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Money as Indicator for the
Natural Rate of Interest

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Abstract

The natural interest rate is of great relevance to central banks, but it is difficult to measure. We show that in a standard microfounded monetary model, the natural interest rate comoves with a transformation of the money demand that can be computed from actual data. The co-movement is of a considerable magnitude and independent of monetary policy. An optimizing central bank that does not observe the natural interest rate can take advantage of this co-movement by incorporating the transformed money demand, in addition to the observed output gap and inflation, into a simple but optimal interest rate rule. Combining the transformed money demand and the observed output gap provides the best information about the natural interest rate.

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Contents

Page

I. INTRODUCTION

The natural interest rate, the real rate of return in the natural economy with flexible prices instead of sticky prices, is of great relevance to modern central banks. For example, interest rate rules like the one proposed by Taylor (1993) suggest that a central bank sets its policy rate equal to the natural interest rate plus the inflation rate when the inflation rate is at its target and the output gap is closed. More generally, monetary policy is considered contractionary if the actual real rate exceeds the natural interest rate, while it is considered expansionary if the actual real rate is below the natural interest rate.

The caveat to these straightforward principles is that the natural interest rate is fairly difficult to measure. Economic theory suggests that at business cycle frequency, the natural interest rate varies over time with shocks to technology, preferences, and absorption. A number of empirical studies show, however, that the uncertainty surrounding estimates of the natural interest rate is substantial, and this uncertainty undermines the usefulness of the natural interest rate as a practical guide to monetary policymakers.[1]

We show that money demand can play a role in measuring the natural interest rate and therefore, in the optimal monetary policy rule. To illustrate this, we revisit a basic microfounded monetary model with a money-in-utility (MIU) specification of money demand. In this model, we derive a transformation of the money demand that can be computed from actual data and that we call the money gap. The money gap co-moves with the natural interest rate because it reflects, among other things, shocks to the marginal utility of consumption. These shocks also alter the time path of natural consumption and, therefore, the natural interest rate. The co-movement between the money gap and the natural interest rate is independent of the specific model used, and we extend the basic model to a quantitative model with a dynamic money demand function to show this.

We find that the correlation between the money gap and the natural interest rate is of considerable magnitude, is independent of monetary policy, and is thus, immune to the Lucas (1976) critique. These are welcome characteristics from the perspective of a central bank that uses the money gap to measure the natural interest rate more accurately. Based on the quantitative model calibrated for the euro area, our estimate of the correlation between the money gap and the natural interest rate is between 0.3 and 0.45. Results for the U.S., indicate a somewhat lower but still notable correlation. The correlation is immune to changes in the monetary policy regime because, by construction, the money gap omits the variation in money demand

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[1]See, e.g., Ferguson (2004).

that results from the opportunity costs and the transaction volume of money demand. This insulates the money gap against the direct and indirect effects of changes in the policy regime, just like the natural interest rate.

A central bank can take advantage of the co-movement between the money gap and the natural interest rate. To demonstrate the appropriate role of the money gap for monetary policy, we suppose that the optimizing central bank selects the coefficients of a typical interest rate rule that is modified to account for the lack of information on the side of central banks. With full information, the interest rate rule incorporates the natural interest rate, the output gap between the actual and the natural level of output, and the inflation rate. However, two of these variables, the natural interest rate and the output gap, are not readily observable in the real world. We operationalize this lack of information in the modified interest rate rule by replacing the natural interest rate with the money gap, which can be computed from the actual data. Furthermore, we constrain the central bank to only observe the output gap conflated with noise.

Our analytical solution of the policy problem implies that the optimizing central bank selects a positive coefficient of the money gap in the modified interest rate rule. The principles underlying the interest rate rule suggest that the central bank, provided it observes the natural interest rate directly, varies the policy rate one-for-one with the natural interest rate to counter the impact of the natural interest rate on aggregate demand. Selecting a positive coefficient of the money gap helps the central bank to approximate this ideal policy if it cannot observe the natural interest rate directly. Furthermore, we find that the optimizing central bank combines the money gap with the observed output gap in a way that provides the best information about the natural interest rate. The central bank considers both the money gap and the observed output gap at the same time because their combination improves upon the information that each indicator provides individually.

These results are fairly robust. We use the basic model to establish them analytically and to develop the core findings. Then, we use the quantitative model that features a dynamic money demand function and that can only be solved numerically to extend our analysis to more sophisticated policy rules. The analytical results hold up well and provide a suitable guideline for the numerical results. We also show that, in the quantitative model, lags of the money gap (rather than leads) co-move tightly with the contemporaneous natural interest rate. Consequently, in the optimal policy rule, the lagged money gap performs better as an indicator of the natural interest rate. We trace the pronounced lead-lag structure between the money gap and the natural interest rate back to the habit formation in consumption, which alters the time

profile of the natural interest rate. Furthermore, while in the real world the information content of the money gap may change over time, our numerical results suggest that, as a rule, the central bank will be better off taking the money gap into account instead of not appropriately responding to the movements in the natural interest rate that the money gap indicates.

It is worth mentioning that our approach considers only a narrow (non-interest bearing) concept of money. However, while there are multiple concepts of money, broadening the model along this dimension would only add to the number of monetary indicator variables that the central bank would combine optimally according to their informational content. Here we focus on only one of possibly many money gaps to portray a link between money demand and the natural interest rate that the literature has not explored so far.

Our study is related to two branches in the literature. The first branch emphasizes that the natural interest rate is of great relevance to central banks and attempts to measure it using the information contained in the money demand. Our study differs from this branch in that we explore a new link between the money demand and the natural interest rate that is complementary to the link that has been explored so far. Furthermore, in contrast to the literature, we derive the consequences of the new link for the optimal monetary policy rule.

Andres, Lopez-Salido, and Nelson (2009) build upon Andres, Lopez-Salido, and Valles (2006) and belong to the first branch of the literature. They show that the money demand contains information about the natural interest rate if the money demand serves as a summary index of unobserved yields and, therefore, is forward looking.[2] Arestis, Chortareas, and Tsoukalas (2010) measure the natural rate of output, which is related to the natural interest rate, also using the information contained in the forward-looking money demand and obtain an estimate of the natural output that is fairly precise. In contrast, Laubach and Williams (2003), Mesonnier and Renne (2007), and Edge, Kiley, and Laforte (2008) measure the natural interest rate without using the information contained in the money demand and find estimates of the natural interest rate that are fairly imprecise and subject to substantial measurement error.

The second branch of the related literature studies the indicator role of money demand when monetary policymakers lack information about the state of the economy (Dotsey and Hornstein (2003), Coenen, Levin, and Wieland (2005), Lippi and Neri (2007), Beck and Wieland (2007), Beck and Wieland (2008), Scharnagl, Gerberding, and Seitz (2010), Unsal, Portillo,

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[2] Andres, Lopez-Salido, and Nelson (2009) suggest that optimal monetary policy in a model with dynamic money demand would be an important topic for future research. Our paper can be considered to be following up on this suggestion.

and Berg (2010)).[3] A common feature of this literature is that it treats the money demand residual as mutually independent of all other structural shocks in the economy. In the typical microfounded model that we use, however, the money demand residual contains one component that also drives the natural interest rate and, therefore, is not mutually independent of all other shocks. A number of authors have mentioned this characteristic of the money demand residual (see, e.g., McCallum and Nelson (1999), Neiss and Nelson (2001), Nelson (2002), Woodford (2003), Favara and Giordani (2009), and Sargent and Surico (2011)), but the implications for the indicator role of money demand have remained largely unexplored.

In the next section, we briefly recap the MIU specification of money demand and isolate the link between the money demand and the natural interest rate that we explore. Section III describes our transformation of money demand, which we call the money gap, and derives the correlation between the money gap and the natural interest rate. In Section IV, we demonstrate the usefulness of the money gap for the optimal monetary policy rule in the basic model. Section V contains the quantitative analysis of the link between the money gap and the natural interest rate and of the consequences for the optimal policy rule in the quantitative model with a dynamic money demand function, and Section VII concludes.

II. BASIC MODEL

We start by examining the link between the money demand and the natural interest rate in a basic New Keynesian model (see Woodford (2003), Gali (2008), or Walsh (2010)), which allows us to present analytical results with closed-form solutions. The basic model constitutes the core of the quantitative model with dynamic money demand and more endogenous amplification of shocks than in the basic model, for which we present numerical results below.

The basic model comprises three types of agents: the representative household, infinitely many firms, and the government. Firms produce intermediate products using labor as the sole input to production and using a technology that is subject to aggregate productivity shocks. Firms sell their product in a monopolistically competitive market, and they optimally adjust their product prices infrequently as in Calvo (1983). The representative household demands intermediate products and aggregate them to the composite consumption good C_t. The government implements monetary policy using a policy rule for the short-term nominal interest rate, which we describe below. Appendix A contains further model details.

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[3]See Berger, Harjes, and Stavrev (2010) for a recent survey.

A. The household and the MIU specification of money demand

The representative household demands money because holding money facilitates transactions. The MIU specification captures this idea of money demand by incorporating real money balances directly into the household's utility function. A general representation of the expected discounted lifetime utility function of the household is

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[u(C_t, \psi_t) + q(M_t/P_t, \varepsilon_t^m) - h(N_t) \right], \ 0 < \beta < 1 , \tag{1}$$

where E_0 is the expectation operator conditional on information at time zero, C_t denotes composite consumption, M_t/P_t denotes end-of-period real money balances, and N_t denotes labor. The functions u and q and h fulfill standard regularity conditions.[4]

The utility function contains two types of preference shocks. The shock ψ_t affects the utility of consumption and thereby, will alter the time path of consumption and the real rate of return. The shock ε_t^m affects the utility of holding real money balances and can be interpreted, among other things, as reflecting exogenous changes in the velocity of money demand. We omit a shock to the disutility from labor, since it would be isomorphic to the aggregate productivity shock. Further below we also consider shocks to the household's discount rate β.

The utility function is additively separable such that real money demand does not influence the utility of consumption directly. We adopt additively separable utility (as do McCallum (2001), Woodford (2003), Ireland (2004), Andres, Lopez-Salido, and Valles (2006), and Andres, Lopez-Salido, and Nelson (2009)) to clearly distinguish the link between the money demand and the natural interest rate that we explore from other links involving money demand examined elsewhere.

The household maximizes its discounted lifetime utility subject to the flow budget constraint

$$P_t C_t + B_t + M_t \geq (1 + i_{t-1}) B_{t-1} + M_{t-1} + (1 - \tau) W_t N_t + D_t - T_t .$$

This constraint ensures that the sum of the expenditures for consumption, $P_t C_t$, and for the financial portfolio, which comprises money and government bonds B_t, does not exceed total

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[4]The function $u(\cdot, \psi)$ is twice continuously differentiable, increasing, and concave in its argument for each value of the shock ψ. The function $q(\cdot, \varepsilon^m)$ is increasing and concave in its argument for each value of the shock ε^m. The function q has a finite satiation level $m^s > 0$ such that, in the steady state, $q_m(m^s, \varepsilon^m) = 0$, with $m = M/P$ and $q_m(m, \varepsilon^m) = \partial q(m, \varepsilon^m)/\partial m$. The limiting value of q_{mm} is finite and negative when m approaches m^s from below (see Woodford (2003), Assumption 6.1, for a discussion). The function $h(\cdot)$ is twice continuously differentiable, increasing, and convex in its argument.

income. Income accrues from the financial portfolio in the previous period $(1 + i_{t-1})B_{t-1} + M_{t-1}$, the labor income net of taxes, $(1 - \tau)W_t N_t$, the profits D_t from firm ownership, and the lump-sum transfer T_t. i_t denotes the nominal interest rate on government bonds.

B.   The money demand and the natural interest rate

Money demand co-moves with the natural interest rate because both variables depend on the marginal utility of consumption. Deriving the first-order conditions of the utility maximization problem with respect to C_t, M_t, and B_t and rearranging them yields the money demand function

$$\frac{q_m(M_t/P_t, \varepsilon_t^m)}{u_c(Y_t - G_t, \psi_t)} = \frac{i_t}{1 + i_t} \; , \tag{2}$$

where we substituted income Y_t for consumption C_t using the identity $Y_t = C_t + G_t$. We use G_t to denote autonomous aggregate demand, including the demand for credit goods and foreign demand.

Real money demand M_t/P_t depends on the opportunity costs of holding money rather than bonds $i_t/(1 + i_t)$, the velocity shock ε_t^m, and the marginal rate of substitution between money and consumption q_m/u_c. Importantly, the money demand function (2) also depends on the two shocks G_t and ψ_t, which affect the relationship between income and the marginal utility of consumption because movements in G_t and ψ_t alter the household's level of consumption and, through the marginal rate of substitution, the household's money demand. Jointly, these two shocks constitute the link between the money demand and the natural interest rate.

