of the above expression.

Endowment Dynamics and the Balassa-Samuelson Effect

Let us first study the role of endowments and how our model relates to the Balassa-Samuelson theory. Our model has the following implications:

- Countries with higher endowment in nontradables have lower exchange rate. This comes from the fact that nontradables are less scarce and, consequently, cheaper.
- In a similar way, countries with higher present discounted value of expected future tradables endowments have higher real exchange rates, since nontradables are relatively scarcer and, therefore, more expensive.

This is related to the Balassa-Samuelson effect.¹⁴ The latter states that countries whose relative productivity advantage in tradables is larger than their relative productivity advantage in nontradables should have higher real exchange rates. Our model's prediction is similar, though not identical.¹⁵

¹⁴ For a presentation of the Balassa-Samuelson effect, see, e.g., Obstfeld and Rogoff (1997), Chapter 4.

¹⁵ It should be noted that the channel of transmission in our model is very different from the Balassa-Samuelson framework. The Balassa-Samuelson theory is based on the assumptions about the labor market, with perfect cross-sector wage competition. In our model, there is no labor market, and the appreciation of the real exchange

(continued)

What matters in our model is not exactly the relative difference in productivity between tradables and nontradables. The key element is the relative difference between today's productivity in nontradables and an annuity of the expected present discounted value of future productivity in tradables. In other words, countries with a relative advantage in their expected present discounted value productivity in tradables larger than their relative current productivity advantage in nontradables should have higher real exchange rates.

This has a similar implication as the Balassa-Samuelson with respect to the relation between productivity growth and real exchange rate dynamics. A country whose advantage in productivity growth for tradables is larger than its advantage in productivity growth for nontradables should see its real exchange rate appreciate.

To illustrate how the predictions of the model differ from the Balassa-Samuelson effect, though, let us see what happens in the following case. All countries have the same productivity in both nontradables and tradables this year. But people know that from next year on, the productivity in the tradable sector will increase more in some countries than in others. What the Balassa-Samuelson theory predicts is that all countries will have the same real exchange rate (since the latter depends only on **current** productivity). Our model, on the other hand, suggests that countries, which will enjoy higher productivity in the tradable sector in future years, should have higher real exchange rate **now**. This stresses an important difference between our model and Balassa-Samuelson's: our model has inter-temporal implications.

This type of consideration may help explain what happened to the U.S. dollar in recent years. High **expectations** of future productivity gains (rather than realized gains) in tradables for the United States should have translated into a higher real exchange rate. This expectation of high productivity gains was at the heart of the "new economy" paradigm. Since these gains were supposed to be largest in high tech sectors such as electronics and telecom –which are tradable goods, the observed strength of the U.S. dollar is consistent with our story. Also, if the 2001 economic downturn was seen as temporary and it was accompanied by little change in expectations of future productivity gains, then the dollar was meant to remain strong.

Another difference with the traditional Balassa-Samuelson effect is that our framework is much richer. The Real exchange rate is thus determined by several factors other than relative productivities. Let us now analyze what these new factors are.

rate stemming from an endowment increase in current or future tradable goods is due to a relative increase in non-traded goods scarcity and the corresponding price increase in nontradables.

Revenue on Past Financial Investment

This term is important in understanding how stock markets can influence real exchange rate. It states that countries with higher income on their past financial investment should enjoy higher real exchange rates.

The analysis of this component will suggest a new transmission channel on the real exchange rate for some parameters, including:

- Gross income on the net foreign position of the country.
- Country's risk structure.

There is a subtlety to analyzing the role of revenue on past financial investment. The corresponding term does not distinguish between two things: the revenues on the country's net foreign position and the revenue on the country's own stocks. This makes comparisons between countries awkward. To put things differently, let us compare the situation of two countries, *A* and *B*. Countries *A* and *B* both have the same GDP. But country *A* has a larger share of its economy quoted on its stock market than country *B* has. As a consequence, the present discounted value sum of country *A*'s future endowments will be smaller than country *B*'s. On the other hand, country *B* should, ceteris paribus, have larger revenues on past financial investment, since a larger share of its economy is marketed.

This illustrates the fact that one cannot analyze the present value of future endowments and the revenues on past financial investments independently. Let us, therefore, rewrite these two terms in a way which will allow comparison between countries.

Proposition. The sum of present discounted value of future endowments and revenue on past financial investment is:

$$\sum_{i=0}^{+\infty} \frac{1}{(1+r)^i} E_t(y_{T,t+i}^h) + (R_0 \omega_{0,t-1}^h + R'_t \omega_{t-1}^h) = \underbrace{\sum_{i=0}^{+\infty} \left[\frac{1}{(1+r)^i} E_t(y_{T,t+i}^h + d'_{t+i} \frac{\Phi^h}{n_t^h N_t}) \right]}_{\text{PDV of production}}$$

$$+ \underbrace{R_0 \omega_{0,t-1}^h + R'_t (\omega_{t-1}^h - S_{t-1}^h)}_{\text{Gross income on net foreign assets}} - \underbrace{\frac{r}{1+r} \sum_{i=1}^{+\infty} \left(\frac{1}{1+r} \right)^i A_{t+i}^{\#} \left(\frac{1}{N_{t+i-1}} \Delta \tilde{\Sigma}_{t+i} \Delta \Phi + \sum_{k=1}^H n_{t+i-1}^k \Psi^k \Delta \tilde{\beta}_{t+i}^k \right)}_{\text{Structural risk cost}}, \frac{\Phi^h}{n_t^h N_t}$$

where $\Phi^h = \left(\begin{array}{c} \phi_j^h \text{ if } j \text{ is a company located in } h \\ 0 \text{ otherwise} \end{array} \right)_{j=1 \dots J}$.

Proof: see Appendix II.

We see that the sum can be divided into three terms, all of them comparable across countries.

The first component is the present discounted value of per capita production (production is split between the country's endowment and the dividends paid by the companies located in the country). This term corresponds to our previous discussion on the Balassa-Samuelson effect.

The second term is the gross income on net foreign assets. It states that a country with a larger gross income on its net foreign position will have a higher real exchange rate. It is related to, but different from, the prediction of traditional models in which countries with net foreign debt should have a depreciated real exchange rate in order to be able to pay back their debts. In such a model, the reason why the country should have a depreciated currency is that it would give it a competitive advantage on world markets, which in turn would translate into a positive trade balance. Here, the mechanism is different. What drives the real exchange rate is the relative scarcity of tradables. A country with larger gross income on its net foreign assets is richer in tradables, and, as a consequence, nontradables will be relatively more expensive.

Finally, it is interesting to note that what matters is not the net foreign position in itself. What matters is the total gross income on the country's net foreign position (this is, the product of net foreign position by the gross return on these assets). If, for any reason (for example a stock-market crash), a creditor country has negative returns on its net foreign position a given year, then its real exchange rate would be lower that year, in spite of its positive net foreign position.

