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Estimating Parameters of Short-Term Real Interest Rate Models

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Estimating Parameters of Short-Term Real Interest Rate Models.[1]

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Abstract

This paper sheds light on a narrow but crucial question in finance: What should be the parameters of a model of the short-term real interest rate? Although models for the nominal interest rate are well studied and estimated, dynamics of the real interest rate are rarely explored. Simple *ad hoc* processes for the short-term real interest rate are usually assumed as building blocks for more sophisticated models. In this paper, parameters of the real interest rate model are estimated in the broad class of single-factor interest rate diffusion processes on U.S. monthly data. It is shown that the elasticity of interest rate volatility—the relationship between the volatility of changes in the interest rate and its level—plays a crucial role in explaining real interest rate dynamics. The empirical estimates of the elasticity of the real interest rate volatility are found to be about 0.5, much lower than that of the nominal interest rate. These estimates show that the square root process, as in the Cox-Ingersoll-Ross model, provides a good characterization of the short-term real interest rate process.

1. Introduction

Modeling and estimating the volatility of interest rates has significant implications in finance, particularly in pricing bonds, options, and other derivatives. While there is some degree of theoretical and empirical consensus about models for the nominal interest rate, only recently has research tended toward the simultaneous analysis of these main components of the nominal interest rate–real interest rate, expected inflation, and inflation risk premia– though, primarily focusing on the latter two (see, for example, Haubrich, Pennacchi, and Ritchken (2012), Ang, Bekaert, and Wei (2008), and Grishchenko and Huang (2012)). Some papers focus on the term-structure of real interest rates, while dynamics of the real interest rate at the short end of the yield curve are barely studied. An *ad hoc* process for the short-term real interest rate is usually assumed as a building block for more sophisticated models.

To my knowledge, this is the first paper that attempts to shed light on a narrow but crucial question in finance: *What should be the parameters of a model of the short-term real interest rate?* By estimating single-factor models for the short-term real interest rate, it is shown that the relationship between the volatility of changes in the interest rate and its level–called the elasticity of interest rate volatility–plays a crucial role in explaining real interest rate dynamics. Model comparison shows that a square root interest rate process (as in Cox, Ingersoll, and Ross (1985)) is enough to capture the dependence of volatility on the level of interest rates. Many models fail to incorporate this feature, though it should an important assumption according to the empirical results of this paper.

A number of interest rate models that are commonly used to price and hedge interest-rate-dependent securities begin with an assumed process for the instantaneous short-term interest rate. These models differ most notably in the volatility structure assumed to govern interest rate movements. Many empirical papers focus on nominal interest rates and do not consider the fact that two major components of the nominal interest rate are the real interest rate and expected inflation.[2] Researchers have developed many models for the short-term nominal interest rate (see the discussion of nominal interest rate models in Dai and Singleton (2000) and Dai and Singleton (2003)),[3] but fewer models were developed for the real interest rate (see, for example, the discussion in Ang, Bekaert, and Wei (2008)).

There is some understanding of the sources of inflation and factors that can influence it, as well as the way policymakers can forecast and control it, though only a small number of papers devote attention to real interest rates. Although theoretical research often assumes that the real interest rate is constant, empirical estimates for the real interest rate process vary between constancy (Fama (1975)), mean-reverting behavior (Hamilton (1985)), and a unit root process (Rose (1988)). There seems to be greater consensus on the fact that the real interest rate variation mainly affects the short end of the term structure and expected inflation and inflation risk premia influence long-term interest rates (see, among others, Fama (1990) and Mishkin (1990)). Ang, Bekaert, and Wei (2008) show that real interest rates are quite variable at short maturities but smooth and persistent at long maturities. Haubrich, Pennacchi, and Ritchken (2012) develop and estimate a model of nominal and real bond yield curves. They show that time-varying volatility is particularly apparent in short-term real rates and expected inflation.

It is typical to follow the standard stochastic discount factor approach and assume that the real interest rate is a function only of fundamentals or of a vector of state variables (see, for example, Ang, Bekaert, and Wei (2008), Chernov and Mueller (2012), and Haubrich, Pennacchi,

[2] Ang, Bekaert, and Wei (2008) show that inflation compensation explains about 80 percent of the variation in nominal rates for both short and long maturities.

[3] A partial listing of theoretical interest rate models includes those by Merton (1973), Brennan and Schwartz (1977, 1980), Vasicek (1977), Dothan (1978), Cox, Ingersoll, and Ross (1980, 1985), Constantinides and Ingersoll (1984), Schaefer and Schwartz (1984), Sundaresan (1984), Feldman (1989), Longstaff (1989), and Longstaff and Schwartz (1992).

and Ritchken (2012)). In these models, the variance of the real interest rate does not depend on the level of interest rate, but instead is assumed to be constant or to have a GARCH structure. This approach allows estimating risk premia, inflation expectations, and various parameters of models, though it suffers from overly simplistic assumptions about the dynamics of the real interest rate.