To show this, we define the natural interest rate r_t^n as the real rate of return that prevails in the natural economy with fully flexible prices. The natural economy constitutes a useful reference for policymakers by indicating the direction in which monetary policy will have to adjust to overcome the inefficient adjustment to shocks under sticky prices.[5] The natural interest rate derives from the Euler equation, which is part of the household's optimality conditions and therefore unaffected by assumptions regarding how firms set prices:

$$\beta E_t \left(\frac{u_c(Y_{t+1}^n - G_{t+1}, \psi_{t+1})}{u_c(Y_t^n - G_t, \psi_t)} (1 + r_t^n) \right) = 1 \; , \tag{3}$$

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[5]The natural interest rate also corresponds to the interest rate in the efficient economy. There are two remaining inefficiencies in the natural economy with flexible prices. First, output is inefficiently low as a result of monopolistically competitive product markets. Second, a non-negative price for money liquidity prevents the economy from reaching money satiation. We assume that the inefficiency from monopolistic competition is offset by an appropriate tax τ on labor income. Moreover, any inefficiency from incomplete money satiation does not affect the natural interest rate because utility is additively separable in consumption and money.

where Y_t^n denotes the level of output in the natural economy. Natural output is a function of the two shocks ψ_t and G_t and of the productivity shock, and this function is determined by the production side of the natural economy. Equation (3) shows that the natural interest rate varies with the two shocks G_t and ψ_t (and their expected future values), which also affect money demand (2). Both the money demand M_t/P_t and the natural interest rate r_t^n are affected by these shocks because they depend on the marginal utility of consumption.

This link between money demand and the natural interest rate does not rely on using the additively separable utility function. Instead, if we were to use a utility function that is non-separable in consumption and the money demand, the natural interest rate additionally depends on velocity shocks (see Chapter 3.2 in Woodford (2003)), while the composition of money demand in terms of shocks does not change. In this case, the money demand holds additional information about the natural interest rate because both variables depend on velocity shocks. Therefore, our approach of using the additively separable instead of a non-separable utility function is likely to downplay the role of the money demand for monetary policy. However, the quantitative implications of non-separable utility often turn out to be small.

C. Linearized model

To further illustrate the link between the money demand and the natural interest rate, we calculate the basic model accurately to the first order at the flexible-price steady state with zero inflation and with real money balances close to satiation (see Appendix A). This yields

$$m_t = \eta_y(Y_t - g_t) - \eta_i i_t + \varepsilon_t^m \,, \tag{4}$$

$$r_t^n = -E_t(1 - L^{-1})(a_t - \tfrac{\omega}{1+\omega}g_t) \,, \tag{5}$$

$$x_t = E_t x_{t+1} - (i_t - E_t \pi_{t+1}) + r_t^n \,, \tag{6}$$

$$\pi_t = \beta E_t \pi_{t+1} + \mu x_t + u_t \,. \tag{7}$$

In what follows, we express variables as percentage deviations from steady state. Equation (4) is the linearized version of money demand (2), with m_t denoting the percentage deviation of real money from its steady state. $\eta_y > 0$ denotes the elasticity of money demand with respect to income, and $\eta_i > 0$ denotes the semi-elasticity of money demand with respect to the nominal interest rate. The shock g_t, which we refer to as the IS shock, summarizes the two shocks G_t and ψ_t and constitutes the link between the money demand and the natural interest rate. This shock is an important source of fluctuations in the natural interest rate in estimated

DSGE models such as Andres, Lopez-Salido, and Nelson (2009) or Arestis, Chortareas, and Tsoukalas (2010).

Equation (5) defines the natural interest rate and is derived from the linearized version of the Euler equation (3) after substituting for natural output and imposing log utility of consumption. The natural interest rate varies with the IS shock g_t, which also perturbs money demand, and with the productivity shock a_t to the technology of intermediate firms. As indicated by the lead operator L^{-1}, the natural interest rate also depends on the expected value of these shocks. Natural output, $Y_t^n = a_t + \frac{1}{1+\omega}g_t$, depends on the same shocks as the natural interest rate. Shocks a_t, u_t, g_t, and ε_t^m are AR(1) processes with the uniform AR coefficient $0 \leq \rho < 1$, and the residuals of these shocks are mutually independent.[6] The parameter $\omega > 0$ summarizes the properties of the production side of the economy (see Appendix A).

Equation (6) is the intertemporal IS equation. To obtain this equation, we linearized the Euler equation in the actual economy and subtracted from it the Euler equation (3) in the natural economy. The IS equation relates the output gap, defined as actual output minus natural output, $x_t = Y_t - Y_t^n$, to the real interest rate gap. The real interest rate gap is the actual real rate, $i_t - E_t \pi_{t+1}$, minus the natural interest rate. If monetary policy moves the actual real rate one-for-one with the natural interest rate, the output gap is perfectly insulated from disturbances in the natural interest rate. If, however, monetary policy cannot establish a one-for-one co-movement, the output gap will contain at least some information about the natural interest rate.

Equation (7) is the New Keynesian Phillips curve (NKPC), and μ denotes its slope with respect to the output gap. The slope is a function of structural parameters (see Appendix A). Following convention, we add the ad hoc cost-push shock u_t to the NKPC. As a consequence of our MIU specification with additive separability, money demand does not appear in the NKPC or the IS equation and, as long as monetary policy does not respond to money demand, remains irrelevant for computing the equilibrium values of inflation and the output gap.

III. MONEY DEMAND AS AN INDICATOR OF THE NATURAL INTEREST RATE

While the natural interest rate is difficult to measure, money demand can play a role in identifying it. A convenient way to extract information about the natural interest rate from money

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[6] We impose the uniform AR coefficient to obtain a policy problem of the central bank that we can handle analytically. In our quantitative analysis below, we use numerical methods and consider various AR coefficients.

demand is to compute the money gap m_t^g defined as the difference between the money demand that is predicted by the endogenous determinants of money demand and the actual money demand,

$$m_t^g = \eta_y Y_t - \eta_i i_t - m_t \ , \tag{8}$$

where $\eta_y Y_t - \eta_i i_t$ are the endogenous determinants of money demand. The money gap resembles a generalized measure of money velocity that is adjusted for short-term nominal interest rates.

The money gap has a number of useful properties for policymakers. One is that it can be readily computed from actual data without the need to identify IS shocks g_t and velocity shocks ε_t^m separately, because standard assumptions about g_t and ε_t^m allow the money demand elasticities η_y and η_i to be recovered from simple quantity regression of actual money demand m_t on income Y_t and the nominal interest rate i_t (e.g., Teles and Uhlig (2010)).

Another useful property of the money gap is that it remains independent of changes in the monetary policy regime. We show this by substituting money demand (4) for m_t into the money gap (8):

$$m_t^g = \eta_y g_t - \varepsilon_t^m \ . \tag{9}$$

The money gap remains independent of changes in the monetary policy regime because the money gap is composed only of exogenous shocks.

Importantly, the money gap also co-moves positively with the natural interest rate. To demonstrate this, we combine equations (5) and (9) and derive the unconditional and contemporaneous correlation between the money gap and the natural interest rate, assuming the uniform AR coefficient across shocks,[7]

$$\mathrm{cor}(m_t^g, r_t^n) = \frac{\omega \eta_y}{\left(\eta_y^2 + \sigma_m^2/\sigma_g^2\right)^{\frac{1}{2}} \left(\omega^2 + (1+\omega)^2 \ \sigma_a^2/\sigma_g^2\right)^{\frac{1}{2}}} \ . \tag{10}$$

The correlation is positive as long as both the standard deviation of velocity shocks σ_m and the standard deviation of productivity shocks σ_a remain finite relative to the standard deviation of IS shocks σ_g.

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[7]The correlation is independent of ρ for the following reasons. Recall that the natural interest rate depends on the expected growth rate of shocks, while the money gap depends on the current value of shocks. Assuming AR(1) processes for the shocks implies that the expected growth rate of a shock is proportional to the current value of this shock. Further assuming the uniform AR coefficient ρ implies that the factor of proportionality is the same across shocks.
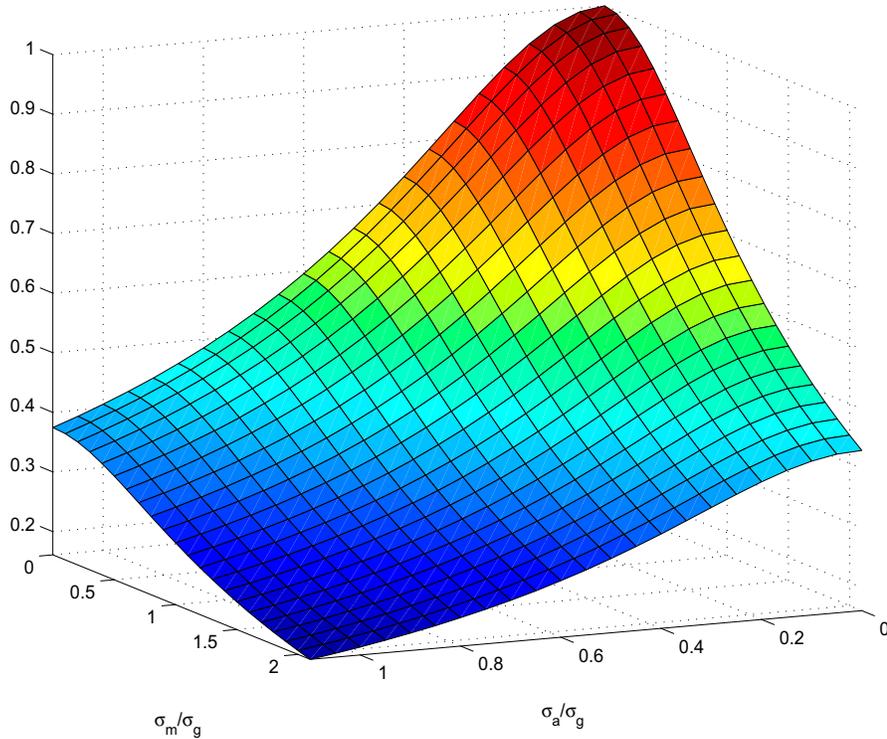
Figure 1. The figure shows the correlation between money gap m_t^g and natural interest rate r_t^n for different values of the relative standard deviation of velocity shocks σ_m/σ_g and of productivity shocks σ_a/σ_g.

Changes in the monetary policy regime do not affect the correlation (10) because, by construction, the money gap and the natural interest rate are independent of monetary policy. This implies that the correlation will not change once monetary policy starts exploiting it, and this is another useful property of the money gap from the perspective of monetary policy-makers. These findings constitute our first main result.

Result 1: The unconditional and contemporaneous correlation between the money gap and the natural interest rate is positive and independent of the monetary policy regime.

Figure 1 contains the correlation for different values of the ratios σ_m/σ_g and σ_a/σ_g, and with η_y and ω set to unity (Section V.C describes the calibration). The correlation between the money gap and the natural interest rate is large when the IS shock is volatile, because the IS shock drives both the money gap and the natural interest rate. In contrast, the correlation is small when the velocity shock is volatile, and this fits well with the conventional view that exogenous movements in velocity are a central impediment to money demand being a useful indicator variable for monetary policy. This conventional view reemerges in our setup be-

cause velocity shocks disturb the information that the money gap contains about the natural interest rate. Nevertheless, the correlation exceeds 0.15 even if we consider extremely volatile velocity shocks, and it is of the order 0.30 to 0.40 if we consider less extreme values for the standard deviations of velocity and productivity shocks.

IV. MONETARY POLICY

What use will a welfare-maximizing central bank make of the money gap? To answer this question, we consider a standard setup in which the central bank objective is to stabilize the variation in inflation and in the output gap. The central bank's loss function weighs these stabilization objectives according to the relative preference $\lambda_x > 0$,

$$\text{var}(\pi_t) + \lambda_x \text{var}(x_t) , \tag{11}$$

denoting the unconditional variance of the generic variable z_t as $\text{var}(z_t)$. As shown by Woodford (2003), this loss function is best understood as reflecting the central bank's desire to adjust the actual economy with sticky prices to the outcome represented by the state of the natural economy with flexible prices.

In the model described by equations (4) to (7), the central bank attempts to stabilize two types of disturbances: shocks to the natural interest rate and cost shocks. A policy rule that ensures that the policy rate i_t will follow closely to the natural interest rate r_t^n stabilizes inflation and the output gap perfectly with respect to shocks to the natural interest rate (Woodford (2003)). In contrast, cost shocks induce a tradeoff between stabilizing inflation versus the output gap, and the central bank's policy rule must be flexible enough to resolve this tradeoff optimally according to the preference λ_x.