The third component reflects the country's risk structure. It relates to the cost in terms of risk of the country's assets. This cost includes the variance of the dividends paid by the country, minus the average risk-hedging benefits provided by these assets. The larger the dividends' variance, the larger the cost. But the larger the risk-hedging benefits the assets provide to the world, the lower this cost. What this fundamentally means is that countries with a more favorable risk-structure will *ceteris paribus* be able to consume more than other countries. Intuitively, if a country *h* possesses risky assets that have high risk-hedging value for other countries, then these other countries will be ready to pay country *h* to own some of these assets. In other words, country *h* will receive real resources from abroad in exchange for the risk-hedging benefits its assets supply.

Consequently, countries with a relatively favorable asset risk-structure will have a higher real exchange rate. This is a potential determinant of the exchange rate that is not present in the literature. It could be tested empirically.

Future Financial Excess Gains

Expectations of future stock-market developments also influence the real exchange rate through this third term. All else being equal, countries that expect higher excess financial gains on their portfolio will have a higher real exchange rate. This comes from the fact that agents of these countries feel wealthier, and consume more as a consequence. The relative price of traded goods is thus lower.

Let us see the mechanism in greater detail. We will see that more risk-averse countries should have lower exchange rates, as should countries, which can hedge risk less efficiently.

Let us now study the term $E_t \left[X'_{t+i+1} (\omega_{t+i}^h - \omega_{t+i}^f) \right]$.

We have:

$$E_t \left(X'_{t+i+1} \omega_{t+i}^h \right) = \frac{r}{1+r} \cdot A_{t+i+1}^{\#} \left(\frac{1}{N_{t+i}} \Phi' \Delta \tilde{\Sigma}_{t+i+1} \Delta + \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}_{t+i+1}^k \Delta \right) \cdot \left[\frac{A_{t+i+1}^{\#}}{A_{t+i+1}^h} \frac{\Phi}{N_{t+i}} + \Delta^{-1} (\tilde{\Sigma}_{t+i+1})^{-1} \left(\frac{A_{t+i+1}^{\#}}{A_{t+i+1}^h} \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}_{t+i+1}^k - \Psi^h \tilde{\beta}_{t+i+1}^h \right) \right] \quad \forall i \geq 1$$

Therefore,

$$E_t \left[X'_{t+i+1} (\omega_{t+i}^h - \omega_{t+i}^f) \right] = \underbrace{\frac{r}{1+r} \cdot A_{t+i+1}^{\#} \left(\frac{1}{N_{t+i}} \Phi' \Delta \tilde{\Sigma}_{t+i+1} \Delta + \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}_{t+i+1}^k \Delta \right)}_{(1)}$$

$$\left[\underbrace{\left(\frac{\Phi}{N_{t+i}} A_{t+i+1}^{\#} + \Delta^{-1} (\tilde{\Sigma}_{t+i+1})^{-1} A_{t+i+1}^{\#} \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}_{t+i+1}^k \right)}_{(2)} \left(\frac{1}{A_{t+i+1}^h} - \frac{1}{A_{t+i+1}^f} \right) + \underbrace{\Delta^{-1} (\tilde{\Sigma}_{t+i+1})^{-1} (-\Psi^h \tilde{\beta}_{t+i+1}^h + \Psi^f \tilde{\beta}_{t+i+1}^f)}_{(3)} \right] \quad \forall i \geq 1$$

Before pursuing the analysis, let us first note that all the J elements of the $J \times 1$ term (1) vector are usually positive. Term (1) indeed corresponds to the expected excess returns at equilibrium. Although this term can theoretically be negative for some values of the parameters, this is very unlikely to be the case in practice, since this would correspond to a negative risk-premium, which is not observed.

Term (2). The first element once again corresponds to the risk premium, which is usually positive. As a consequence, term (2) reflects the fact that more risk-averse countries should have a lower real exchange rate¹⁶. Indeed, if agents in country h are more risk-averse than

¹⁶ In the analysis, we assume that the elements of Δ are positive (this is usually the case since dividends are usually serially auto-correlated).

these in f , then they hold fewer risky assets. Consequently, they will earn less in excess financial gains in the future, and they will, therefore, consume less today. This has a negative effect on the real exchange rate of h relative to f .

Term (3) comes from the fact that countries for which assets offer more risk-hedging benefits will possess more risky assets. The consequence is that the country will expect to enjoy higher excess financial gains. It will, therefore, consume more now, and the country will have a higher real exchange rate.

To conclude, countries with larger expected financial gains should have a higher real exchange rate. The magnitude of these financial gains depends on parameters such as risk aversion and risk-hedging opportunities. More risk-averse countries should have lower exchange rates, as should countries which hedge risk less efficiently.

Discount Rate

The more patient the country, the less it will consume today. Thus, the more valuable tradable goods are relative to nontradables. Consequently, more patient countries will, *ceteris paribus*, have a lower real exchange rate¹⁷.

Consumption Volatility (Precautionary Saving)

Another way the stock market can influence real exchange rate is through the effect of risky assets on consumption volatility. In order to protect themselves against the volatility of future consumption, agents will engage in precautionary saving. The larger this effect is, the less the agents will consume, and the lower the exchange rate. Purchasing risky assets, be it for risk-premium exploitation or risk-hedging motives, will influence the volatility of the agent's consumption. Let us now study this in greater detail.

Proposition

$$\text{var}(c_t^h) = \underbrace{\frac{r}{1+r} \Psi^{h2} \left(\theta_t^h - \tilde{\beta}_t^h, (\tilde{\Sigma}_t)^{-1} \tilde{\beta}_t^h \right)}_{\text{undiversifiable part of idiosyncratic risk}} + \underbrace{\left(\frac{r}{1+r} \frac{A_t^{\#}}{A_t^h} \right)^2 \left(\frac{1}{N_{t-1}} \Delta \tilde{\Sigma}_t \Delta \Phi + \sum_{k=1}^H n_{t-1}^k \Psi^k \Delta \tilde{\beta}_t^k \right)' \Delta^{-1} (\tilde{\Sigma}_t)^{-1} \Delta^{-1} \left(\frac{1}{N_{t-1}} \Delta \tilde{\Sigma}_t \Delta \Phi + \sum_{k=1}^H n_{t-1}^k \Psi^k \Delta \tilde{\beta}_t^k \right)}_{\text{premium exploitation risk}}$$

¹⁷ A similar effect is found in Stockman (1987).

Proof: see Appendix II.

This proposition shows that consumption volatility has two sources. One source is the part of the endowment risk that the country could not hedge using risky assets. The other one is the extra-risk associated with the purchase of risky assets for premium exploitation purposes. This second term also depends on the risk aversion of the country relative to the world average risk aversion. This reflects the fact that at equilibrium less risk-averse countries will own a larger share of the risky assets available in the world.