A number of theoretical models of the short-term interest rate have been built. Canonical term structure models imply dynamics for the short-term riskless rate that can be nested in a single-factor stochastic differential equation of the form: $dr = \kappa(\mu - r)dt + \sigma r^{\gamma} dz$, where *r* is the interest rate and dz is the Brownian motion. An important volatility structure parameter that distinguishes models from each other is the elasticity of volatility with respect to the level of interest rates, γ. While other parameters are parts of the linear structure of the interest rate model, the elasticity of volatility of the interest rate adds a non-linearity component.

Studies of the nominal interest rate dynamics show a relatively high level of elasticity of interest rate volatility in the U.S. In the class of single-factor term structure models, a famous result is that of Chan, Karolyi, Longstaff, and Sanders (CKLS, 1992), who compare a series of models for the short-term 1-month Treasury-Bill nominal interest rate over the period 1964 through 1989 for the U.S. They found that an elasticity of volatility with respect to the interest rate level, γ, of 1.5 is required to properly model the nominal interest rate dynamics. Bliss and Smith (1998) provide a re-examination of the CKLS (1992) results and find the elasticity of interest rate volatility to be around 1 if the structural changes in monetary policy in the 1980s are properly taken into account. Empirical estimates of the elasticity of volatility vary among countries. Nowman (1997) shows that the volatility of the short-term interest rate is highly sensitive to its level in the U.S. (the elasticity is about 1.5), while it is not in the U.K. (the elasticity is about 0.28). More advanced estimation methods found lower levels of elasticity of volatility of the nominal interest rate in the U.S. (Episcopos (2000) and Andersen and Lund (1997)). Evidence for other countries is mixed (Episcopos (2000), Hiraki and Takezawa (1997)).

Much less has been done in the analysis of the real interest rate dynamics. The major problem here is that real interest rates are not directly observed. In the U.S., Treasury Inflation-Protection Securities (TIPS), “real” bonds, are issued in terms of 5, 10, and 30 years and, therefore, do not allow extracting short-term inflation expectations. Furthermore, TIPS did not start trading until 1997 and had considerable liquidity problems during the first few years, making a consistent analysis of real interest rates for the entire interest rate history of the U.S. almost impossible.

In theory, the Fisher equation tells us that the nominal interest rate is simply the sum of the real interest rate and expected inflation. When inflation is stochastic, the Fisher equation is extended by inflation risk premia and other “higher-order” components, related to nonlinearities, when calculating inflation expectations (see the discussion in Sarte (1998)).

The problem is more complex with longer-term real interest rates and different econometric methods have been applied to estimate real interest rates and their term structure. Older papers simply used projected ex-post real interest rates on instrumental variables (Mishkin (1981) and Huizinga and Mishkin (1986)). Hamilton (1985), Fama and Gibbons (1982), and Burmeister, Wall, and Hamilton (1986) use ARIMA models and identify expected inflation and real interest rates under the assumption of rational expectations using the Kalman filter. Ang, Bekaert, and Wei (2008) were the first to establish a comprehensive set of stylized facts regarding the term structure of real interest rates. They found that the term structure of real interest rates has a fairly flat shape and that the real short-term interest rate is negatively correlated with both expected and unexpected inflation.

Another problem for calculating the real interest rate is expected inflation. There are a variety of methods for forecasting inflation and evaluating inflation expectations. The most popular are: time-series ARIMA models; regressions based on the Phillips curve; term structure models that include linear, non-linear, and arbitrage-free specifications; and survey-based measures. Ang, Bekaert, and Wei (2007) examine the forecasting power of these four methods and show that surveys outperform other methods for the U.S. To calculate real interest rates, this paper uses two major expected inflation surveys in the U.S.—the Michigan Survey of Consumer Attitudes and Behavior (MICH), which surveys a cross-section of the population on their inflation expectations, and the Livingston Survey, which surveys economists from industry, government, banking, and academia.