Interest rate rules like the one proposed by Taylor (1993) condense this fairly broad optimal policy rule into a simple feedback rule of the form

$$i_t = r_t^n + \phi_x x_t + \phi_\pi \pi_t . \tag{12}$$

This rule prescribes that the policy rate i_t should respond one-for-one to the natural interest rate r_t^n. Thus, the central bank stabilizes inflation and the output gap perfectly with respect to shocks to the natural interest rate. In addition, the policy rate responds to both inflation and the output gap and therefore allows the central bank to resolve the tradeoff following cost shocks in the optimal fashion. Simple interest rules like equation (12) are interesting in their

own right but also have been shown to replicate the fully optimal monetary policy, which is typically considerably more complicated, reasonably well in a wide array of models.

A main caveat to the interest rate rule (12) is, however, that both the natural interest rate and the output gap are difficult to measure. As a consequence, practical applications of this interest rate rule often replace the time varying natural interest rate by a constant intercept term.[8] Along similar lines, the output gap is often estimated using statistical filters of the actual level of output instead of using information about natural output. While such a statistical approach to the output gap likely contains some useful information, this information will be conflated with considerable measurement error, or noise.

Our setup allows us to explore whether an optimizing central bank will exploit the correlation between money gap and natural interest rate to cope with the difficulty of measuring the natural interest rate and the output gap. We confine our analysis to a simple policy rule of the form

$$i_t = \phi_m m_t^g + \phi_x \tilde{x}_t + \phi_\pi \pi_t \ , \tag{13}$$

or slight modifications thereof, and operationalize the information barriers of real-world central banks as follows. First, the central bank does not observe the natural interest rate. However, instead of replacing the time-varying natural interest rate by a constant intercept, the central bank can respond to the money gap m_t^g. If the central bank finds the money gap a useful indicator variable, it will select a non-zero policy coefficient ϕ_m. Second, the central bank does not observe the output gap. Rather, it observes the measure $\tilde{x}_t$ that conflates the output gap with a noise shock, $\tilde{x}_t = Y_t - Y_t^n + \xi_t$. The noise shock ξ_t is exogenous, serially correlated with coefficient ρ, and mutually independent of all other shocks. Finally, the central bank observes inflation without any error.

A. Optimal policy coefficients and their interpretation

The policy problem of the central bank is to select the policy coefficients ϕ_m, ϕ_x, and ϕ_π of the simple rule (13) that minimize the loss (11) subject to the model's rational expectation equilibrium. This equilibrium is derived from the IS equation (6), the NKPC (7), the money gap (9), and the policy rule (13). In what follows, we set $\omega = 1$ and, without loss of generality, normalize the standard deviation of IS shocks σ_g to unity. To simplify, the baseline model as-

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[8]Economic theory predicts that the natural interest rate varies with shocks to preferences, technology, and absorption. Estimates of these shocks obtained from DSGE models are fairly volatile. Therefore, we consider it inadequate to replace the natural interest rate by a constant intercept term.

sumes no productivity shocks, $\sigma_a = 0$, and no serially correlated shocks, $\rho = 0$. We disregard these and other assumptions later on when we consider several model extensions.

The policy problem yields the optimal coefficients (see Appendix B)

$$\phi_m^\star = \frac{1}{2}\frac{\eta_y}{\eta_y^2 + \sigma_m^2} \, , \quad \phi_x^\star = \frac{1}{4}\frac{\sigma_m^2}{\sigma_\xi^2(\eta_y^2 + \sigma_m^2)} \, , \quad \phi_\pi^\star = \frac{\mu}{\lambda_x}\left(\frac{\sigma_\xi^2(\mu^2 + \lambda_x)}{\sigma_u^2}\phi_x^\star + \phi_x^\star + 1\right) \, . \quad (14)$$

We discuss the optimal coefficient of the money gap $\phi_m^\star$ first. This coefficient is positive if the income elasticity of money demand η_y is finite and positive and if velocity shocks have finite standard deviation σ_m. To summarize:

Result 2: The optimal policy coefficient of the money gap $\phi_m^\star$ is positive.

The central bank incorporates the money gap into its simple rule because the money gap indicates the natural interest rate, and knowing the natural interest rate allows the central bank to better stabilize inflation and the output gap. To see this, consider for a moment the model without velocity shocks, $\sigma_m = 0$. Combining the money gap (9) and the natural interest rate (5) shows that the money gap is proportional to the natural interest rate, $m_t^g = 2\eta_y r_t^n$. Furthermore, without velocity shocks, the coefficient of the money gap reduces to $\phi_m^\star = 1/(2\eta_y)$ such that $\phi_m^\star m_t^g = r_t^n$. The two remaining policy coefficients are $\phi_x^\star = 0$ and $\phi_\pi^\star = \mu/\lambda_x$. Substituting all this into the policy rule (13) yields

$$i_t = r_t^n + (\mu/\lambda_x)\pi_t \, .$$

The positive coefficient $\phi_m^\star$ implies that, effectively, the central bank moves the policy rate one-for-one with the natural interest rate. Assuming white-noise shocks, the IS equation reduces to $x_t = -i_t + r_t^n$ and illustrates that moving i_t one-for-one with r_t^n perfectly insulates the output gap, and therefore inflation, from shocks to the natural interest rate.

In the presence of velocity shocks, $\sigma_m > 0$, the coefficient $\phi_m^\star$ in (14) declines to below its value without velocity shocks. In this case, the coefficient actually corresponds to the OLS estimate of the regression $r_t^n = \phi_m m_t^g + e_t$, with e_t denoting the projection error:[9]

$$\phi_m^\star = \mathrm{cov}(m_t^g, r_t^n)/\mathrm{var}(m_t^g) \, .$$

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[9] To see that $\phi_m^\star = \phi_m^{ols}$, depart from $\phi_m^{ols} = \mathrm{cov}(m_t^g, r_t^n)/\mathrm{var}(m_t^g)$. Use the money gap (9) and the natural interest rate (5) to obtain $\mathrm{cov}(m_t^g, r_t^n) = \eta_y\sigma_g^2/2$ and $\mathrm{var}(m_t^g) = \eta_y^2\sigma_g^2 + \sigma_m^2$. Thus, $\phi_m^{ols} = \phi_m^\star = \frac{1}{2}\eta_y/(\eta_y^2 + \sigma_m^2)$ normalizing $\sigma_g = 1$.

The central bank reacts strongly to the money gap when money gap and natural interest rate co-move tightly. However, velocity shocks disturb the useful information contained in the money gap by reducing the co-movement and increasing $\text{var}(m_t^g)$. Therefore, if the central bank reacts to the money gap, the velocity shocks will distort inflation and the output gap, and this is detrimental to welfare. While the central bank accounts for these factors by scaling down its response to the money gap, it continues to select a positive $\phi_m^\star$.

The remaining two coefficients $\phi_x^\star$ and $\phi_\pi^\star$ govern, among other things, the optimal policy response to cost shocks u_t. A positive cost shock pushes inflation up and the output gap down and creates a tradeoff for the central bank. The optimal resolution of this tradeoff determines the magnitude of $\phi_\pi^\star$ relative to $\phi_x^\star$, which is evident from the coefficients in (14). However, cost shocks leave the central bank with a degree of freedom because they do not determine the magnitude of $\phi_x^\star$. Our next result explains how the central bank uses this degree of freedom optimally.

Result 3: The optimizing central bank combines the money gap and the observed output gap in a way that yields the strongest signal about the natural interest rate.

In other words, a central bank interested in the natural interest rate selects the coefficient $\phi_x^\star$ to supplement the information that the money gap holds about the natural interest rate. There are two aspects to this result. First, to see how the observed output gap can help at all, consider the IS equation. Adding the noise shock to both sides of this equation yields $\tilde{x}_t = -i_t + r_t^n + \xi_t$. Clearly, the observed output gap co-moves with the natural interest rate as long as the policy rate does not move one-for-one with the natural interest rate.

Second, the magnitude of $\phi_x^\star$ is determined by the information that the observed output gap holds about the natural interest rate in addition to the information that the money gap holds about the natural interest rate. Consider, for illustrative purposes, a modified version of the policy problem in which the central bank's only objective is to stabilize the output gap. The informational assumptions are the same as in the original policy problem. Further, let the central bank employ a policy rule with only the money gap and the observed output gap. The modified policy problem is to

$$\min_{\phi_m, \phi_x} \quad \text{var}(x_t) \quad \text{subject to} \quad i_t = \phi_m m_t^g + \phi_x \tilde{x}_t \ , \tag{15}$$

and subject to the model's rational expectation equilibrium derived from the IS equation (6) and the money gap (9). It is easy to verify that the modified and the original policy problem yield exactly the same optimal coefficients $\phi_m^\star$ and $\phi_x^\star$.

The modified policy problem admits interpreting the coefficient $\phi_x^\star$ along the lines of a signal extraction problem using two indicators for the natural interest rate: the money gap and the observed output gap. To rewrite the central bank objective $\mathrm{var}(x_t)$, substitute $i_t = \phi_m m_t^g + \phi_x \tilde{x}_t$ for the policy rate into the IS equation and use the definition $\tilde{x}_t = x_t + \xi_t$ and the independence of the noise shock ξ_t, which yields

$$\mathrm{var}(x_t) = \left(\frac{1}{1+\phi_x}\right)^2 \mathrm{var}\left(r_t^n - \phi_m m_t^g\right) + \left(\frac{\phi_x}{1+\phi_x}\right)^2 \mathrm{var}(\xi_t) . \tag{16}$$

This way of writing the central bank objective recovers the OLS interpretation of the money gap coefficient, since minimizing $\mathrm{var}(x_t)$ requires minimizing $\mathrm{var}\left(r_t^n - \phi_m m_t^g\right)$ with respect to ϕ_m, and this yields the OLS estimator.

In addition to the money gap, however, the central bank uses the observed output gap as an indicator of the natural interest rate. The coefficient $\phi_x^\star$ that minimizes $\mathrm{var}(x_t)$ determines the relative weight attached to each indicator, depending on the particular indicator's usefulness:

$$\phi_x^\star = \mathrm{var}(r_t^n - \phi_m m_t^g)/\mathrm{var}(\xi_t) .$$

The observed output gap is a useful indicator when it contains only a small amount of noise. In this case, $\mathrm{var}(\xi_t)$ is small and the central bank selects a high value of $\phi_x^\star$. Conversely, the central bank selects a low value of $\phi_x^\star$ if $\mathrm{var}\left(r_t^n - \phi_m^\star m_t^g\right)$ is small relative to $\mathrm{var}(\xi_t)$ because small errors $e_t = r_t^n - \phi_m^\star m_t^g$ make the money gap a useful indicator. Generally, the central bank considers both indicators at the same time by selecting some intermediate value for $\phi_x^\star$. Thereby, it improves upon the signal that each indicator can provide individually, because the projection error e_t and the noise ξ_t are independent of each other.[10]

The ratio of policy coefficients provides a compact summary of the relative usefulness of the money gap and the observed output gap as indicators of the natural interest rate:

$$\phi_m^\star/\phi_x^\star = 2\eta_y \left(\sigma_\xi/\sigma_m\right)^2 . \tag{17}$$

When the noise in the observed output gap is volatile relative to velocity shocks, the central bank attaches a relatively large weight to the money gap. Conversely, when velocity shocks

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[10]Unless e_t and ξ_t are perfectly positively correlated, the central bank is always better off considering both the money gap and the observed output gap to extract the strongest signal about the natural interest rate.

and, thereby, projection errors are volatile, the money gap is uninformative about the natural rate, and the central bank attaches a relatively small weight to the money gap.[11]

The related literature has focused mainly on whether or not money demand contains useful information about the current level of output and about the output gap. In contrast, our central finding is that money demand contains useful information about a different variable, namely, the natural interest rate.[12] Interestingly, we also find that the central bank utilizes the output gap, which it measures with error, as yet another indicator of the natural interest rate. Both of these findings are consistent with the view that the natural interest rate, but less so the output gap, constitutes the crucial variable for policymakers in New Keynesian models. Intuitively, in these models, a central bank that pursues a simple interest rate rule does not need to know the output gap in order to stabilize it.

B. Extensions of the basic model

Extending the basic model by adding serially correlated shocks or a central bank preference for stabilizing the nominal interest rate leaves our results intact. Moreover, while adding productivity shocks tends to reduce the money gap's relative usefulness without eliminating it, allowing for shocks to the household's discount rate could add to the relative usefulness of the money gap or reduce it.