The consequence of this for the real exchange rate is that countries with a larger undiversifiable part of idiosyncratic risk should have lower real exchange rate (the risk premium exploitation term does not matter for the real exchange rate, since the corresponding component in $A_t^{h^2} \text{var}(c_t^h) - A_t^{f^2} \text{var}(c_t^f)$ is equal to zero). Intuitively, this comes from the fact that these countries will save more in order to protect themselves against the future variability of their income. Consequently, the price of nontradables in terms of tradables will be higher.

C. Exchange Rate Volatility

Let us now turn to the volatility of the real exchange rate. It will allow us to study how the stock market influences real exchange-rate volatility. This has received a lot of non-analytical attention, but was rarely studied in a general equilibrium framework. In our framework, exchange rate volatility depends on the asymmetries of the countries. They include asymmetries in endowments' shocks and on shocks on dividends, but also in the risk-hedging opportunities of the two countries.

In order to make the analysis simpler, we will focus on the volatility of the logarithm of the real exchange rate, $rer_t = \log(RES_t)$ ¹⁸.

$$\text{Var}(rer_t) = \text{Var} \left[\left(A_t^f c_{NT,t}^f - A_t^h c_{NT,t}^h \right) + \left(-A_t^f c_{T,t}^f + A_t^h c_{T,t}^h \right) \right].$$

¹⁸ It is possible to study the volatility of the real exchange rate directly, at a heavy cost in terms of notation and complexity. Since this has no obvious pay-off, we will stick to the analysis in logarithmic terms. Note that the variance of the real exchange rate is a direct function of the variance of its logarithm. We indeed have:

$\text{Var}(RES_t) = \text{Var}[\exp(rer_t)]$. Under our normality assumption, we can use the fact that for any Gaussian variable x_t , we have: $E_{t-1}[\exp(Ax_t)] = \exp[AE_{t-1}(x_t)] + \frac{A^2}{2} \text{Var}_{t-1}(x_t)$. After some algebra, the following expression can then be derived: $\text{Var}_{t-1}(RES_t) = \text{Var}_{t-1}(rer_t) \left\{ 2 - \exp[E_{t-1}(rer_t)] - \frac{1}{4} \text{Var}_{t-1}(rer_t) \right\}$. This formula could be useful for empirical applications of the model.

Plugging the expressions for consumption in and noting that terms that are non-stochastic at time $t-1$ play no role in the variance, we get:

$$Var(rer_t) = Var \left[\begin{aligned} & \left(A_t^f y_{NT,t}^f - A_t^h y_{NT,t}^h \right) + \frac{r}{1+r} \sum_{i=0}^{+\infty} \left\{ \frac{1}{(1+r)^i} E_{t-1} \left(A_t^h y_{T,t+i}^h - A_t^f y_{T,t+i}^f \right) \right\} \\ & + \frac{r}{1+r} R'_t \left(A_t^h \omega_{t-1}^h - A_t^f \omega_{t-1}^f \right) \end{aligned} \right]$$

After some algebra, we find the following expression for the variance of rer (the proof is given in Appendix II):

$$Var(rer_t) = Var \left[\begin{aligned} & \left(A_t^f \varepsilon_{NT,t}^f - A_t^h \varepsilon_{NT,t}^h \right) + \frac{r}{1+r} \left(A_t^h \psi^h \varepsilon_{T,t}^h - A_t^f \psi^f \varepsilon_{T,t}^f \right) + \frac{r}{1+r} \eta'_t \left(\tilde{\Sigma}_t \right)^{-1} \left(-\Psi^h A_t^h \tilde{\beta}_t^h + \Psi^f A_t^f \tilde{\beta}_t^f \right) \end{aligned} \right]$$

The variance of the real exchange rate is determined by the variance and covariance of the difference between the two countries in the innovations in nontradables and tradables endowments, as well as the shocks on dividends¹⁹. To put it simply, the larger the shocks and the more asymmetric the countries, the higher the real exchange rate volatility. This is in particular true for the last term. It shows that the impact of dividend shocks is larger the more different the risk-hedging benefits for the two countries. This is due to the fact that an important difference between risk-hedging benefits translates into very different holdings in risky assets. As a consequence, shocks in the returns on risky assets will have a larger impact on the real exchange rate.

It is straightforward to develop the above expression into the fundamental parameters of the model (all that needs to be done is to develop the variance into a sum of covariances). This would allow a closed-form solution for the variance of the real exchange rate in terms of the parameters of the model. Such an expression could be useful for empirical applications of the model.

This also shows that the real exchange rate depends on structural parameters of the stock market. It is, to my knowledge, the only closed-form solution of the volatility of the exchange rate as a function of the parameters of the stock market. It could be used for empirical analysis.

¹⁹ All the shocks are magnified by the coefficient of risk aversion. One should indeed recall that real exchange rate is derived from consumption patterns, which depend on risk aversion.

IV. CONCLUSION

We develop a model with stock markets in which a closed-form solution for the real exchange rate is derived. Our model allows for a given number of risky assets, which form an incomplete market. Risky-asset prices and allocations of risky assets among countries are determined endogenously. It is one of the very few existing models of the role of stock markets on exchange rate determination, in which risky-asset prices are endogenous. The risk-free rate is exogenous, so our model is an intermediate step toward a full general equilibrium. Our framework allows an analysis of how fundamental parameters, such as the variance and covariance of the risky assets or demographic variables affect the real exchange rate. We contrast the predictions of the model with the Balassa-Samuelson effect. We also suggest a new transmission channel of the real exchange rate for parameters such as income on net foreign assets, risk aversion, and risk-hedging opportunities. To test some of these propositions empirically would be interesting tasks for future research. The fact that we have closed-form solutions for all variables in the model should greatly facilitate future empirical work on the role of stock markets in real exchange rate determination.

Appendix I. Some Useful Expressions

Let us also give briefly here the expression of some variables in function only of parameters and asset prices. This will be useful to assess the impact of changes of these parameters on the real exchange rate.

- The equilibrium expected excess returns are:

$$E_t X_{t+1} = \frac{r}{1+r} \cdot A_{t+1}^\# \left(\Sigma_{t+1} \frac{S_t}{N_t} + \sum_{h=1}^H n_t^h \Psi^h \beta_{t+1}^h \right), \quad \text{where } A_t^\# = \left(\sum_{h=1}^H \frac{n_t^h}{A_t^h} \right)^{-1}.$$

In terms of fundamental parameters and asset prices, this gives:

$$E_t X_{t+1} = \frac{r}{1+r} \cdot A_{t+1}^\# \Pi_t^{-1} \left(\Delta \tilde{\Sigma}_{t+1} \Delta \cdot \frac{\Phi}{N_t} + \sum_{h=1}^H n_t^h \Psi^h \Delta \tilde{\beta}_{t+1}^h \right),$$

where Π_t and Δ are the $J \times J$ diagonal matrixes with $P_{j,t}$ and Λ_j ($j=1 \dots J$) on their diagonals respectively.