To summarize, a lot has been done in the field of nominal interest rate modeling, while the dynamics of the real interest rate are rarely studied. Ang, Bekaert, and Wei (2007) recently documented some stylized facts about the real interest rate dynamics, though some basic questions about the dynamics of the real interest rate are still to be answered. This paper proposes an answer to one of them: What should be the parameters of a model of the short-term real interest rate? This paper estimates parameters of the real interest rate model in the broad class of single-factor continuous interest rate diffusion processes. The empirical estimates show that the key parameters of the nominal and real interest rate models differ substantially. The major difference comes from the volatility structure of these models, mainly related to the elasticity of interest rate volatility, which is estimated to be much lower for the real interest rate model. The empirical estimates of this paper document the fact that the square root process, as in the Cox, Ingersoll, and Ross (1985) model, provides a good characterization of the short-term real interest rate process.

The remainder of paper is organized as follows. Section 2 discusses different theoretical single-factor short-term interest rate models. Section 3 provides the estimation methodology, data description, and empirical results. In Section 4, potential implications of the results of this paper are discussed. Section 5 concludes.

2. Models of the Short-Term Interest Rate

In this section, I briefly discuss canonical models that can be nested in the broad class of single-factor continuous interest rate diffusion processes. To model the interest rate dynamics, it is common to consider a continuous-time diffusion process defined by:

(1) $$dr = \kappa(\mu - r)dt + \sigma r^{\gamma} dz,$$

where r is the continuous (real) interest rate and dz is the Brownian motion.

This continuous-time model can be represented as the following discrete-time analog:

(2) $$r_{t+1} - r_t = \alpha + \beta r_t + \varepsilon_{r,t+1},$$

(3) $$E_t[\varepsilon_{r,t+1}^2] = \sigma^2 r_t^{2\gamma},$$

where r_t is the (real) interest rate and $\varepsilon_{r,t+1}$ is the *iid* shock with the variance $\sigma^2 r_t^{2\gamma}$. In this model, α represents a drift, β is the parameter of mean-reversal, σ is the variance level, and γ is a measure of the dependence of volatility on the interest rate level, or the elasticity of the interest rate volatility. This general version of the model comprises nine special cases that impose restrictions on the values of α, β, σ, and γ (Table 1).

Models	Model name	Parameters				Degrees of freedom
		α	β	σ	γ	
Model 1	Unrestricted	-	-	-	-	0
Model 2	CEV	0	-	-	-	1
Model 3	$\alpha = 0, \beta = 0$	0	0	-	-	2
Model 4	Merton	-	0	-	0	2
Model 5	Vasicek	-	-	-	0	1
Model 6	GBM	0	-	-	1	2
Model 7	CIR-SR	-	-	-	0.5	1
Model 8	Dothan	0	0	-	1.0	3
Model 9	Brennan-Schwartz	-	-	-	1	1
Model 10	CIR-VR	0	0	-	1.5	3

Table 1. Parameter restrictions (degrees of freedom) imposed by alternative models of the short-term interest rate.

Model 1 is an "unrestricted" version of the single-factor interest rate diffusion processes in discrete time, estimated by CKLS (1992). Model 2 is the constant elasticity of variance (CEV) process introduced by Cox (1975) and by Cox and Ross (1976). Model 3 is a version of the constant elasticity of variance (CEV) process with $\alpha = 0$ and $\beta = 0$. Model 4 is used in Merton (1973) to derive a model of discount bond prices. Model 5 is the Ornstein-Uhlenbeck process used by Vasicek (1977) in deriving an equilibrium model of discount bond prices. Model 6 is the geometric Brownian motion (GBM) process. Model 7 is the square root (SR) process, which appears in Cox, Ingersoll, and Ross (CIR, 1985). Model 8 is used by Dothan (1978) in valuing discount bonds. Model 9 is used by Brennan and Schwartz (1980) in deriving a numerical model for convertible bond prices. Model 10 is introduced by Cox, Ingersoll, and Ross (1980) in their study of variable-rate (VR) securities.

3. Empirical Estimates

3.1. The Real Interest Rate.

Many theoretical models use a certain interest rate process as an assumption. From a theoretical standpoint, many canonical models mentioned above do not require interest rates to be positive, implying a better fit with real interest rates. A theoretical calculation of the real interest rate is usually based on the stochastic discount factor approach. To satisfy the no-arbitrage condition, the real price of an arbitrary financial instrument must adhere to the law of one price:

$$P_t = E_t(M_{t+1}P_{t+1}), \qquad (4)$$

where M_{t+1} is a real stochastic discount factor, P_t is the price level, and E_t is the conditional expectation operator at time *t*.

As the nominal and real stochastic discount factors are connected through inflation, under standard assumptions of log-normality, the one-period nominal interest rate can be expressed as:

$$R_t = r_t + E_t(\pi_{t+1}) + Cov_t(m_{t+1}, \pi_{t+1}) - \frac{1}{2}Var_t(\pi_{t+1}), \qquad (5)$$

where R_t is the nominal interest rate, r_t is the real interest rate, π_{t+1} is inflation in period $t+1$, and m_{t+1} is a log of the real stochastic discount factor. This equation is different from the standard Fisher equation through the third and four terms, which account for the inflation premium and Jensen's inequality "higher-order" term, respectively.