1. Serially correlated shocks

A more general model allows for serially correlated shocks. Assuming a uniform AR coefficient $\rho \in (0,1)$, the optimal policy coefficient of the money gap becomes

$$\phi_m^\star = \frac{1}{2} \left(1 - \rho\right) \frac{\eta_y}{\eta_y^2 + \sigma_m^2} \ .$$

When the velocity shock ε_t^m is serially correlated, σ_m^2 denotes the variance of the residual to ε_t^m instead of the variance of ε_t^m itself, and the same holds true for the other shocks.

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[11]The mapping between the variances of projection errors and velocity shocks is $\sigma_e^2 = \frac{1}{4}\sigma_m^2/(\eta_y^2 + \sigma_m^2)$. We obtain it by plugging the money gap (9) and the natural interest rate (5) into $e_t = r_t^n - \phi_m^\star m_t^g$ and computing variances.

[12]We sidestep the role that the money demand plays in informing about the output by eliminating the transaction motive of money demand from the money gap. We do this because the literature has largely concluded that this role is minor.

The optimizing central bank continues to select a positive coefficient $\phi_m^\star$ of the money gap, and Result 2 generalizes to the case of serially correlated shocks. Quantitatively, $\phi_m^\star$ decreases the more serially correlated the shocks are. This happens because the household anticipates that serially correlated shocks impact current as well as future consumption and smoothes its consumption path. Thus, there is less need for the natural interest rate to respond to, say, IS shocks. Unlike the natural interest rate, however, the money gap's response to IS shocks does not depend on the serial correlation of shocks, and the central bank reduces its reaction to the money gap to avoid overreaction. Nevertheless, serially correlated shocks do not reduce the information that the money gap holds about the natural interest rate, because the composition of the money gap in terms of IS and velocity shocks remains unchanged.

The same logic applies to the coefficient $\phi_x^\star$ of the observed output gap. Accordingly, despite serially correlated shocks, the ratio of policy coefficients $\phi_m^\star/\phi_x^\star$ still corresponds to equation (17), and Result 3, which pertains to the relative usefulness of the money gap and the observed output gap, extends to the model with serially correlated shocks.

2. Interest rate stabilization

A more general model also allows for a central bank preference for stable nominal interest rates. This extension is one way to capture that, at times, the natural interest rate can be too volatile for the nominal interest rate to follow it.[13] The central bank may then have to relax the otherwise tight co-movement between the nominal interest rate and the natural interest rate, and this may affect the usefulness of the money gap as an indicator of the natural rate.

We analyze this issue by introducing the preference for stable nominal interest rates into the central bank objective function,[14]

$$\mathrm{var}(\pi_t) + \lambda_x \mathrm{var}(x_t) + \lambda_i \mathrm{var}(i_t) , \quad \lambda_i > 0 . \tag{18}$$

Solving the policy problem of the central bank with $\lambda_i > 0$ (maintaining $\rho = 0$) yields the optimal policy coefficient of the money gap:

$$\phi_m^\star = \frac{1}{2} \frac{\lambda_x \eta_y}{(\lambda_x + \lambda_i)(\eta_y^2 + \sigma_m^2) + 0.25\lambda_i \sigma_m^2 [\mu^2/\sigma_u^2 + 1/\sigma_\xi^2]} .$$

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[13]One reason for this is the zero lower bound on nominal interest rates.

[14]Alternatively, we could capture the preference for stable interest rates by introducing the lagged nominal interest rate into the policy rule. We pursue this approach in the quantitative model below to save having to deal with the additional state variable in our analytical solutions.

This coefficient is still positive, and thus extends Result 2, which pertains to the indicator role of the money gap, to the more general case of a central bank that prefers stable nominal interest rates. Quantitatively, $\phi_m^\star$ decreases in λ_i, because the denominator increases in λ_i. This relationship reflects that the central bank moves the nominal interest rate less than one-for-one with the natural interest rate when it prefers stable nominal interest rates. Moreover, dividing the money gap coefficient by the output gap coefficient for the case $\lambda_i > 0$ yields exactly the same ratio as before, showing that Result 3 also applies to this model extension.

3. Productivity shocks

Yet another way to extend the model is to allow for productivity shocks. As Figure 1 illustrates, the correlation between the money gap and the natural interest rate drops in the presence of productivity shocks. This reflects the fact that these shocks affect the natural interest rate but not the money gap. The question is whether and how this changes the usefulness of the money gap for monetary policy.

The optimal policy coefficients of the money gap and the observed output gap for the case $\sigma_a > 0$ (maintaining $\lambda_i = \rho = 0$) are

$$\phi_m^\star = \frac{1}{2} \frac{\eta_y}{\eta_y^2 + \sigma_m^2} \quad , \quad \phi_x^\star = \frac{1}{4} \frac{\sigma_m^2}{\sigma_\xi^2 \left(\eta_y^2 + \sigma_m^2 \right)} + \frac{\sigma_a^2}{\sigma_\xi^2} . \tag{19}$$

Productivity shocks leave the money gap coefficient untouched, generalizing Result 2 to the model with productivity shocks. To see the reason for this, consider again the modified policy problem (15). As shown above, the central bank sets the coefficient ϕ_m to the OLS estimate of the regression $r_t^n = \phi_m m_t^g + e_t$. While productivity shocks inject additional variation into r_t^n, this variation is orthogonal to the money gap and, therefore, does not alter the optimal money gap coefficient. Instead, productivity shocks are absorbed into the error e_t and increase its variance.[15]

The increase in the error variance matters for the central bank's choice of ϕ_x, however. The increased $\text{var}(e_t)$ tilts the central bank's response towards the observed output gap, as the variance of the noise ξ_t associated with the output gap remains unchanged. Accordingly, $\phi_x^\star$ increases with the variance of productivity shocks, as shown in (19) and illustrated by the ra-

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[15]It is straightforward to show using $\phi_x^\star = \sigma_e^2/\sigma_\xi^2$ that $\sigma_e^2 = \frac{1}{4}\sigma_m^2/(\eta_y^2 + \sigma_m^2) + \sigma_a^2$.

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$$\frac{\phi_m^\star}{\phi_x^\star} = 2\eta_y \, \frac{\sigma_\xi^2}{\sigma_m^2 + 4\sigma_a^2(\eta_y^2 + \sigma_m^2)} \, .$$

In words, the central bank puts relatively less weight on the money gap when productivity shocks are volatile. Result 3 still holds in this extension of the model, but the relative usefulness of the money gap diminishes as gyrations in productivity become larger.

4. Discount factor shocks

The last extension of the basic model we consider is a stochastic discount factor, namely a shock to the discount rate β of the household. This shock makes the household discount the expected future utility at a different rate, at least temporarily. In equilibrium, this shock will affect the natural interest rate and, therefore, the indicator role of the money gap. Depending on the magnitudes, allowing for a discount factor shock could add to the usefulness of the money gap for policymakers or reduce it.

A convenient approach to model the discount factor (DF) shock is to introduce co-movement among the IS shock g_t, the velocity shock ε_t^m, and the productivity shock a_t. This co-movement reflects that the discount rate β, and hence the DF shock, affects all terms of the period utility function uniformly.[16] We include the productivity shock because it is isomorphic to a shock to the disutility from work $h(N_t)$ that expands the labor supply. The new parameter $b \in [0, 1]$ governs the amount of the co-movement among these shocks (see Appendix B). A value of b equal to unity transforms the IS shock g_t into the DF shock, with any movement in g_t uniformly affecting all terms of the period utility function, while both the velocity shock and the productivity shock also vary independently of g_t. Intermediate values of b make g_t a weighted average of the IS shock and the DF shock, and b equal to zero recovers the original IS shock.

Solving the policy problem of the central bank with $b \in [0, 1]$ (maintaining $\sigma_a > 0$ and $\lambda_i = \rho = 0$) yields the optimal policy coefficients for the money gap and the observed output gap,

$$\phi_m^\star = \frac{1}{2} \frac{\eta_y(1-b)(1+b)}{\eta_y^2(1-b)^2 + \sigma_m^2} \quad , \quad \phi_x^\star = \frac{1}{4} \frac{\sigma_m^2(1+b)^2}{\sigma_\xi^2 \left[\eta_y^2(1-b)^2 + \sigma_m^2 \right]} + \frac{\sigma_a^2}{\sigma_\xi^2} \, .$$

These coefficients depend on the new parameter b, but are otherwise very similar to the policy coefficients that we obtained for the case of including productivity shocks.

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[16]Alternatively, we could introduce a completely new shock to the discount rate β, but this would yield a less parsimonious representation of the basic model than our approach.
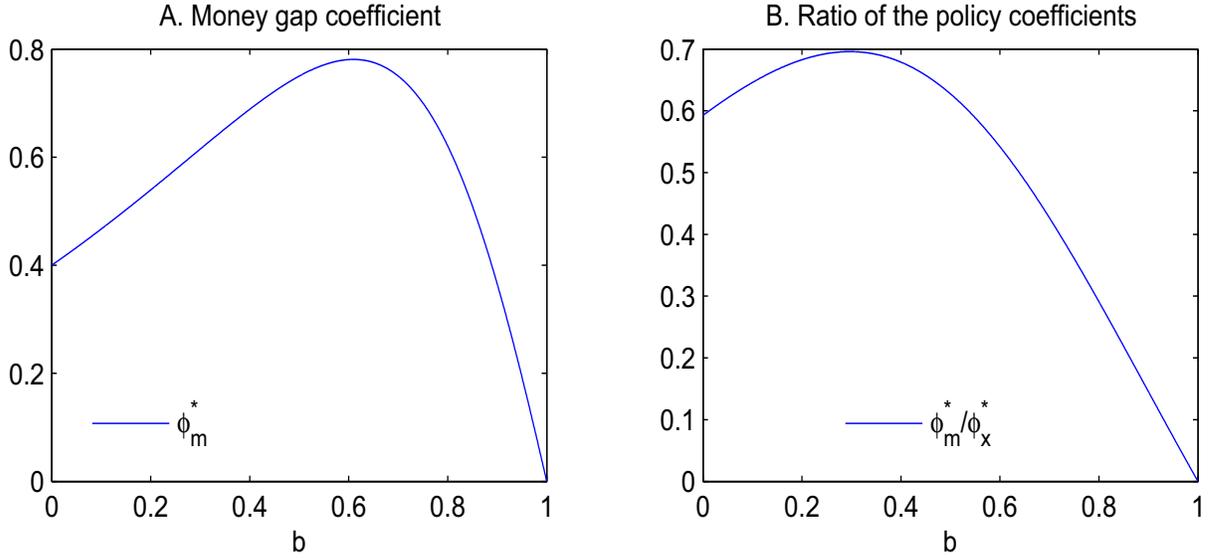
Figure 2. Panel A shows the optimal policy coefficient of the money gap in the basic model with a discount factor shock as a function of b. Panel B shows the ratio of the money gap and the observed output gap coefficient as a function of b. The calibration is $\eta_y = 1$, $\sigma_m = \frac{1}{2}$, $\sigma_a = \frac{1}{2}$, and $\sigma_\xi = \frac{2}{3}$, normalizing σ_g to unity. For more details on the calibration, see the Section V.C.

The Panel A of Figure 2 shows that the money gap coefficient increases for small to intermediate values of b and decreases for extremely large values of b. To understand this pattern, note that the OLS interpretation of the money gap coefficient in the basic model still applies, $\phi_m^\star = \text{cov}(m_t^g, r_t^n)/\text{var}(m_t^g)$, but now both the covariance and the variance depend on b and decline by raising b (see Appendix B). The money gap coefficient increases for the intermediate values of b, because the covariance declines more slowly than the variance.

The declining variance of the money gap is related to the money demand (2). While the IS shock affects the money demand through the marginal utility of consumption, the DF shock does not. Accordingly, transforming the IS shock gradually into the DF shock reduces the impact of g_t on the money demand, and this reduces the variance of the money gap. The covariance $\text{cov}(m_t^g, r_t^n)$ declines more slowly than the variance of the money gap because raising b increases the impact of g_t on the natural interest rate. Raising b transforms the IS shock that operates along both the intratemporal and the intertemporal margin into the DF shock that operates only along the intertemporal margin, and the intertemporal DF shock turns out to affect the natural interest rate more.

The Panel B of Figure 2 shows the ratio $\phi_m^\star/\phi_x^\star$ of policy coefficients, as a measure of the relative usefulness of the money gap versus the observed output gap. The central bank tends to

respond more to the observed output gap for large values of b because in this case the money gap turns into a more noise indicator of the natural interest rate. However, raising b from small to intermediate values, the central bank actually finds the money gap increasingly useful relative to the observed output gap. Overall, we conclude that the money gap will remain a useful indicator for policymakers under plausible extensions of the basic model.

V. A CALIBRATED QUANTITATIVE MODEL

With a number of factors influencing the indicator role of the money gap for monetary policy, the question becomes how relevant the money gap would be in a model that represents real world economies more realistically. To this end, we consider a quantitative model with richer dynamics and calibrate it for the euro area and the U.S.