Proof: see Appendix II.

$$E_t \left(X'_{t+i+1} \omega_{t+i}^h \right) = \frac{r}{1+r} \cdot A_{t+i+1}^\# \left(\frac{1}{N_{t+i}} \Phi' \Delta \tilde{\Sigma}_{t+i+1} \Delta + \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}'_{t+i+1} \Delta \right) \cdot \left[\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \frac{\Phi}{N_{t+i}} + \Delta^{-1} \left(\tilde{\Sigma}_{t+i+1} \right)^{-1} \left(\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}_{t+i+1}^k - \Psi^h \tilde{\beta}_{t+i+1}^h \right) \right] \quad \forall i \geq 1$$

- $S_t = (\phi_j P_{j,t})_{j=1 \dots J}$.

or $S_t = \Pi_t \Phi$, where Π_t is the $J \times J$ diagonal matrix with $P_{j,t}$ on the diagonal, and Φ the $J \times 1$ vector of $\phi_{j,t}$.

- $\beta_{t+1}^h = \left[\frac{\Lambda_j}{P_{j,t}} \tilde{\beta}_{j,t+1}^h \right]_{j=1 \dots J}$,

or $\beta_{t+1}^h = \Pi_t^{-1} \Delta \tilde{\beta}_{t+1}^h$, where Π_t and Δ are the $J \times J$ diagonal matrixes with $P_{j,t}$ and Λ_j ($j=1 \dots J$) on their diagonals respectively.

- $\Sigma_{(i,j)t+1} = \frac{\Lambda_j \Lambda_i}{P_{j,t} \cdot P_{i,t}} \tilde{\Sigma}_{(i,j)t+1}$.

In matrix form, this gives: $\Sigma_{t+1} = \Pi_t^{-1} \Delta \tilde{\Sigma}_{t+1} \Delta \Pi_t^{-1}$.

Proofs: all the proofs are given in Appendix II.

Appendix II. Proofs

The solution of the model can be found in Mercereau (2003). We only give here the proofs of the new propositions.

It should be kept in mind that throughout the paper primed variables denote the transpose of the corresponding vector. (e.g., Z' is the transpose of vector Z).

Portfolio holding with ARIMA stochastic processes

Proposition. The risky assets portfolio is given by:

$$\omega_t^h = \underbrace{\frac{1+r}{rA_{t+1}^h} \Sigma_{t+1}^{-1} E_t X_{t+1}}_{\text{risk premium exploitation}} - \Psi^h \underbrace{\Sigma_{t+1}^{-1} \beta_{t+1}^h}_{\text{risk hedging}} \quad \text{equation (A1)}^{20}.$$

Proof. We guess the equilibrium portfolio allocation, and then we verify that it is solution to the problem.

The guess is given by equation (A1).

We have to verify that it satisfies the rearranged Euler equation:

$$E_t (X_{j,t+1}) = A_{t+1}^h \text{cov}_t [R_{j,t+1}; c_{t+1}^h].$$

Let us first recall the expression of consumption:

²⁰ Remember that Ψ^h is the present discounted value of endowment income for country implied by a unit endowment innovation. Formally, let ψ_i^h be the i^{th} coefficient in the moving average representation of the endowment process for country h. That is, $E_t (y_{t+i}^h) - E_t (y_{t+i}^h) = \psi_i^h \varepsilon_t^h$. Then, we define

$$\Psi^h = \sum_{s=0}^{+\infty} \frac{1}{(1+r)^s} \psi_s^h.$$

$$c_t^h = \frac{r}{1+r} \left\{ \underbrace{\sum_{i=0}^{+\infty} \frac{1}{(1+r)^i} E_t(y_{t+i}^h)}_{(1) \text{ labor income wealth}} + \underbrace{R_0 \omega_{0,t-1}^h + R'_t \omega_{t-1}^h}_{(2) \text{ current financial income}} + \underbrace{\sum_{i=1}^{+\infty} \frac{1}{(1+r)^i} E_t(X'_{t+i} \omega_{t+i-1}^h)}_{(3) \text{ future financial excess gains}} \right. \\ \left. - \underbrace{\frac{1}{r \cdot A_t^h} \text{Ln}[\delta^h \cdot (1+r)]}_{(4) \text{ consumption tilting}} - \underbrace{\frac{A_t^h}{2} \sum_{i=1}^{+\infty} \frac{1}{(1+r)^i} \text{var}_t c_{t+i}^h}_{(5) \text{ precautionary savings}} \right\}$$

Plugging consumption into the above rearranged Euler equation yields:

$$E_t(X_{j,t+1}) = A_{t+1}^h \text{cov}_t [R_{j,t+1}; \Psi^h y_{t+1}^h + R'_{t+1} \omega_t^h].$$

This can be rewritten in matrix form as:

$$\frac{A_{t+1}^h r}{1+r} [\Sigma_{t+1} \omega_t^h + \Psi^h \beta_{t+1}^h] = E_t X_{t+1},$$

$$\text{or } \omega_t^h = \underbrace{\frac{1+r}{r A_{t+1}^h} \Sigma_{t+1}^{-1} E_t X_{t+1}}_{\text{risk premium exploitation}} - \underbrace{\Psi^h \Sigma_{t+1}^{-1} \beta_{t+1}^h}_{\text{risk hedging}}, \text{ which was our initial guess.}$$

Equilibrium excess returns and asset prices

Proposition. The equilibrium asset prices are given by:

$$\Pi_t = \sum_{i=1}^{+\infty} \left(\frac{1}{1+r} \right)^i \left[E_t(d_{t+i}) - \frac{r}{1+r} \cdot A_{t+i}^{\#} \left(\frac{1}{N_{t+i-1}} \Delta \tilde{\Sigma}_{t+i} \Delta \cdot \Phi + \sum_{h=1}^H n_{t+i-1}^h \Psi^h \Delta \tilde{\beta}_{t+i}^h \right) \right].$$

Proof. The proof includes three steps:

- 1- Guess the equilibrium asset prices.
- 2- Use the market clearing condition for risky asset to derive the expression of the equilibrium expected excess returns.
- 3- Verify that the guessed asset prices satisfy the expression found for expected excess returns.

1- Guess: the equilibrium asset prices are given by the above proposition.