For short horizons, it is typical to assume that the interest rate is risk-free and the inflation risk premium is negligible (see, for example, Ang, Bekaert, and Wei (2007)). Also, if interest rates are small, second-order components that come from Jensen's inequality are insignificant. Therefore, the canonical Fisher equation would hold for short horizons and the calculation of the real interest rate boils down to subtracting the expected inflation from the nominal interest rate:

(6) $$r_t = R_t - E_t(\pi_{t+1}).$$

In this paper, I study short term (3 months) interest rates and assume that there is only a negligible inflation risk in it. I intentionally do not attempt to decompose the nominal interest rate into other components, as they are very small for the short-term interest rate and any procedure for estimating the risk premia would demand prior *ad hoc* assumptions about the structure of the real interest rate model. Instead, I focus on estimating the model of the short-term real interest rate using only data on 3-month Treasury-Bill interest rates and expected inflation.

3.2. Data

The real interest rate is calculated using the standard Fisher equation (6). For the short-term nominal interest rate, I use the 3-month Treasury-Bill interest rate included in the Federal Reserve's weekly H.15 release (monthly data is available from January 1934 to December 2012).

While there are many models of inflation expectations, the necessity of extended historical data on inflation expectations limits choice options. A typical approach of using TIPS for measuring expected inflation would not work either, as TIPS are issued in terms of 5, 10, and 30 years and, therefore, do not allow extracting short-term inflation expectations. Ang, Bekaert, and Wei (2007) show that surveys outperform other forecasting methods. Therefore, two inflation expectation surveys are used in this paper to measure expected inflation: (1) monthly data from University of Michigan Inflation Expectation survey (MICH) available from the St. Louis Fed database and (2) the Livingston Survey from the Philadelphia Fed database. MICH data is available from January 1978 to December 2012 on a monthly basis. As the Livingston

Survey data is available from 1954 to 2012 only on a semiannual basis, a linear interpolation is used to transform data into monthly series.

Dynamics of expected inflation, nominal interest rates, and real interest rates are presented in Figures 1 and 2. Both surveys provide similar dynamics of real interest rates. Since 1947, the dynamics of real interest rates was usually between -5 percent and 5 percent (Figure 1) and the dynamics of real interest rates looks more like a random process without clear trends, although expected inflation and nominal interest rates have historical trends and were influenced by economic policies. In the early 80s, inflation was high and Paul Volker, the chairman of the Federal Reserve, implemented the policy of high interest rates, pushing real interest rates up. Since the beginning of the 2008 crisis, nominal interest rates fell almost to zero, while inflation expectations were rather volatile, leading to substantial volatility in real interest rates.