A. Quantitative model

The quantitative model enriches the basic model by adding habit persistence and price indexation. Firms index prices to the rate of inflation in the previous period if they cannot adjust their prices optimally. This yields a hybrid NKPC that relates a polynomial with leads and lags in the inflation rate to the output gap and cost shocks. Similarly, habit formation in consumption yields a hybrid intertemporal IS equation that relates a polynomial with leads and lags in the output gap to the actual real rate and the natural interest rate. We consider the same shocks as in the basic model, including productivity shocks, and allow for various serial correlation across shocks. The Appendix A contains further details about the quantitative model.

A final and especially important generalized feature of the quantitative model is habit formation in the household's stock of real money demand. Habit formation in real money demand works exactly like habit formation in consumption and extends the static money demand function used earlier to a dynamic money demand function.[17] A dynamic money demand function matters greatly from an empirical point of view because the literature emphasizes the strong partial adjustment component of estimated money demand functions.

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[17]We take the MIU specification of dynamic money demand using habit formation from Weber (2008). Two alternative specifications of dynamic money demand are money adjustment costs and the assumption that adjusting money balances creates disutility. Goldfeld (1973) and Laidler (1990) use the adjustment cost specification, whereas Chari, Christiano, and Eichenbaum (1995), Christiano and Gust (1999), Nelson (2002), and Andres, Lopez-Salido, and Nelson (2009) use the disutility specification.

B. Dynamic money demand function and the money gap

Calculated to the first order, the dynamic money demand function contains leads and lags in real money demand and relates them to the nominal interest rate and to a measure of income:

$$m_t - \phi m_{t-1} - \varepsilon_t^m = \phi \beta E_t[m_{t+1} - \phi m_t - \varepsilon_{t+1}^m] + \theta_y I(Y_t, g_t) - \theta_i i_t . \qquad (20)$$

The degree of habit formation $\phi \in [0,1)$ in the money demand, which can differ from habit formation in consumption $\eta \in [0,1)$, governs the dynamics of money demand. The short-run elasticities of the money demand are $\theta_k = (1 - \phi)(1 - \phi\beta)\eta_k$, with $k = y, i$, and thus are proportional to the long-run elasticities η_k. As before, ε_t^m denotes velocity shocks. Habit formation in the money demand gives expected future money demand a role in the money demand function. Expected future money demand matters because the household anticipates that today's money demand will become tomorrow's habit. The measure of income that determines money demand (20) is equal to $I(Y_t, g_t) = E_t(1 - \eta\beta L^{-1})[(1 - \eta L)Y_t - g_t]/((1 - \eta\beta)(1 - \eta))$. It is a two-sided filter of output and the IS shock, which derives from assuming habit formation in consumption.

We proceed along the same lines as before to extract the money gap as the difference between the predicted money demand and the actual money demand. This time, however, we incorporate the dynamic money demand function. Iterating equation (20) forward yields a representation of the money demand function that emphasizes the partial adjustment (see Appendix C),

$$m_t = \phi m_{t-1} + (1 - \phi)m_t^p - m_t^g . \qquad (21)$$

Actual money demand m_t partially adjusts at rate ϕ from past money holdings m_{t-1} to predicted money demand m_t^p. The remaining part of actual money demand is the money gap m_t^g.

The predicted money demand is an infinite weighted sum over current and expected future endogenous determinants of the money demand, with weights decaying at rate $\phi\beta$,

$$m_t^p = (1 - \phi\beta)E_t \sum_{s=0}^{\infty} (\phi\beta)^s [\eta_y I(Y_{t+s}, 0) - \eta_i i_{t+s}] . \qquad (22)$$

As in the basic model, the measure of income $I(Y_t, 0)$ in m_t^p omits the IS shock g_t. Likewise, ignoring habit formation in consumption and in money demand, $\phi = \eta = 0$, the predicted money demand m_t^p contains only contemporaneous variables because all forward-looking variables vanish. In general, however, the predicted money demand refers to the infinite hori-

zon and therefore, aligns well with the quantity theory of money demand, which prevails on average and over long periods of time.

Combining the partial adjustment representation (21) and the predicted money demand (22) yields the money gap in the quantitative model (see Appendix C):

$$m_t^g = \eta_y \, E_t \left(\frac{(1-\phi)(1-\phi\beta)}{1-\phi\beta L^{-1}} \right) \left(\frac{1-\eta\beta L^{-1}}{(1-\eta)(1-\eta\beta)} \right) g_t - \varepsilon_t^m \, . \tag{23}$$

As before, the money gap is composed of IS and velocity shocks. With the money gap composed only of shocks, it continues to remain unperturbed by shifts in the monetary policy regime and is thus, immune to the Lucas critique.

The main difference between the money gap in the basic model and the money gap in the quantitative model is that IS shocks enter the money gap in the quantitative model in a much richer fashion. Both the dynamics in money demand ϕ and the habit formation in consumption η affect the composition of the money gap in terms of shocks, except for the case $\eta = \phi$ when these parameters exactly offset each other.

Evidently, ϕ and η are crucial parameters to determine the relevance of the money gap for monetary policymakers in the enriched model. For instance, in the extreme case in which ϕ is near unity and η is equal to zero, the money gap attaches only a small weight to IS shocks. In this case, the household determines the money demand mostly by the force of habit and only to a very small degree by transaction and opportunity cost motives. Consequently, the money demand elasticities with respect to $I(Y_t, g_t)$ and i_t are minuscule, and the money demand (20) is basically self referential. Conversely, when η is near unity and ϕ is equal to zero, IS shocks are the predominant driving force of the money gap.

C. Calibration

Our approach explores a plausible support range for the key parameters that matter most for the indicator role of the money gap, while holding other parameters fixed at reasonable point estimates. Specifically, the degree of dynamics in the money demand is in the range $\phi \in [0.3, 0.7]$. This range comprises estimates obtained from DSGE models with MIU specifications of dynamic money demand and from reduced-form time series models estimated with macro and micro data (see Appendix D). Onatski and Williams (2004) and Smets and Wouters (2003) estimate a degree of habit formation in consumption of 0.4 and 0.57, respec-

tively, using euro area data from Fagan, Henry, and Mestre (2005), and we center the range for η around 0.5, $\eta \in [0.3, 0.7]$.

Furthermore, we consider a wide range of values for the relative standard deviation of velocity shocks, $\sigma_m / \sigma_g \in [\frac{1}{4}, 2]$, to capture an important distinction in the data for the U.S. and the euro area. Whereas velocity shocks in the U.S. tend to be considerably more volatile than IS shocks, velocity shocks in the euro area tend to be considerably less volatile than IS shocks (see Appendix D). We return to this observation below. Finally, the range for the relative standard deviation of noise shocks is $\sigma_\xi / \sigma_g \in [\frac{1}{3}, 1]$ (see Appendix D).

The other parameters are calibrated to reasonable point estimates. A subjective discount rate β of 0.99 implies an annual real interest rate of close to three percent. The long-run income elasticity of money demand η_y is unity. The (absolute) long-run interest rate semi-elasticity of money demand η_i is 10, following Dotsey and Hornstein (2003). We use 0.32 for the degree of price indexation $\kappa \in [0, 1)$ and 0.46 for the household's relative risk aversion σ. Both values are from Onatski and Williams (2004). The slope of the hybrid NKPC is equal to 0.114, which we computed from the deep parameters reported in Appendix D.

The standard deviation of productivity shocks σ_a is equal to $\frac{1}{2}\sigma_g$, while the standard deviation of cost shocks σ_u is equal to $\frac{1}{3}\sigma_g$. These values are broadly consistent with empirical evidence for both the U.S. and the euro area. We set σ_g to 0.01, though the exact value does not matter for our results. As a benchmark calibration, all shocks except cost shocks u_t are serially correlated with AR coefficient equal to 0.9, but we check for the robustness of our results using various AR coefficients. Cost shocks are often estimated with considerably less serial correlation than the other shocks, and therefore we set the AR coefficient of cost shocks to 0.2.

VI. QUANTITATIVE RESULTS

The results of the calibrated quantitative model confirm that the money gap co-moves with the natural interest rate and suggest that the money gap matters for monetary policy, but there are a number of unexpected twists. Our calibration approach yields a large correlation between the money gap and the natural interest rate. Interestingly, lags of the money gap co-move most strongly with the natural interest rate because the money gap tends to lead the natural interest rate by several quarters. Similarly, the role of the money gap as an indicator in monetary policy is substantial, and this role is more pronounced in an economy closer to the

euro area than the U.S. This is due to the fact that money demand in the euro area tends to be less subject to shocks in velocity. As a consequence, the information contained in the money gap tends to be of higher quality in the euro area than in the U.S.

A. Correlation between the money gap and the natural interest rate

The money gap and the natural interest rate co-move independently of the monetary policy regime, even in the quantitative model. This happens because the money gap (23) still consists only of shocks though its dynamics in terms of these shocks are richer in the quantitative model than in the basic model. Likewise, the natural interest rate consists only of shocks because monetary policy still cannot influence the natural economy with flexible prices in the quantitative model.

At the same time, the richer dynamics in the quantitative model suggest that the money gap and the natural interest rate co-move also in terms of their leads and lags. We account for this possibility and compute the dynamic correlation $\mathrm{cor}(m_{t+s}^g, r_t^n)$ using $s = \cdots -1, 0, 1, \cdots$, where $s = 0$ yields the contemporaneous correlation. We explore the correlation over the full support range for the key parameters ϕ, η, and σ_m, which we calibrated in Section V.C, using a sampling procedure that randomly draws from uniform distributions with parameter-specific support.[18] We then report the median correlation and the 10% and 90% quantile at each s for a large number of such draws.

Figure 3 plots the dynamic correlation between the money gap and the natural interest rate for the calibration with $\sigma_m \leq \sigma_g$, which resembles the euro area. The median value of the contemporaneous correlation is 0.1. As in the basic model, this correlation emerges from IS shocks, which affect both the money gap and the natural interest rate through the marginal utility of consumption. However, more striking than the magnitude of the contemporaneous correlation is the pronounced lead and lag pattern. The lagged money gap co-moves considerably stronger with the contemporaneous natural interest rate than the contemporaneous money gap or its leads. For example, the money gap lagged by two, three, or four quarters has a correlation with the contemporaneous natural interest rate of around 0.40. Thus, in the quantitative model, the money gap leads the natural interest rate pronouncedly.

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[18]The standard deviation σ_ξ of the noise ξ_t in the output gap that the central bank observes does not affect the correlation because the correlation is independent of monetary policy.
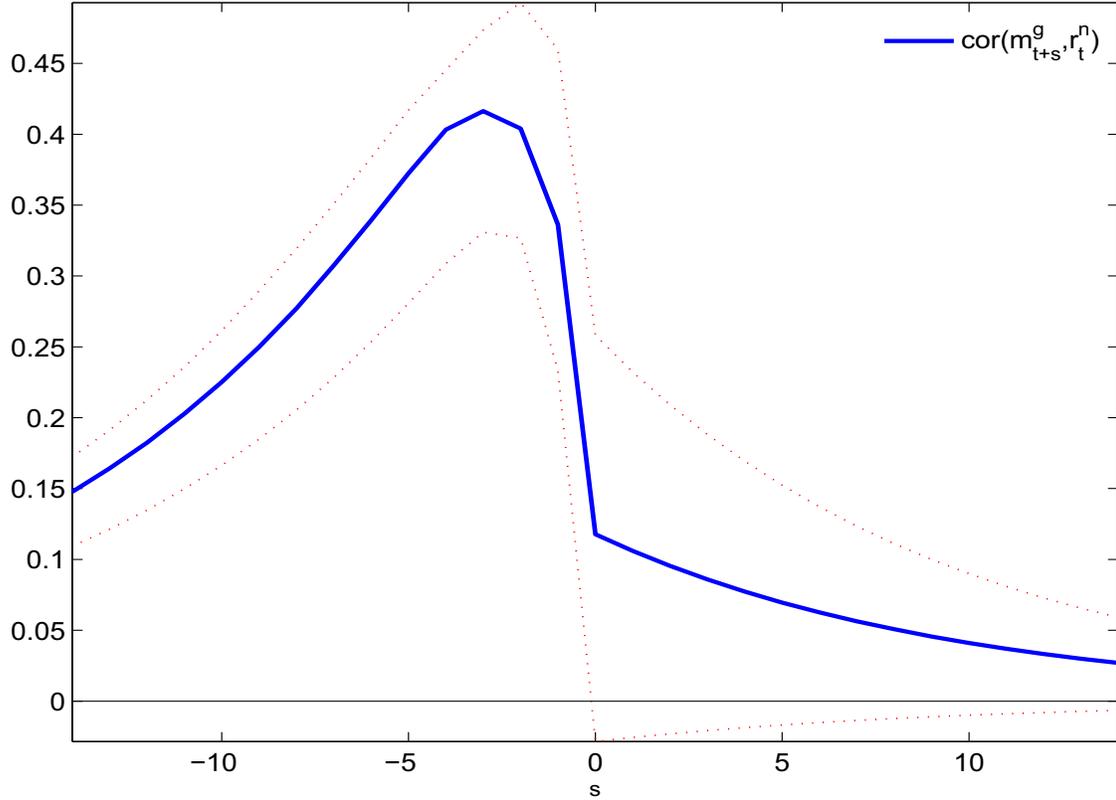
Figure 3. Median dynamic correlation between money gap and natural interest rate jointly with the 10% and 90% quantile, at each s. We draw parameters ϕ, η, and σ_m/σ_g independently from uniform distributions with support range $[0.3, 0.7]$, $[0.3, 0.7]$, and $[1/4, 1]$, respectively. Quantiles are computed from 1,000 draws. All other parameters are calibrated as described in Section V.C.