2- Let us use the market clearing condition for risky assets to derive the expression of the equilibrium expected excess returns:

$$\sum_{h=1}^H N_t n_t^h \omega_t^h = S_t.$$

Recalling that $\omega_t^h = \underbrace{\frac{1+r}{rA_{t+1}^h} \Sigma_{t+1}^{-1} E_t X_{t+1}}_{\text{risk premium exploitation}} - \Psi^h \underbrace{\Sigma_{t+1}^{-1} \beta_{t+1}^h}_{\text{risk hedging}}$, and rearranging the terms, we find

the equilibrium excess returns:

$$E_t X_{t+1} = \frac{r}{1+r} A_{t+1}^{\#} \left(\Sigma_{t+1} \frac{S_t}{N_t} + \sum_{h=1}^H n_t^h \Psi^h \beta_{t+1}^h \right).$$

Let us now find the expression of excess returns in terms of fundamentals of the model (e.g., the second moments of dividends innovations rather than of returns) and of asset prices. Let us first derive such an expression for each of its terms:

- $S_t = (\phi_j P_{j,t})_{j=1 \dots J}$.

$S_t = \Pi_t \Phi$, where Π_t is the $J \times J$ diagonal matrix with $P_{j,t}$ on the diagonal, and Φ the $J \times 1$ vector of $\phi_{j,t}$.

- $\beta_{t+1}^h = [\text{cov}_t(y_{t-1}^h; R_{j,t+1})]_{j=1 \dots J}$.

$$\beta_{t+1}^h = \left[\text{cov}_t \left(\varepsilon_{t+1}^h; \frac{d_{j,t+1} + P_{j,t+1}}{P_{j,t}} \right) \right]_{j=1 \dots J}$$

Using our guessed asset prices, this gives:

$$\beta_{t+1}^h = \left[\Lambda_j \text{cov}_t \left(\varepsilon_{t+1}^h; \frac{\eta_{j,t+1}}{P_{j,t}} \right) \right]_{j=1 \dots J}$$

$$\beta_{t+1}^h = \left[\frac{\Lambda_j}{P_{j,t}} \tilde{\beta}_{j,t+1}^h \right]_{j=1 \dots J},$$

or $\beta_{t+1}^h = \Pi_t^{-1} \Delta \tilde{\beta}_{t+1}^h$, where Π_t and Δ are the $J \times J$ diagonal matrixes with $P_{j,t}$ and Λ_j ($j = 1 \dots J$) on their diagonals respectively.

- $\Sigma_{t+1} = [\text{cov}_t(R_{j,t+1}; R_{i,t+1})]_{(i,j)=(1 \dots J)^2}$.

$$\Sigma_{(i,j)t+1} = \text{cov}_t \left(\frac{d_{j,t+1} + P_{j,t+1}}{P_{j,t}}; \frac{d_{i,t+1} + P_{i,t+1}}{P_{i,t}} \right).$$

Using our guess for asset prices, this yields:

$$\Sigma_{(i,j),t+1} = \frac{\Lambda_j \Lambda_i}{P_{j,t} P_{i,t}} \text{cov}_t(\eta_{j,t+1}; \eta_{i,t+1}),$$

$$\text{or } \Sigma_{(i,j),t+1} = \frac{\Lambda_j \Lambda_i}{P_{j,t} P_{i,t}} \tilde{\Sigma}_{(i,j),t+1}.$$

In matrix form, this gives:

$$\Sigma_{t+1} = \Pi_t^{-1} \Delta \tilde{\Sigma}_{t+1} \Delta \Pi_t^{-1}.$$

- Bringing all the pieces together, we get the following expression for expected excess returns:

$$E_t X_{t+1} = \frac{r}{1+r} A_{t+1}^\# \Pi_t^{-1} \left(\frac{1}{N_t} \Delta \tilde{\Sigma}_{t+1} \Delta \Phi + \sum_{h=1}^H n_t^h \Psi^h \Delta \tilde{\beta}_{t+1}^h \right).$$

3 - Verify that the guessed asset prices satisfy the expression found for expected excess returns

By definition, the expected excess return of asset j is:

$$E_t X_{j,t+1} = E_t \left[\frac{P_{j,t+1} + d_{j,t+1}}{P_{j,t}} - (1+r) \right],$$

$$\text{or } E_t X_{j,t+1} = E_t \left[\frac{P_{j,t+1} + d_{j,t+1} - (1+r) P_{j,t}}{P_{j,t}} \right].$$

But using our guess for the asset prices, we have:

$$E_t \left[\frac{P_{j,t+1} + d_{j,t+1} - (1+r) P_{j,t}}{P_{j,t}} \right] = -\frac{r}{1+r} A_{t+1}^\# \left[\frac{1}{N_t} (\Lambda_k \Lambda_j \tilde{\Sigma}_{(k,j),t+1})_{k=(1 \dots J)} \cdot \Phi + \sum_{h=1}^H n_t^h \Psi^h (\Lambda_j \tilde{\beta}_{j,t+1}^h) \right].$$

Put in matrix format, this yields:

$$E_t X_{t+1} = \frac{r}{1+r} A_{t+1}^\# \Pi_t^{-1} \left(\frac{1}{N_t} \Delta \tilde{\Sigma}_{t+1} \Delta \Phi + \sum_{h=1}^H n_t^h \Psi^h \Delta \tilde{\beta}_{t+1}^h \right).$$

This what we had found in part 2 for the expected excess returns. The proof is, therefore, complete. \square

Other expressions for portfolio holdings

- *Proposition.* $\omega_t^h = \frac{A_{t+1}^\#}{A_{t+1}^h} \left(\frac{S_t}{N_t} \right) + \Sigma_{t+1}^{-1} \left[\frac{A_{t+1}^\#}{A_{t+1}^h} \sum_{k=1}^K (n_t^k \Psi^k \beta_{t+1}^k) - \Psi^h \beta_{t+1}^h \right].$

Proof. We had:

$$\omega_t^h = \underbrace{\frac{1+r}{rA_{t+1}^h} \Sigma_{t+1}^{-1} E_t X_{t+1}}_{\text{risk premium exploitation}} - \underbrace{\Psi^h \Sigma_{t+1}^{-1} \beta_{t+1}^h}_{\text{risk hedging}},$$

$$\text{and } E_t X_{t+1} = \frac{r}{1+r} A_{t+1}^{\#} \left(\Sigma_{t+1} \frac{S_t}{N_t} + \sum_{k=1}^H n_t^k \Psi^k \beta_{t+1}^k \right).$$

$$\text{As a consequence, } \omega_t^h = \frac{A_{t+1}^{\#}}{A_{t+1}^h} \frac{S_t}{N_t} + \Sigma_{t+1}^{-1} \frac{A_{t+1}^{\#}}{A_{t+1}^h} \sum_{k=1}^H (n_t^k \Psi^k \beta_{t+1}^k) - \Psi^h \Sigma_{t+1}^{-1} \beta_{t+1}^h,$$

$$\text{which gives: } \omega_t^h = \frac{A_{t+1}^{\#}}{A_{t+1}^h} \left(\frac{S_t}{N_t} \right) + \Sigma_{t+1}^{-1} \left[\frac{A_{t+1}^{\#}}{A_{t+1}^h} \sum_{k=1}^H (n_t^k \Psi^k \beta_{t+1}^k) - \Psi^h \beta_{t+1}^h \right].$$