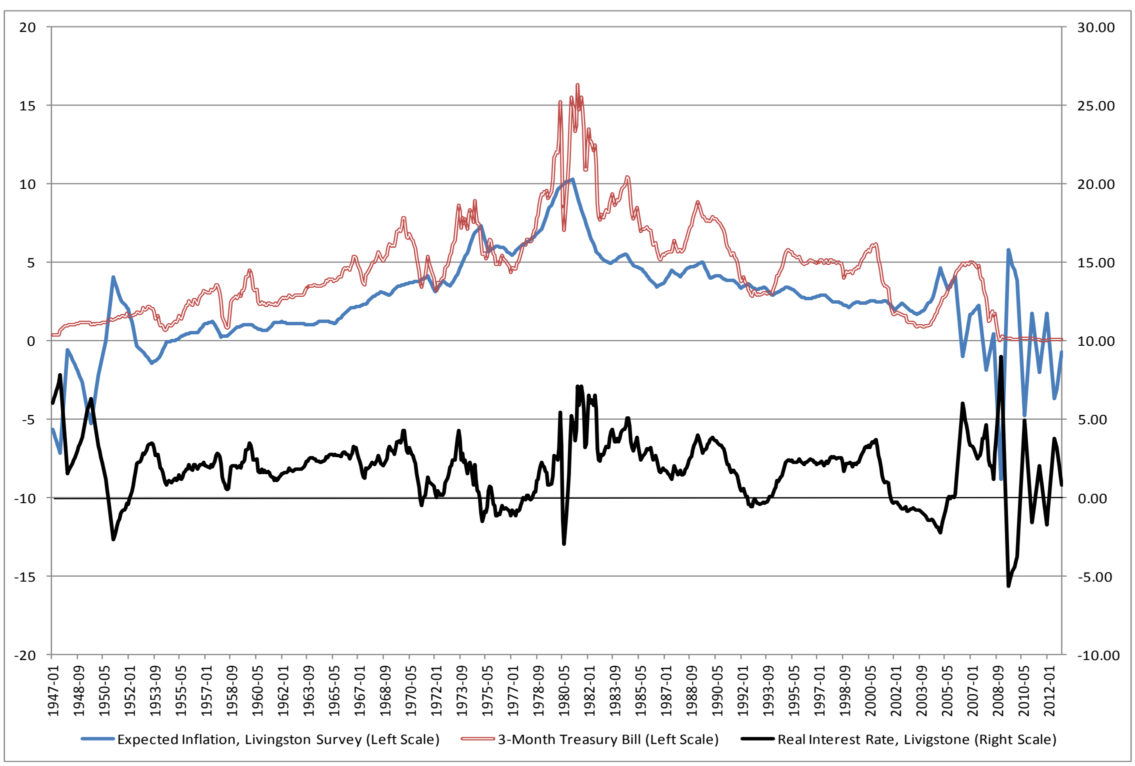

Figure 1. Expected inflation (Livingston Survey), 3-Month Treasury-Bill rate, and real interest rate, Jan 1947-Dec 2012.

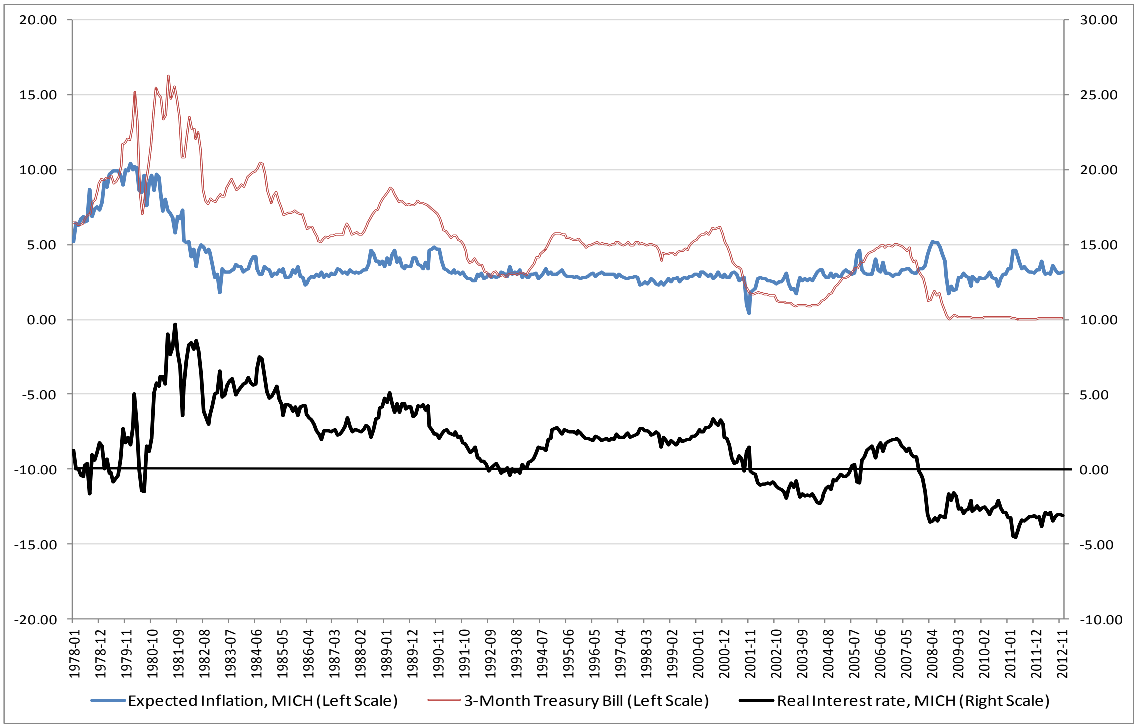

Figure 2. Expected inflation (MICH Survey), 3-Month Treasury-Bill rate, and real interest rate, Jan 1978-Dec 2012.

3.3. Empirical Estimates

I begin by estimating a single-equation model for the short-term interest rate of the form:

(7) $$r_{t+1} - r_t = \alpha + \beta r_t + \varepsilon_{r,t+1},$$

(8) $$E[\varepsilon_{r,t+1}^2] = \sigma^2 r_t^{2\gamma},$$

where r_t is the real interest rate and $\varepsilon_{r,t+1}$ is a shock.