The reason for this is that the peak response of the money gap to the IS shock occurs on impact, while the peak response of the natural interest rate occurs with a delay. With habit formation in consumption, the household attempts to smooth both the level and the (quasi) change in consumption such that these variables will respond only gradually to the IS shock. This leads to a hump-shaped response in the natural interest rate, and the hump shape delays the peak response of the natural interest rate. As a result, the money gap leads the natural interest rate.

Figure 3 also indicates that the ranges that we calibrate for the key parameters ϕ, η, and σ_m create considerable variation around the median correlation, reading off $\mathrm{cor}(m_{t+s}^g, r_t^n)$ at each lead or lag. We gauge the impact of the different parametrizations from the 10% and 90% quantiles surrounding the median correlation. The correlation between the lagged money gap

and the natural interest rate is significant and large, and this finding is robust across parametrizations.

B. Optimal simple policy rules with money demand

Following the by now familar pattern, we ask what coefficients a welfare-maximizing central bank would select in the quantitative model for a simple interest rule that incorporates the money gap. We adopt the same informational assumptions as in Section IV. The central bank minimizes a quadratic loss function obtained from expanding the lifetime utility of the representative household in the quantitative model accurately to the second order (see Appendix A).[19] The loss penalizes variation in the (quasi) change of inflation and the output gap,

$$L = (1 - \beta)E_0 \sum_{t=0}^{\infty} \beta^t \{ (\pi_t - \kappa \pi_{t-1})^2 + \lambda_x(x_t - \delta x_{t-1})^2 + \lambda_i i_t^2 \} , \qquad (24)$$

where δ is a function of η, i.e., the habit formation in consumption, and other deep parameters. The preference weight λ_x is also a function of deep parameters and equals 0.019 in our calibration. With $\lambda_i > 0$, the central bank prefers stable nominal interest rates. Following Woodford (2003), we set λ_i to five times the value for the weight λ_x, which yields 0.095.

To capture the dynamic characteristics of the money gap, the central bank optimizes a policy rule that encompasses either the contemporaneous or lagged money gap, the observed output gap, and inflation:

$$i_t = \phi_{m,j} m_{t-j}^g + \phi_x \tilde{x}_t + \phi_\pi \pi_t , \quad j \geq 0. \qquad (25)$$

As before, $\tilde{x}_t$ denotes the observed output gap that is equal to the actual output gap x_t conflated with noise ξ_t. The central bank selects the policy coefficients $\phi_{m,j}, \phi_x$, and ϕ_π to minimize the loss function (24) conditional on the rational expectation equilibrium of the quantitative model. While this policy problem is of exactly the same form as the one analyzed in Section IV, we now resort to numerical optimization because of the more complicated structure of the quantitative model.

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[19]Weber (2008) shows that loss function (24) is proportional to the utility-based loss function in the quantitative model only after adding the term $\lambda_m(m_t - \phi m_{t-1} - \varepsilon_t^m)^2$. Here, we impose $\lambda_m = 0$ but our quantitative results are insensitive to working with the utility-based weight $\lambda_m > 0$, which is a function of deep parameters.

Table 1. Optimal policy coefficients (annualized) for euro area calibration.

σ_ξ/σ_g	$\phi_{m,0}^\star$	$\phi_{m,1}^\star$	$\phi_x^\star$	$\phi_\pi^\star$	% change in L
$1/3$	0.329		0.354	1.613	-5.644
$2/3$	0.327		0.083	1.659	-6.687
1	0.327		0.037	1.667	-6.883
$1/3$		0.295	0.341	1.611	-5.799
$2/3$		0.294	0.080	1.657	-6.865
1		0.294	0.035	1.665	-7.066

Notes: The policy rule with annualized nominal interest rate and annualized inflation rate is $4i_t = [4\phi_{m,j}]m_{t-j}^g + [4\phi_x]\tilde{x}_t + [\phi_\pi]4\pi_t$, and reported coefficients are those in square brackets. The last column reports the percentage change in L relative to the case with a zero coefficient for the money gap. For reference, the zero coefficient of the money gap yields the optimized policy rule $4i_t = 0.185\tilde{x}_t + 1.914(4\pi_t)$ using $\sigma_\xi/\sigma_g = \frac{2}{3}$.

C. Quantifying the optimal policy coefficients

Following up on our analytical results, we explore the magnitude of the optimal coefficient $\phi_{m,j}^\star$ for the money gap and its relation to the coefficient $\phi_x^\star$ for the observed output gap. To avoid numerical overflow, we set the key parameters ϕ, η, σ_m, and σ_ξ to specific values rather than drawing them from distributions, and check for robustness in the next section. Both the degree of dynamics in the money demand ϕ and the degree of habit formation in consumption η are equal to 0.5. In this case, the money gap (23) remains a contemporaneous function of shocks, as in the basic model. The relative standard deviation of velocity shocks σ_m/σ_g is $\frac{1}{2}$, which is a conservative choice given that estimates obtained from euro area data put the ratio closer to $\frac{1}{3}$ (see Appendix D). Finally, we consider three values for the relative standard deviation of noise shocks σ_ξ/σ_g, namely $\frac{1}{3}$, $\frac{2}{3}$, and unity, to shed light on how sensitive the money gap coefficient reacts to changing the indicator quality of the observed output gap.

Table 1 reports the coefficients optimized for the policy rule (25). The second column shows that the optimal coefficient $\phi_{m,0}^\star$ of the rule with the contemporaneous money gap ($j = 0$) is positive, in line with our analytical results, and around 0.33. The money gap coefficient falls to a value close to zero when we incorporate the natural interest rate in addition to the money gap into the policy rule (not shown in the table). This finding confirms that the money gap serves as an indicator variable of the natural interest rate also in the quantitative model. The last column of Table 1 shows that using a policy rule that incorporates the money gap reduces

the welfare loss from incomplete stabilization by about six percent, relative to using a policy rule that is optimized for the case without the money gap.[20]

The optimal coefficients of the money gap and the observed output gap of the rule with the contemporaneous money gap ($j = 0$) react as expected to increasing the amount of noise σ_ξ/σ_g in the observed output gap. Table 1 shows that when the output gap is measured less accurately, the money gap coefficient changes only marginally. However, the output gap co-efficient decreases substantially from 0.35 to 0.04 because the central bank tilts its response towards the best available indicator of the natural interest rate. These results also mirror those by Smets (2002) and Rudebusch (2001), namely that output gap uncertainty reduces the re-sponse to the output gap when the central bank pursues a simple but optimal rule.[21]

Figure 4 plots the coefficient $\phi_{m,0}^\star$ and the ratio $\phi_{m,0}^\star/\phi_x^\star$ for different values of the standard deviation of velocity shocks relative to noise shocks, σ_m/σ_ξ, and for various degrees of dy-namics ϕ in the money demand (scales are inverted for reasons of display). The coefficient $\phi_{m,0}^\star$ in Panel A is positive and increases when the relative variability of velocity shocks falls, confirming Result 2 earlier. Varying the dynamics in the money demand has ambiguous ef-fects on the money gap coefficient depending on the values of σ_m/σ_ξ, but the quantitative impact of ϕ seems limited overall. In Panel B, the ratio $\phi_{m,0}^\star/\phi_x^\star$ is higher, the more informa-tive the money gap is relative to the output gap, i.e., the smaller σ_m/σ_ξ is. This panel extends the rationale underlying equation (17) to the quantitative model and reaffirms Result 3. Again, changing the dynamics in the money demand ϕ has ambiguous but quantitatively small ef-fects on the ratio of optimal policy coefficients.

D. The quantitative role of money demand in the U.S.

The results for the euro area extend to a calibration representing the U.S., even though the more unstable money demand there makes the money gap a somewhat less attractive indicator

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[20]To assess the welfare consequences of the money gap further, we compute the equivalent reduction in steady state consumption that makes the household indifferent between the steady state and the economy with shocks and a particular monetary policy. Adding the money gap to the policy rule closes 12.2 percent of the consumption-equivalent gap between the best attainable monetary policy, i.e., the fully optimal policy under commitment, which we obtain from Weber (2008), and the optimal policy rule that excludes the money gap.

[21]More generally, Swanson (2000) shows that certainty equivalence of policy coefficients does not apply to re-stricted simple policy rules. Rather, policymakers should attenuate their reaction to the variable about which un-certainty has increased, while they should respond more aggressively to those variables about which uncertainty has not changed.
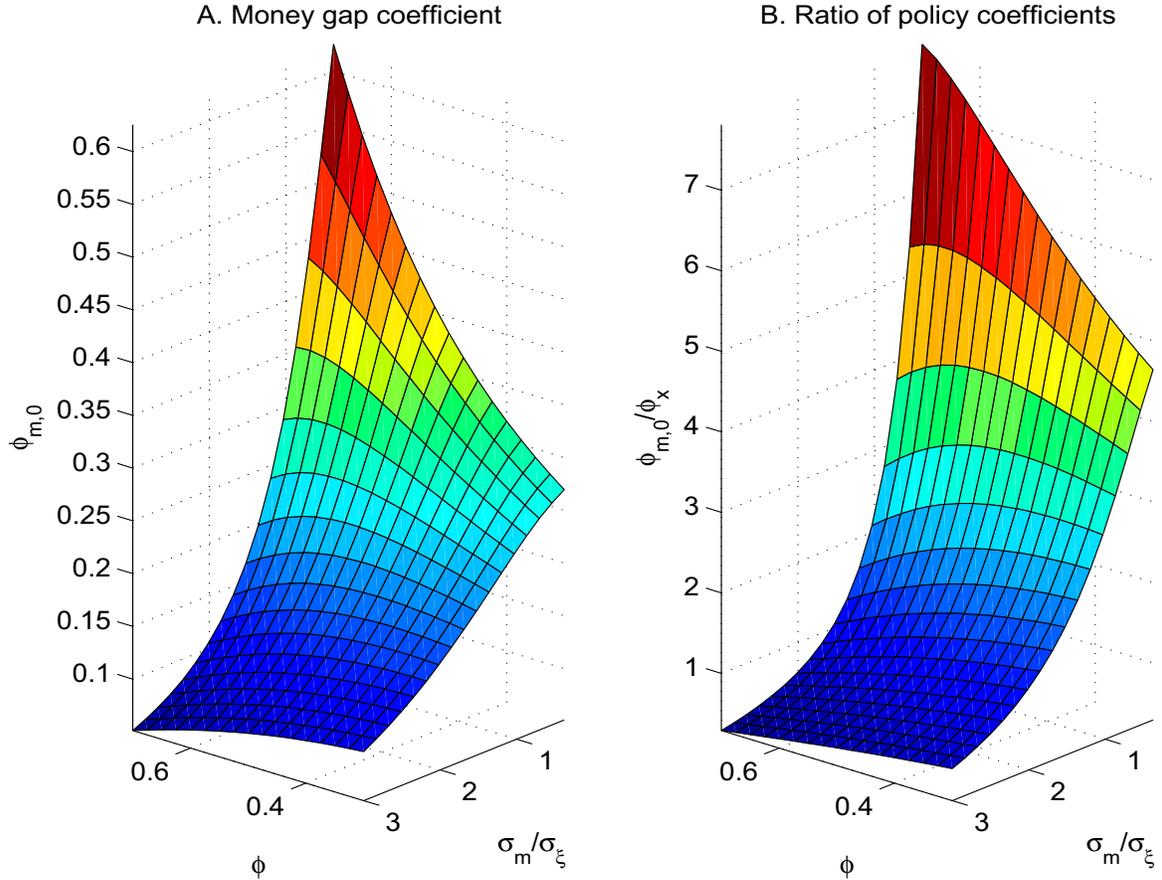
Figure 4. Panel A shows the optimal money gap coefficient of the rule with the contemporaneous money gap as a function of dynamics in money demand ϕ and the standard deviation of velocity shocks relative to noise shocks. Panel B shows the ratio of the optimal policy coefficients $\phi_{m,0}^\star/\phi_x^\star$ as a function of the same factors. We vary σ_m/σ_ξ by fixing σ_ξ at $\frac{2}{3}\sigma_g$ and varying σ_m accordingly.

of the natural interest rate. Our model treats changes in money velocity as exogenous and attributes them to velocity shocks. Accordingly, for the U.S., we calibrate a large standard deviation for velocity shocks using $\sigma_m > \sigma_g$.