- *Proposition.* $\omega_t^h = \Pi_t \left[\frac{A_{t+1}^{\#}}{A_{t+1}^h} \frac{\Phi}{N_t} + \Delta^{-1} (\tilde{\Sigma}_{t+1})^{-1} \left(\frac{A_{t+1}^{\#}}{A_{t+1}^h} \sum_{k=1}^H n_t^k \Psi^k \tilde{\beta}_{t+1}^k - \Psi^h \tilde{\beta}_{t+1}^h \right) \right].$

Proof. We have: $\omega_t^h = \frac{A_{t+1}^{\#}}{A_{t+1}^h} \frac{S_t}{N_t} + \Sigma_{t+1}^{-1} \frac{A_{t+1}^{\#}}{A_{t+1}^h} \sum_{k=1}^H (n_t^k \Psi^k \beta_{t+1}^k) - \Psi^h \Sigma_{t+1}^{-1} \beta_{t+1}^h.$

But $S_t = \Pi_t \Phi,$

$$\beta_{t+1}^h = \Pi_t^{-1} \Delta \tilde{\beta}_{t+1}^h,$$

and $\Sigma_{t+1} = \Pi_t^{-1} \Delta \tilde{\Sigma}_{t+1} \Delta \Pi_t^{-1}.$

Plugging this in, it yields:

$$\omega_t^h = \frac{A_{t+1}^{\#}}{A_{t+1}^h} \Pi_t \frac{\Phi}{N_t} + \Pi_t \Delta^{-1} (\tilde{\Sigma}_{t+1})^{-1} \Delta^{-1} \Pi_t \left(\frac{A_{t+1}^{\#}}{A_{t+1}^h} \sum_{k=1}^H n_t^k \Psi^k \Pi_t^{-1} \Delta \tilde{\beta}_{t+1}^k - \Psi^h \Pi_t^{-1} \Delta \tilde{\beta}_{t+1}^h \right).$$

And we, therefore, have:

$$\omega_t^h = \Pi_t \left[\frac{A_{t+1}^{\#}}{A_{t+1}^h} \frac{\Phi}{N_t} + \Delta^{-1} (\tilde{\Sigma}_{t+1})^{-1} \left(\frac{A_{t+1}^{\#}}{A_{t+1}^h} \sum_{k=1}^H n_t^k \Psi^k \tilde{\beta}_{t+1}^k - \Psi^h \tilde{\beta}_{t+1}^h \right) \right] \square$$

Other propositions

- *Proposition.*

$$E_t \left(X'_{t+i+1} \omega_{t+i}^h \right) = \frac{r}{1+r} \cdot A_{t+i+1}^\# \left(\frac{1}{N_{t+i}} \Phi' \Delta \tilde{\Sigma}_{t+i+1} \Delta + \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}'_{t+i+1} \Delta \right) \cdot \left[\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \frac{\Phi}{N_{t+i}} + \Delta^{-1} \left(\tilde{\Sigma}_{t+i+1} \right)^{-1} \left(\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}_{t+i+1}^k - \Psi^h \tilde{\beta}_{t+i+1}^h \right) \right] \quad \forall i \geq 1$$

Proof. By definition, $X_{j,t+i+1} = \frac{P_{j,t+i+1} + d_{j,t+i+1} - (1+r)P_{j,t+i}}{P_{j,t+i}}$.

Therefore, $X'_{t+i+1} = \left(P_{j,t+i+1} + d_{j,t+i+1} - (1+r)P_{j,t+i} \right)'_{j=1 \dots J} \Pi_{t+i}^{-1}$.

Moreover, $\omega_{t+i}^h = \Pi_{t+i} \left[\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \frac{\Phi}{N_{t+i}} + \Delta^{-1} \left(\tilde{\Sigma}_{t+i+1} \right)^{-1} \left(\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}_{t+i+1}^k - \Psi^h \tilde{\beta}_{t+i+1}^h \right) \right]$,

$$E_t \left(X'_{t+i+1} \omega_{t+i}^h \right) = \frac{r}{1+r} \cdot A_{t+i+1}^\# \left(\frac{1}{N_{t+i}} \Phi' \Delta \tilde{\Sigma}_{t+i+1} \Delta + \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}'_{t+i+1} \Delta \right).$$

Thus,

$$\left[\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \frac{\Phi}{N_{t+i}} + \Delta^{-1} \left(\tilde{\Sigma}_{t+i+1} \right)^{-1} \left(\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}_{t+i+1}^k - \Psi^h \tilde{\beta}_{t+i+1}^h \right) \right] \quad \forall i \geq 1$$

and

$$E_t \left(X'_{t+i+1} \omega_{t+i}^h \right) = E_t \left\{ \left(P_{j,t+i+1} + d_{j,t+i+1} - (1+r)P_{j,t+i} \right)'_{j=1 \dots J} \cdot \left[\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \frac{\Phi}{N_{t+i}} + \Delta^{-1} \left(\tilde{\Sigma}_{t+i+1} \right)^{-1} \left(\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}_{t+i+1}^k - \Psi^h \tilde{\beta}_{t+i+1}^h \right) \right] \right\}.$$

We can, therefore, go back to our proof for expected excess returns, and we find:

$$E_t \left(X'_{t+i+1} \omega_{t+i}^h \right) = \frac{r}{1+r} \cdot A_{t+i+1}^\# \left(\frac{1}{N_{t+i}} \Phi' \Delta \tilde{\Sigma}_{t+i+1} \Delta + \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}'_{t+i+1} \Delta \right) \cdot \left[\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \frac{\Phi}{N_{t+i}} + \Delta^{-1} \left(\tilde{\Sigma}_{t+i+1} \right)^{-1} \left(\frac{A_{t+i+1}^\#}{A_{t+i+1}^h} \sum_{k=1}^H n_{t+i}^k \Psi^k \tilde{\beta}_{t+i+1}^k - \Psi^h \tilde{\beta}_{t+i+1}^h \right) \right] \quad \forall i \geq 1 \quad \square$$