I follow CKLS (1992) and use the GMM to estimate the model, a logical choice for the estimation of the single-factor interest rate processes. GMM estimators are consistent even if the errors are conditionally heteroskedastic, which is important in our case, as the variance of the interest rate process, $E[\varepsilon_{r,t+1}^2]$, depends on the level of interest rates. Also, the justification for the

GMM procedure only requires that the distribution of interest rate changes be stationary and ergodic and that the relevant expectations exist.

For comparison, I estimate both the real and nominal interest rate models.[4] As expected inflation data from two surveys are available for different periods, empirical estimates are provided for two samples: from January 1978 to December 2012 and from January 1947 to December 2012. As only the MICH survey has monthly data, estimates for the January 1978-December 2012 period should be considered to be the most robust.

First, for comparison purposes, the process for the nominal interest rate is estimated (Table 1). The estimates of α and β are not statistically different from zero, consistent with no-arbitrage assumptions. The estimate for σ is very small. The estimate of the elasticity of the volatility of the nominal interest rate, γ, for the 1978-2012 period is about 1.8, which is consistent with the CKLS (1992) finding of about 1.5 for the 1964-1989 period. Such a high level of γ explains that the nominal interest rate becomes much more volatile when the level of interest rates is high.

Second, the process for the real interest rate, with the expected inflation taken from the MICH and Livingston surveys, is estimated. Results are similar for both real interest rate data series. The estimates of α are not statistically different from zero, which explains the absence of drift in the real interest rate dynamics. The estimate of the mean-reversal parameter, β, is very small and negative, which is consistent with standard properties of interest rate processes. The estimates for σ are very close to zero, meaning that the variance level is relatively small, which is consistent with the observation of Ang, Bekaert, and Wei (2008).

The estimates show that the real interest rate process has a much lower value for the

[4] The nominal interest rate model is the same as the real interest rate model, expect for the use of the nominal interest rate instead of the real.

elasticity of the interest rate volatility, γ, than the nominal interest rate process. The estimated elasticities of volatility of the real interest rate are about 0.55 and 0.47 with standard errors of about 0.2 for both data series. These results are striking, as they are much smaller than 1.8 for nominal interest rates and a canonical value of about 1.5.

As the Livingston survey of inflation expectations has data available starting from 1947, I estimate the nominal and real interest rate models from January 1947 to December 2012 separately. The estimates confirm the finding that the nominal interest rate process has a very high γ but the real interest rate process has a much lower one. Estimates on a full data set for the nominal interest rates from 1934 to 2012 confirm the high levels of γ of about 1.58 for nominal interest rates.

	Parameters			
	α	β	σ	γ
Period: 1978/01-2012/12				
Nominal interest rate model	0.025	-0.008	0.013***	1.820***
s.e.	*0.046*	*0.012*	*0.007*	*0.252*
t-stat	*0.549*	*-0.678*	*1.829*	*7.229*
Real interest rate model (MICH survey of inflation expectations)	0.000	-0.025***	0.047**	0.545***
s.e.	*0.000*	*0.011*	*0.034*	*0.219*
t-stat	*0.846*	*-2.233*	*1.376*	*2.487*
Real interest rate model (Livingston survey of inflation expectations)	0.000	-0.008	0.032**	0.468***
s.e.	*0.000*	*0.019*	*0.023*	*0.234*
t-stat	*-0.335*	*-0.425*	*1.378*	*1.996*
Period: 1947/01-2012/12				
Nominal interest rate model	0.040	-0.009	0.022***	1.595***
s.e.	*0.031*	*0.009*	*0.009*	*0.219*
t-stat	*1.287*	*-1.061*	*2.359*	*7.281*
Real interest rate model (Livingston survey of inflation expectations)	0.000	-0.010	0.045**	0.606***
s.e.	*0.000*	*0.017*	*0.028*	*0.193*
t-stat	*-0.192*	*-0.579*	*1.624*	*3.134*
Period: 1934-2012				
Nominal interest rate model	0.025	-0.007	0.022***	1.588***
s.e.	*0.020*	*0.007*	*0.009*	*0.218*
t-stat	*1.226*	*-0.951*	*2.362*	*7.285*

Table 2. GMM estimation results of the single-equation real and nominal interest rate models.