Figure 5 plots the median dynamic correlation between the money gap and the natural interest rate for the U.S. calibration (dashed line) and the euro area calibration (solid line). As in Figure 3, we draw σ_m/σ_g from a uniform distribution with range $[1, 2]$ for the U.S. calibration, as opposed to the range $[1/4, 1]$ for the euro area calibration. Both correlations in the figure are positive, but the euro area correlation is almost twice as large as the U.S. correlation, which has a peak value of about 0.25 rather than the 0.4 for the euro area. Our model assigns a lower correlation to the U.S. because more volatile velocity shocks disturb the information that the money gap holds about the natural interest rate. However, the pronounced

lead-lag pattern, by which the money gap leads the natural interest rate over several quarters, prevails also for the U.S. calibration.
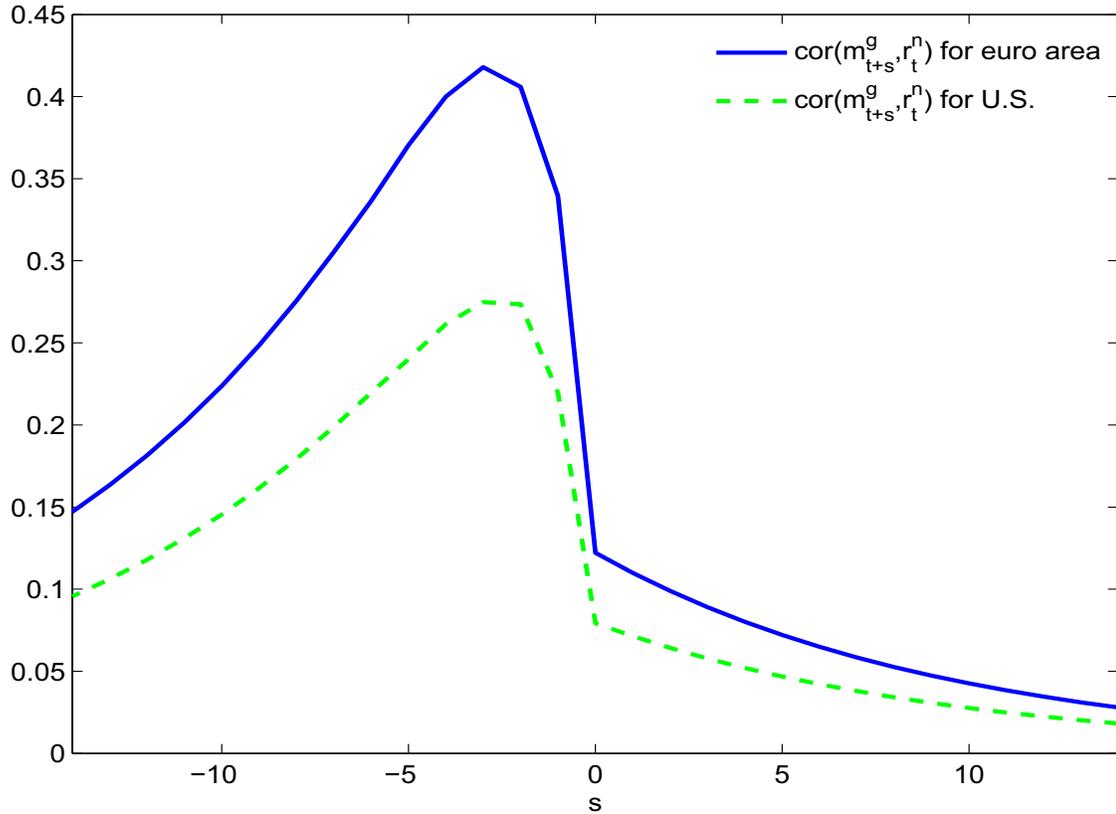
Figure 5. Median dynamic correlation between the money gap and the natural interest rate for the U.S. calibration with $\sigma_m/\sigma_g \in [1, 2]$ and for the euro area calibration with $\sigma_m/\sigma_g \in [1/4, 1]$. For each calibration, the median correlation is computed from 1,000 draws. We draw parameters ϕ and η independently from uniform distributions with support range $[0.3, 0.7]$. All other parameters are calibrated as described in Section V.C.

Table 2 reports the optimized coefficients for policy rule (25), and these coefficients come from repeating the quantitative policy analysis using the U.S. calibration with $\sigma_m = 1.5\sigma_g$ rather than $\sigma_m = \frac{1}{2}\sigma_g$. The contemporaneous money gap coefficient is positive, even though it is smaller than for the euro area. The smaller response of the optimizing central bank to the money gap reflects that the money gap holds less information about the natural interest rate. In contrast, the coefficient of the observed output gap increases relative to the euro area calibration. The central bank puts more weight on the observed output gap because the output

Table 2. Optimal policy coefficients (annualized) for U.S. calibration.

σ_ξ/σ_g	$\phi^\star_{m,0}$	$\phi^\star_{m,1}$	$\phi^\star_x$	$\phi^\star_\pi$	% change in L
$1/3$	0.127		0.648	1.759	-1.776
$2/3$	0.126		0.148	1.835	-2.189
1	0.126		0.065	1.845	-2.267
$1/3$		0.113	0.641	1.759	-1.812
$2/3$		0.113	0.146	1.835	-2.246
1		0.113	0.064	1.845	-2.328

Notes: See Table 1 and the main text for explanation.

gap is an indicator of the natural interest rate whose quality improved relative to the quality of the money gap.

E.   Robustness checks

As a first check, we examine the policy rule (25) when we incorporate the money gap with a lag of one quarter, $j = 1$. This is an interesting check because the natural interest rate co-moves more with the lagged money gap than with the contemporaneous money gap. The calibration is for the euro area, but results for the U.S. are similar. Also, considering longer lags does not change the results significantly.

The main difference between the policy rule with the contemporaneous money gap and the policy rule with the lagged money gap is that the optimal coefficient of the lagged money gap is below the optimal coefficient of the contemporaneous money gap (see the bottom panel of Table 1 for the euro area calibration and of Table 2 for the U.S. calibration). Likewise, the optimal coefficient of the observed output gap in the policy rule with $j = 1$ is below the optimal coefficient of the observed output gap in the policy rule with $j = 0$, and the analog is true for the inflation coefficient (the ambiguities in Table 2 arise from rounding the exact coefficients).

This follows from the central bank preference for stable nominal interest rates. Compared to the contemporaneous money gap, the better indicator quality of the lagged money gap improves the central bank's stabilization of inflation and the output gap with respect to disturbances in the natural interest rate, as is evident from the greater reduction in the loss function L (see last column in Table 1 and 2). The central bank defers some of this improved stabilization of inflation and the output gap to stabilize nominal interest rates more, and it achieves

this by reacting less to the lagged money gap.[22] Furthermore, the central bank reacts less to the observed output gap because this variable depreciates as an indicator of the natural interest rate, relative to the lagged money gap. Consequently, the inflation coefficient must fall to ensure that the central bank continues to react optimally to cost shocks.

As a second robustness check, we examine a policy rule that incorporates the lagged nominal interest rate in addition to the money gap, the observed output gap, and inflation. Optimizing the coefficients of this rule for the case of $\sigma_\xi / \sigma_g = \frac{2}{3}$ yields

$$i_t = 1.135\, i_{t-1} + 0.008\, m_t^g + 0.004\, \tilde{x}_t + 0.406\, \pi_t\ .$$

The result is a superinertial policy rule with the coefficient of the lagged interest rate exceeding unity, as in Rotemberg and Woodford (1999). Superinertial policy behavior implies that, for all bounded paths of $m_t^g, \tilde{x}_t$, and π_t, the nominal interest rate will become infinitely large (Woodford (1999)). However, given the central bank's commitment to its policy rule, the private sector adjusts to a stable equilibrium and the central bank achieves better stabilization outcomes than without superinertia. With interest rate smoothing, the central bank engineers a relatively large response to inflation. At the same time, it responds about twice as much to the money gap than to the observed output gap, confirming that the money gap remains a useful indicator variable for the central bank.

Our last robustness checks concern the sensitivity of the policy coefficients with respect to the properties of shocks. First, when we vary the amount of serial correlation ρ_m of the velocity shock between 0.75 and 0.999, while keeping the variance of the velocity shock constant, the money gap coefficient varies between 0.3 and 0.4 and thus remains fairly constant.

Second, when we vary the amount of serial correlation ρ_g of the IS shock, the money gap coefficient gradually increases for values of ρ_g below 0.9, while it gradually decreases for values of ρ_g above 0.9. In the limit of ρ_g towards unity, the natural interest rate ceases to depend on the IS shock and the money gap coefficient approaches zero. However, this case does not

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[22] Without a preference for stable nominal interest rates, i.e., using $\lambda_i = 0$ instead of $\lambda_i > 0$, the coefficient $\phi_{m,1}^\star$ equals 0.168 and does exceed the coefficient $\phi_{m,0}^\star$, which is equal to 0.121. These numbers are for the euro area calibration and with σ_ξ / σ_g equal to $\frac{2}{3}$. Another difference between the policy rule with $j = 0$ and the policy rule with $j = 1$ is that the latter rule is history dependent, while the former rule is not history dependent. The impact of history dependence is difficult to isolate from the improved indicator quality of the lagged money gap. However, for the case of $\lambda_i = 0$, the OLS estimate from running the simple regression $r_t^n = \phi_{m,1} m_{t-1}^g + e_t$ is very close to $\phi_{m,1}^\star$. Thus, history dependence considerations do not seem to push $\phi_{m,1}^\star$ far away from what the indicator role of the lagged money gap would suggest.

seem of practical relevance because the IS shock is an important driving force of the natural interest rate in estimated DSGE models.

Finally, when we gradually transform the IS shock into the DF shock by raising b from zero to unity, the money gap coefficient increases for small to intermediate values of b, much like in the Panel A of Figure 2 for the basic model. To sum up, the usefulness of the money gap appears reasonably insensitive to changing important dimensions of our quantitative model and of the monetary policy rule.

VII. Conclusion

The natural interest rate is an important landmark of the stance of monetary policy, but it is difficult to measure. Employing a conventional microfounded monetary model with a money-in-utility specification of money demand, we show that the natural interest rate co-moves with a transformation of the money demand, which we call the money gap.

The money gap has a number of interesting characteristics from the perspective of policymakers. Unlike the natural interest rate, the money gap can be readily computed from the actual data. Furthermore, the correlation between the money gap and the natural interest rate is of considerable magnitude, and the money gap tends to lead the natural interest rate by several quarters. Finally, the link between the money gap and the natural interest rate is independent of the monetary policy regime. These characteristics do not depend on the specific model used and are also the same using a quantitative model with a dynamic money demand function.

Our main result is that a central bank that optimizes a simple interest rule without observing the natural interest rate will incorporate the money gap in addition to the observed output gap and inflation. The information contained in the money gap about the natural interest rate allows the nominal interest rate to adjust and stabilize the economy. Another result is that in the view of policymakers, the money gap augments the information contained in the observed output gap, and the weight attached to either reflects the relative quality of the money gap as an indicator of the natural interest rate.

These results support the view that money demand can play a relevant informational role in aiding monetary policymakers, even though the mechanism through which money demand becomes relevant does not rely on the more conventional notion of a longer-term correlation between money growth and inflation. An interesting extension to the approach we use here

would be to pursue a joint approach, following Svensson and Woodford (2003), to the central bank's signal extraction and optimal monetary policy problem.

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APPENDIX A. BASIC MODEL AND QUANTITATIVE MODEL

We briefly recap the main features of the quantitative model, which we take from Giannoni and Woodford (2005) and Weber (2008). Then, we provide its equations, calculated to the first order, and parameters. Finally, we show how to obtain the basic model as a special case of the quantitative model.