- *Proposition.* The sum of present discounted value of future endowments and revenue on past financial investment is:

$$\sum_{i=0}^{+\infty} \frac{1}{(1+r)^i} E_t(y_{T,t+i}^h) + (R_0 \omega_{0,t-1}^h + R'_t \omega_{t-1}^h) = \underbrace{\sum_{i=0}^{+\infty} \left[\frac{1}{(1+r)^i} E_t(y_{T,t+i}^h + d'_{t+i} \frac{\Phi^h}{n_t^h N_t}) \right]}_{\text{PDV of production}} + \underbrace{R_0 \omega_{0,t-1}^h + R'_t (\omega_{t-1}^h - S_{t-1}^h)}_{\text{Returns on net foreign assets}}$$

$$- \underbrace{\frac{r}{1+r} \cdot \sum_{i=1}^{+\infty} \left(\frac{1}{1+r} \right)^i A_{t+i}^{\#} \left(\frac{1}{N_{t+i-1}} \Delta \tilde{\Sigma}_{t+i} \Delta \Phi + \sum_{h=1}^H n_{t+i-1}^h \Psi^h \Delta \tilde{\beta}_{t+i}^h \right)}_{\text{Structural risk cost}} \cdot \frac{\Phi^h}{n_t^h N_t}$$

Proof. We have: $(R_0 \omega_{0,t-1}^h + R'_t \omega_{t-1}^h) = R_0 \omega_{0,t-1}^h + R'_t (\omega_{t-1}^h - S_{t-1}^h) + R'_t S_{t-1}^h$, where S_t^h is the $J \times 1$ vector of per capita market valuation of companies located in country h (i.e.,

$$S_{j,t}^h = \frac{S_{j,t}}{n_t^h N_t} = \frac{\phi_j P_{j,t}}{n_t^h N_t} \text{ if asset } j \text{ is quoted in country } h; = 0 \text{ otherwise.}$$

$$R'_t S_{t-1}^h = \left(\frac{P_{j,t} + d_{j,t}}{P_{j,t-1}} \right)_{j=1 \dots J} \cdot S_{t-1}^h.$$

$$\text{But } P_t = \sum_{i=1}^{+\infty} \left(\frac{1}{1+r} \right)^i \left[E_t(d_{t+i}) - \frac{r}{1+r} A_{t+i}^{\#} \left(\frac{1}{N_{t+i-1}} \Delta \tilde{\Sigma}_{t+i} \Delta \Phi + \sum_{h=1}^H n_{t+i-1}^h \Psi^h \Delta \tilde{\beta}_{t+i}^h \right) \right].$$

Plugging this into the above equation and rearranging the terms yields:

$$R'_t S_{t-1}^h = \sum_{i=0}^{+\infty} \left(\frac{1}{1+r} \right)^i E_t(d_{t+i}) \frac{\Phi^h}{n_t^h N_t} - \frac{r}{1+r} \sum_{i=1}^{+\infty} \left(\frac{1}{1+r} \right)^i A_{t+i}^{\#} \left(\frac{1}{N_{t+i-1}} \Delta \tilde{\Sigma}_{t+i} \Delta \Phi + \sum_{h=1}^H n_{t+i-1}^h \Psi^h \Delta \tilde{\beta}_{t+i}^h \right) \cdot \frac{\Phi^h}{n_t^h N_t}$$

, where $\Phi^h = \left(\begin{array}{c} \phi_j^h = \phi_j \text{ if } j \text{ is a company located in } h \\ 0 \text{ otherwise} \end{array} \right)_{j=1 \dots J}$.

As a consequence, the sum of present discounted value of future endowments and revenue on past financial investment is:

$$\sum_{i=0}^{+\infty} \frac{1}{(1+r)^i} E_t(y_{T,t+i}^h) + (R_0 \omega_{0,t-1}^h + R'_t \omega_{t-1}^h) = \underbrace{\sum_{i=0}^{+\infty} \left[\frac{1}{(1+r)^i} E_t(y_{T,t+i}^h + d'_{t+i} \frac{\Phi^h}{n_t^h N_t}) \right]}_{\text{PDV of production}} + \underbrace{R_0 \omega_{0,t-1}^h + R'_t (\omega_{t-1}^h - S_{t-1}^h)}_{\text{Returns on net foreign assets}}$$

$$- \underbrace{\frac{r}{1+r} \cdot \sum_{i=1}^{+\infty} \left(\frac{1}{1+r} \right)^i A_{t+i}^{\#} \left(\frac{1}{N_{t+i-1}} \Delta \tilde{\Sigma}_{t+i} \Delta \Phi + \sum_{h=1}^H n_{t+i-1}^h \Psi^h \Delta \tilde{\beta}_{t+i}^h \right)}_{\text{Structural risk cost}} \cdot \frac{\Phi^h}{n_t^h N_t}$$

- *Proposition.*

$$\begin{aligned} \text{var}(c_t^h) &= \underbrace{\frac{r}{1+r} \Psi^{h2} \left(\theta_t^h - \tilde{\beta}_t^h, (\tilde{\Sigma}_t)^{-1} \tilde{\beta}_t^h \right)}_{\text{undiversifiable part of idiosyncratic risk}} \\ &+ \underbrace{\left(\frac{r}{1+r} \frac{A_t^\#}{A_t^h} \right)^2 \left(\frac{1}{N_{t-1}} \Delta \tilde{\Sigma}_t \Delta \Phi + \sum_{k=1}^H n_{t-1}^k \Psi^k \Delta \tilde{\beta}_t^k \right)' \Delta^{-1} (\tilde{\Sigma}_t)^{-1} \Delta^{-1} \left(\frac{1}{N_{t-1}} \Delta \tilde{\Sigma}_t \Delta \Phi + \sum_{k=1}^H n_{t-1}^k \Psi^k \Delta \tilde{\beta}_t^k \right)}_{\text{premium exploitation risk}} \end{aligned}$$

Proof. We have $\text{var}(c_t^h) = \left(\frac{r}{1+r} \right)^2 (\Psi^h)^2 \left[\text{var}(\varepsilon_t^h) - \beta_t^h, \Sigma_t^{-1} \beta_t^h \right] + \frac{1}{(A_t^h)^2} \text{EX}'_t \Sigma_t^{-1} \text{EX}_t$.

But we also know that:

$$\beta_t^h = \Pi_{t-1}^{-1} \Delta \tilde{\beta}_t^h,$$

$$\Sigma_t = \Pi_{t-1}^{-1} \Delta \tilde{\Sigma}_t \Delta \Pi_{t-1}^{-1},$$

and $E_{t-1} X_t = \frac{r}{1+r} A_t^\# \Pi_{t-1}^{-1} \left(\frac{1}{N_{t-1}} \Delta \tilde{\Sigma}_t \Delta \Phi + \sum_{h=1}^H n_{t-1}^h \Psi^h \Delta \tilde{\beta}_t^h \right)$.