*Note: *** indicates coefficients significant at the 5% level, ** indicates coefficients significant at the 10% level, ** indicates coefficients significant at the 15% level. Coefficients σ and γ are assumed to be non-negative.*

3.4. Model Comparison

In this section, I compare the "unrestricted" model for the real interest rate with the nine other standard nested models discussed before. Table 3 reports parameter estimates, their standard errors, asymptotic t-statistics, and GMM minimized criterion (χ^2) values for each of the nine nested models. Each model imposes restriction(s) on the parameters of the interest rate model, influencing estimates of other parameters. The χ^2 goodness-of-fit test shows the "validity" of each model and the restrictions it imposes. The model comparison shows that the major parameter that influences the goodness of fit of the model is the parameter of the elasticity of volatility of the interest rate. The χ^2-test suggests that the CIR-VR, Brennan-Schwartz, and Merton are misspecified and can be rejected at the 90% confidence level. These are followed by the Vasicek, GBM, "$\alpha = 0, \beta = 0$", and Dothan models, all of which have lower χ^2 values.

These estimates show that the CIR-SR model provides a good characterization of the short-term real interest rate process. The estimates of this model show that, if γ is pinned down to be 0.5, the estimate of α is not statistically different from zero (which explains the absence of a drift in the real interest rate dynamics) and a mean-reversal coefficient, β, is slightly negative (explaining the mean-reversal dynamics of the real interest rate). These facts are consistent with the stylized facts about real interest rates, established by Ang, Bekaert, and Wei (2008).

		α	β	σ	γ	d.f.	χ^2	*P-value*
Model 1	Unrestricted	0.000	-0.025***	0.047**	0.545***	0	-	-
	s.e.	*0.000*	*0.011*	*0.034*	*0.219*			
	t-stat	*0.846*	*-2.233*	*1.376*	*2.487*			
Model 2	CEV	**0.000**	-0.021***	0.055**	0.603***	1	0.941	0.332
	s.e.		*0.010*	*0.041*	*0.230*			
	t-stat		*-2.066*	*1.321*	*2.624*			
Model 3	$\alpha=0, \beta=0$	**0.000**	**0.000**	0.055*	0.603***	2	4.271	0.118
	s.e.			*0.039*	*0.277*			
	t-stat			*1.081*	*1.991*			
Model 4	Merton	0.000	**0.000**	0.005***	**0.000**	2	4.821	0.090
	s.e.	*0.000*		*0.001*				
	t-stat	*0.130*		*6.768*				
Model 5	Vasicek	0.000	-0.023***	0.006***	**0.000**	1	2.611	0.106
	s.e.	*0.000*	*0.011*	*0.001*				
	t-stat	*0.732*	*-2.072*	*7.434*				
Model 6	GBM	**0.000**	-0.018**	0.173***	**1.000**	2	3.494	0.174
	s.e.		*0.010*	*0.020*				
	t-stat		*-1.766*	*8.860*				
Model 7	CIR-SR	0.000	-0.025***	0.040***	**0.500**	1	0.048	0.827
	s.e.	*0.000*	*0.011*	*0.005*				
	t-stat	*1.020*	*-2.309*	*8.653*				
Model 8	Dothan	**0.000**	**0.000**	0.165***	**1.000**	3	6.096	0.107
	s.e.			*0.020*				
	t-stat			*8.061*				
Model 9	Brennan-Schwartz	0.000	-0.018**	0.174***	**1.000**	1	3.401	0.065
	s.e.	*0.000*	*0.010*	*0.020*				
	t-stat	*0.334*	*-1.766*	*8.827*				
Model 10	CIR-VR	**0.000**	**0.000**	0.610***	**1.500**	3	9.280	0.026
	s.e.			*0.079*				
	t-stat			*7.719*				

Table 3. GMM estimates of alternative models for the short-term real interest rates, Jan 1978-Dec 2012.

Note 1. The MICH survey of inflation expectations and 3-month Treasury-Bill interest rates are used to compute real interest rates. Number of degrees of freedom (d.f.) are equal to the number of restrictions in the nested model.

*Note 2. *** indicates coefficients significant at the 5% level, ** indicates coefficients significant at the 10% level, ** indicates coefficients significant at the 15% level. Coefficients* σ *and* γ *are assumed to be non-negative.*

Note 3. Restrictions imposed by each model are in bold.

3.5. Structural Breaks

Many empirical studies conclude that a change in the Federal Reserve's monetary policy during the Volker period led to changes or structural breaks in interest rate processes. To test this hypothesis, I introduce a dummy variable, D_t, that equals unity for monthly observations from October 1979 through September 1982 (as in Bliss and Smith (1988)) and zero otherwise. The model takes the form:

$$r_{t+1} - r_t = (\alpha + \delta_1 D_t) + (\beta + \delta_2 D_t) r_t + \varepsilon_{r,t+1}, \tag{9}$$

$$E[\varepsilon_{r,t+1}^2] = (\sigma + \delta_3 D_t)^2 r_t^{2(\gamma + \delta_4 D_t)}, \tag{10}$$

where parameters δ_1, δ_2, δ_3, and δ_4 represent marginal changes during the 1979-1982 period of α, β, σ, and γ, respectively. As four additional parameters are introduced into the model, for GMM estimation purposes the orthogonal vector of instruments is extended by the corresponding series of the dummy variables and their products with other variables. The model is estimated on the real interest rates data series from January 1947 to December 2012, based on the Livingston survey of inflation expectations.[5]

[5] The MICH survey started only in 1978 and, therefore, does not have enough data points to consistently evaluate the existence of the structural break in the data.

	α	β	σ	γ	δ_1	δ_2	δ_3	δ_4
	0.000	-0.008	0.076***	0.849***	0.014**	-0.291***	0.001***	-0.171
s.e.	*0.000*	*0.017*	*0.032*	*0.118*	*0.007*	*0.119*	*41.739*	*1.388*
t-stat	*-0.220*	*-0.495*	*2.367*	*7.168*	*1.909*	*-2.444*	*0.000*	*-0.123*

Table 4. Test for structural breaks in the models of the short-term real interest rate, Jan 1947-Dec 2012.

Note 1. The Livingston survey of inflation expectations is used.
*Note 2. *** indicates coefficients significant at the 5% level, ** indicates coefficients significant at the 10% level, ** indicates coefficients significant at the 15% level. Coefficients* σ *and* γ *are assumed to be non-negative.*

The empirical results are striking (Table 4). The estimates of α and β are not statistically different from zero. At the same time, estimates show that there was a statistically significant change in the value of these parameters between October 1979 and September 1982. The drift parameter α increased slightly ($\delta_1 = 0.014$). Parameter β became substantially smaller $(\delta_2 = -0.291)$, reflecting more active mean-reversing dynamics of the real interest rate, which can be explained by an aggressive policy of the Federal Reserve during this period. While there was a statistically significant positive change in σ, it was very small $(\delta_3 = 0.0001)$. It is important to notice that there was no statistically significant change in the volatility structure of the real interest rates during this period, which is consistent with the CLKS (1992) estimates for the nominal interest rate model.

4. Potential Implications

The empirical results of this paper are important as building blocks for more sophisticated interest rate models. Modeling dynamics of the real interest rate simultaneous with dynamics of inflation would give a better perspective on the volatility of the nominal interest rate dynamics. The key findings of this paper are the estimates of the parameters of the volatility structure of the real interest rate model. The results of this paper can be extended and applied to different multi-factor models of interest rates with implications on bond and option pricing.

One of the potential applications of the results of this paper is the improvement of TIPS pricing. The estimated square root process for the real interest rate can be incorporated into a model of the term structure of real interest rates, expected inflation, and inflation risk premia, similar to Haubrich, Pennacchi, and Ritchken (2012) and Grishchenko and Huang (2012). Haubrich, Pennacchi, and Ritchken (2012) construct the model with an *ad hoc* assumption that the real interest rate process has a volatility structure that does not depend on the level of the interest rate. Somewhat similar assumptions are used in Grishchenko and Huang (2012). Both papers have important empirical implications for pricing TIPS. Using the estimated process for the short-term real interest rate of this paper, one might better understand the inflation risk premium for longer maturities and pricing of inflation-protected securities.

Real interest rates might play an important role in understanding the connection between yields on Treasury-Bill and the Federal Funds rate. Piazzesi (2005) shows that nominal bond yields respond to policy decisions of the Federal Reserve and vice versa and, therefore, suggests that models of the yield curve should take into account monetary policy actions of the Federal Reserve. As the Federal Reserve changes its nominal interest rate in response to changes in inflation and other macroeconomic variables, incorporating dynamics of the real interest rate from this paper in Piazzesi's framework might provide a better understanding of the connection between different short-term interest rates.

All of these applications are left for future research.

5. Conclusion

While parameters of nominal interest rate models are well studied, not much is done in the field of real interest rates. This paper demonstrates that a canonical level of the parameter of the elasticity of nominal interest rate volatility of about 1.5 cannot be applied to the real interest rate model. Instead, the empirical estimates of this paper on U.S. data show that the short-term real interest rate has a much lower level of elasticity of interest rate volatility in the class of single-factor diffusion processes.

Using the 3-month Treasury-Bill interest rate and inflation expectations data, time series for real interest rates are constructed. The empirical estimates of this paper found the elasticity of the real interest rate volatility to be about 0.5, consistent with the square root single-factor diffusion process. The model comparison confirms that the Cox, Ingersoll, and Ross (1985) model provides a good characterization of the short-term real interest rate process. The analysis of structural changes during the Volcker disinflation period did not confirm the existence of a structural break in the volatility structure of the real interest rate model.

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