Plugging in the variance expression and rearranging the terms yields:

$$\begin{aligned} \text{var}(c_t^h) &= \underbrace{\frac{r}{1+r} \Psi^{h2} \left(\theta_t^h - \tilde{\beta}_t^h, (\tilde{\Sigma}_t)^{-1} \tilde{\beta}_t^h \right)}_{\text{undiversifiable part of idiosyncratic risk}} \\ &+ \underbrace{\left(\frac{r}{1+r} \frac{A_t^\#}{A_t^h} \right)^2 \left(\frac{1}{N_{t-1}} \Delta \tilde{\Sigma}_t \Delta \Phi + \sum_{k=1}^H n_{t-1}^k \Psi^k \Delta \tilde{\beta}_t^k \right)' \Delta^{-1} \tilde{\Sigma}_t^{-1} \Delta^{-1} \left(\frac{1}{N_{t-1}} \Delta \tilde{\Sigma}_t \Delta \Phi + \sum_{k=1}^H n_{t-1}^k \Psi^k \Delta \tilde{\beta}_t^k \right)}_{\text{premium exploitation risk}} \end{aligned}$$

- *Proposition.*

$$\text{Var}(rer_t) = \text{Var} \left[\left(A_t^f \varepsilon_{NT,t}^f - A_t^h \varepsilon_{NT,t}^h \right) + \frac{r}{1+r} \left(A_t^h \psi^h \varepsilon_{T,t}^h - A_t^f \psi^f \varepsilon_{T,t}^f \right) + \frac{r}{1+r} \eta_t' \tilde{\Sigma}_t^{-1} \left(-\Psi^h A_t^h \tilde{\beta}_t^h + \Psi^f A_t^f \tilde{\beta}_t^f \right) \right]$$

Proof. The translation of endowment terms into endowment innovation terms is straightforward. But the case of portfolio is slightly more complex.

$$\text{Var} \left[R'_t \left(A_t^h \omega_{t-1}^h - A_t^f \omega_{t-1}^f \right) \right] = \text{Var} \left[\eta_t' \Lambda \Pi_{t-1}^{-1} \left(A_t^h \omega_{t-1}^h - A_t^f \omega_{t-1}^f \right) \right], \text{ because the other components of } R'_t \text{ are non stochastic.}$$

One uses the expression for portfolio holding:

$$\omega_t^h = \Pi_t \left[\frac{A_{t+1}^{\#}}{A_{t+1}^h} \frac{\Phi}{N_t} + \Delta^{-1} \left(\tilde{\Sigma}_{t+1} \right)^{-1} \left(\frac{A_{t+1}^{\#}}{A_{t+1}^h} \sum_{k=1}^H n_t^k \Psi^k \tilde{\beta}_{t+1}^k - \Psi^h \tilde{\beta}_{t+1}^h \right) \right].$$

All the terms in $(A_t^h \omega_{t-1}^h - A_t^f \omega_{t-1}^f)$ cancel out, except $\Lambda^{-1} \left(\tilde{\Sigma}_t \right)^{-1} \left(-\Psi^h A_t^h \tilde{\beta}_t^h + \Psi^f A_t^f \tilde{\beta}_t^f \right)$.

We, therefore, have:

$$Var(rer_t) = Var \left[\left(A_t^f \varepsilon_{NT,t}^f - A_t^h \varepsilon_{NT,t}^h \right) + \frac{r}{1+r} \left(A_t^h \psi^h \varepsilon_{T,t}^h - A_t^f \psi^f \varepsilon_{T,t}^f \right) + \frac{r}{1+r} \eta'_t \left(\tilde{\Sigma}_t \right)^{-1} \left(-\Psi^h A_t^h \tilde{\beta}_t^h + \Psi^f A_t^f \tilde{\beta}_t^f \right) \right]$$

Appendix III. Summary of Main Notation

Endowment

Description	Notation	Dimension
Endowment	y_t^h	1x1
Endowment innovation	ε_t^h	1x1
Present value multiplier on an endowment innovation	Ψ^h	1x1

Asset pricing

Risk-free interest rate	$R_0 = 1 + r$	1x1
Dividends paid by company j	$d_{j,t}$	1x1
Dividends innovation	$\eta_{j,t}$	1x1
Present value multiplier on a dividends innovation	Λ_j	1x1
Diagonal matrix with Λ_j on the diagonal	$\Delta = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \Lambda_J \end{pmatrix}$	$J \times J$
Stock prices	$P_t = (P_{j,t})_{j=1,\dots,J}$	$J \times 1$
Diagonal matrix with $P_{j,t}$ on the diagonal	$\Pi_t = \begin{pmatrix} P_{1,t} & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & P_{J,t} \end{pmatrix}$	$J \times J$
Gross returns	R_t	$J \times 1$
Gross return of stock j	$R_{j,t} = \frac{d_{j,t} + P_{j,t}}{P_{j,t-1}}$	1x1
Excess returns	X_t	$J \times 1$
Excess return of stock j	$X_{j,t} = (R_{j,t} - R_0)$	1x1

Covariance between endowments and assets

Variance-covariance matrix of dividends innovations	$\tilde{\Sigma}_t = [\text{cov}_{t-1}(\eta_{i,t}; \eta_{j,t})]_{(i,j)=(1..J)}$	$J \times J$
Covariance between endowments' and dividends' innovations	$\tilde{\beta}_t = [\text{cov}_{t-1}(\varepsilon_t^h, \eta_{j,t})]_{(h,j)=(1..H; 1..J)}$ $\tilde{\beta}_t = [\tilde{\beta}_t^h]_{h=1..H}$	$J \times H$
Variance-covariance matrix of asset returns	$\Sigma_t = [\text{cov}_{t-1}(R_{i,t}; R_{j,t})]_{i,j=1..J}$	$J \times J$
Covariance between endowments and asset returns	$\beta_t = [\text{cov}_{t-1}(y_t^h, R_{j,t})]_{(h,j)=(1..H; 1..J)}$	$J \times H$

Assets holdings and valuation

Risk-free asset holding	$\omega_{0,t}^h$	1×1
Risky asset holding by domestic agent	$\omega_t^h = (\omega_{j,t}^h)_{j=1}^J$	$J \times 1$
Number of shares	$\Phi = (\phi_j)_{j=1..J}$	$J \times 1$
Assets valuation	$S_t = (S_{j,t} = \phi_j P_{j,t})_{j=1..J}$	$J \times 1$
Total valuation of the stock market of country h	$e_t^h = \sum_{j \in \left\{ \begin{smallmatrix} \text{country h's} \\ \text{stocks} \end{smallmatrix} \right\}} \frac{S_{j,t}}{n_t^h N_t}$	1×1

Other

World population	N_t	1×1
Country h share of world population	n_t^h	1×1
Coefficient of absolute risk aversion	A^h	1×1
	$A_t^h \equiv \left(\sum_{h=1}^H \frac{n_t^h}{A_t^h} \right)^{-1}$	1×1
Rate of time preference	δ^h	1×1

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