Description of quantitative model

The model comprises three types of agents: infinitely many firms, a representative household, and a government. The firm indexed by j uses technology $Y_{jt} = A_t f(N_{jt})$ to produce quantity Y_{jt} with N_{jt} hours of labor, and $f(.)$ is increasing and concave. Productivity shock $A_t > 0$ is exogenous and has a positive mean. The firm sells its product in a monopolistically competitive market and thus has pricing power. It can adjust its price P_{jt} only infrequently with probability α, as in Calvo (1983). When the firm cannot adjust its price, it indexes its price to inflation in the previous period, $P_{jt} = P_{jt-1} \pi_{t-1}^\kappa$, where κ denotes the degree of indexation.

The representative household and the government demand the products of the firms and aggregate them according to the Dixit-Stiglitz aggregator with a constant elasticity of substitution between products equal to $\theta > 1$. Aggregate household consumption and aggregate autonomous consumption are denoted C_t and G_t, respectively. The household derives utility from holding money and from consumption, maintains a financial portfolio that comprises money and bonds, and supplies labor. The expected lifetime utility function of the household exhibits habit formation in consumption and in the real money demand,

$$E_0 \sum_{t=0}^{\infty} \beta^t [u(C_t - \eta C_{t-1}, \psi_t) + q(M_t/P_t - \phi M_{t-1}/P_{t-1}, \varepsilon_t^m) - h(N_t)] . \tag{26}$$

The government implements monetary policy using a policy rule for the short-term nominal interest rate, which we describe in the main text. Moreover, the government levies lump sum taxes, a labor income tax, and issues bonds.

The model's equilibrium is inefficient for three reasons. First, output is inefficiently low as a result of monopolistically competitive product markets. Second, the constraint on firms' price setting drives a wedge between the firms' desired price and the firms' constrained optimal price, and this wedge distorts allocation. Third, despite the fact that money is supplied at zero

social costs, a non-negative price of money liquidity prevents the economy from reaching money holdings satiation.

Quantitative model equations

We calculate the quantitative model to the first order around the zero inflation steady state with money holdings close to satiation. We subtract the linearized model with flexible prices from the linearized model with sticky prices. The model with flexible prices corresponds to the natural economy. Quantitative model equations are, apart from the policy rule, as follows:

$$\bar{x}_t = E_t \bar{x}_{t+1} - (1 - \eta \beta) \sigma (i_t - E_t \pi_{t+1} - r_t^n)$$

$$\bar{x}_t = (x_t - \eta x_{t-1}) - \eta \beta E_t (x_{t+1} - \eta x_t)$$

$$\pi_t - \kappa \pi_{t-1} = \mu [(x_t - \delta x_{t-1}) - \delta \beta E_t (x_{t+1} - \delta x_t)] + \beta E_t (\pi_{t+1} - \kappa \pi_t) + u_t$$

$$m_t - \phi m_{t-1} - \varepsilon_t^m = \phi \beta E_t [m_{t+1} - \phi m_t - \varepsilon_{t+1}^m] + \theta_y I(Y_t, g_t) - \theta_i i_t$$

$$I(Y_t, g_t) = E_t (1 - \eta \beta L^{-1})[(1 - \eta L)Y_t - g_t]/((1 - \eta \beta)(1 - \eta))$$

$$r_t^n = -\varphi E_t [(1 - L^{-1})(1 - \eta \beta L^{-1})\{(1 - \eta L)Y_t^n - g_t\}]$$

$$E_t [\varphi (1 - \eta \beta L^{-1})(1 - \eta L) + \omega] Y_t^n = \varphi E_t (1 - \eta \beta L^{-1}) g_t + (1 + \omega) a_t$$

$$x_t = Y_t - Y_t^n .$$

The first two equations represent the intertemporal IS equation using habit formation in consumption and subtracting the intertemporal IS equation for the natural economy. We subsume the preference shock ψ_t and the autonomous demand shock G_t as the IS shock g_t.[23] The third equation is the NKPC with price indexation, habit formation in consumption, and amended ad hoc cost shocks u_t. The fourth and fifths equations represent the dynamic money demand function, which is described in the main text. The last three equations define the natural interest rate, the natural level of output, and the output gap as the difference between actual and natural output, respectively.

---

[23]The autonomous demand shock has a zero mean such that $Y = C$ in steady state.

As in Chapter 3 and 5 of Woodford (2003) and as in Giannoni and Woodford (2005), we define the parameters as

$$\sigma = -\frac{u_c}{Y u_{cc}} > 0 , \quad \varphi^{-1} = (1 - \eta\beta)\sigma ,$$

$$\nu = \frac{h_{NN}\bar{N}}{h_N} , \quad \chi = \frac{f}{\bar{N}f'} , \quad \omega_p = -\frac{\bar{Y}f''}{(f')^2} , \quad \omega_w = \nu\chi , \quad \omega = \omega_p + \omega_w .$$

ω_p reflects by how much higher output increases prices conditional on wages, while ω_w reflects by how much higher output increases wages conditional on prices. The parameters that determine the slope of the NKPC μ are

$$\check{\chi} = \frac{\omega + \varphi(1 + \eta^2\beta)}{\beta\varphi} , \quad \vartheta = \frac{\beta}{2}\left(\check{\chi} + \sqrt{\check{\chi}^2 - 4\eta^2\beta^{-1}}\right) , \quad \delta = \eta\vartheta^{-1} , \quad \Xi_p = \frac{(1 - \alpha\beta)(1 - \alpha)}{\alpha}(1 + \theta\omega_p)^{-1}$$

and $\mu = \frac{\varphi\eta}{\delta}\Xi_p$. The short-run money demand elasticities are $\theta_k = (1 - \phi)(1 - \phi\beta)\eta_k$, with $k = y, i$. The long-run money demand elasticities are $\eta_y = \frac{(1 - \eta\beta)(1 - \eta)Y u_{cc}(1 - \beta)}{(1 - \phi\beta)(1 - \phi)m q_{mm}} > 0$ and $\eta_i = -\frac{(1 - \eta\beta)u_c\beta}{(1 - \phi\beta)(1 - \phi)m q_{mm}} > 0$, using $u_{cc} < 0$ and $q_{mm} < 0$.

Central bank loss function

Giannoni and Woodford (2005) derive the welfare-based loss function for the New Keynesian model with habit formation in consumption and price indexation, like in our quantitative model, and Weber (2008) extends this loss function to the case of habit formation in money demand. The welfare-based weight attached to stabilizing the output gap is $\lambda_x = \delta_0\Xi_p/\theta$, using $\delta_0 = \vartheta\varphi$. Furthermore, the weight attached to stabilizing nominal interest rates λ_i is related to the central bank's concern with the zero lower bound on nominal interest rates. While a linear-quadratic policy problem of the form we consider does not allow a nonlinear constraint, such as the zero lower bound, to be imposed directly, Woodford (2003) imposes a zero lower bound indirectly by imposing the alternative constraint that the likelihood of the nominal interest rate taking values below zero is small. In our setup, the central bank's preference for stable nominal interest rates also implies this alternative constraint.

Basic model as special case of quantitative model

The basic model constitutes the core of the quantitative model without habit formation in consumption and in real money demand and without price indexation. Accordingly, the equations (4) to (7) of the basic model in the main text can be recovered as the special case $\eta = \phi = \kappa = 0$ of the quantitative model in Appendix A. The restriction $\eta = 0$ also implies $\delta = 0$.

Thus, the slope of the NKPC μ in the basic model reduces to

$$\mu = \frac{(1-\alpha\beta)(1-\alpha)}{\alpha}\left(\frac{\omega+\sigma^{-1}}{1+\theta\omega_p}\right),$$

because $\frac{\eta}{\delta} = \frac{\omega+\varphi}{\varphi} = \frac{\omega+\sigma^{-1}}{\sigma^{-1}}$ computing the limit with $\eta = 0$ and $\delta = 0$. Furthermore, in the basic model, we impose the restriction $\sigma = 1$, which corresponds to log utility of consumption.

APPENDIX B. EQUILIBRIUM AND POLICY COEFFICIENTS IN THE BASIC MODEL

We proceed in two steps. First, we solve the basic model's rational expectation equilibrium conditional on the central bank's policy rule. Second, we minimize the central bank's loss function by choosing the coefficients of the policy rule conditional on the solution for the rational expectation equilibrium.

Solving the rational expectation equilibrium

We rearrange the equations (4) to (7) and the policy rule (13) in the main text by substituting for r_t^n and Y_t^n in terms of shocks and by using the definition $\tilde{x}_t = x_t + \xi_t$:

$$
\begin{aligned}
E_t x_{t+1} + E_t \pi_{t+1} &= x_t + i_t + (1-\rho)a_t - (1-\rho)\frac{\omega}{1+\omega}g_t \\
\beta E_t \pi_{t+1} &= \pi_t - \mu x_t - u_t \\
0 &= -i_t + \phi_m m_t^g + \phi_x x_t + \phi_\pi \pi_t + \phi_x \xi_t \\
0 &= -m_t^g + \eta_y g_t - \varepsilon_t^m .
\end{aligned}
$$

Then we express the model in matrix notation,

$$
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ E_t i_{t+1} \\ E_t m_{t+1}^g \end{pmatrix}
=
\begin{bmatrix} 1 & 0 & 1 & 0 \\ -\mu & 1 & 0 & 0 \\ \phi_x & \phi_\pi & -1 & \phi_m \\ 0 & 0 & 0 & -1 \end{bmatrix}
\begin{pmatrix} x_t \\ \pi_t \\ i_t \\ m_t^g \end{pmatrix}
+
\begin{bmatrix} 1-\rho & 0 & -\frac{(1-\rho)\omega}{1+\omega} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_x \\ 0 & 0 & \eta_y & -1 & 0 \end{bmatrix}
\begin{pmatrix} a_t \\ u_t \\ g_t \\ \varepsilon_t^m \\ \xi_t \end{pmatrix},
$$

and summarize it as $AE_t y_{t+1} = By_t + Cs_t$. Matrix B depends on the policy coefficients ϕ_m, ϕ_x, and ϕ_π. Shocks s_t evolve according to $s_t = Ds_{t-1} + Fv_t$, using $D = \rho I_n$, where I_n denotes the identity matrix with dimension n equal to number of shocks. Matrix $F = \text{diag}(\sigma_a, \sigma_u, \sigma_g, \sigma_m, \sigma_\xi)$

contains non-negative diagonal elements. Vector v_t contains the shock residuals, which are mutually independent and serially uncorrelated, with zero mean and unit variance.

To solve for the vector of endogenous variables y_t as a function of shocks, we postulate a solution of the form $y_t = \Gamma s_t$ and solve for Γ using the method of undetermined coefficients. We then plug the postulated solution into the model $AE_t y_{t+1} = By_t + Cs_t$ and compute the conditional expectation $E_t s_{t+1}$ to obtain $\rho A \Gamma s_t = B\Gamma s_t + Cs_t$. Imposing $s_t = 1$ and solving for Γ yields

$$\Gamma(\phi_m, \phi_x, \phi_\pi) = (\rho A - B)^{-1} C \,,$$

provided $(\rho A - B)$ is invertible. Matrix Γ depends on the policy coefficients ϕ_m, ϕ_x, and ϕ_π because B and C depend on them.

Solving for policy coefficients

The central bank solves a linear-quadratic policy problem by choosing the policy coefficients to minimize its quadratic loss function subject to the linear equilibrium constraints. Formally, the policy problem is

$$\min_{\phi_m, \phi_x, \phi_\pi} \mathrm{var}(\pi_t) + \lambda_x \mathrm{var}(x_t) + \lambda_i \mathrm{var}(i_t), \quad \text{subject to}$$

$$y_t = \Gamma(\phi_m, \phi_x, \phi_\pi) s_t$$

$$s_t = Ds_{t-1} + Fv_t \,, \quad s_{-1} \text{ given.}$$

Here, $\lambda_x > 0$ and $\lambda_i \geq 0$. We rearrange this problem as follows. First, we rearrange the loss function as $L = \mathrm{var}(\pi_t) + \lambda_x \mathrm{var}(x_t) + \lambda_i \mathrm{var}(i_t) = E[y_t' W y_t]$, using the weighting matrix $W = \mathrm{diag}(\lambda_x, 1, \lambda_i, 0)$ and denoting the unconditional expectation as E. We then use $y_t = \Gamma s_t$ to rearrange the scalar L according to

$$L = E[s_t' \Gamma' W \Gamma s_t] = E[\mathrm{trace}(s_t' \Gamma' W \Gamma s_t)] = \mathrm{trace}(\Gamma' W \Gamma E[s_t s_t']) \,,$$

and obtain $E[s_t s_t'] = (1 - \rho^2)^{-1} FF'$ from the recursive law of motion of shocks. Thus,

$$L(\phi_m, \phi_x, \phi_\pi) = (1 - \rho^2)^{-1} \mathrm{trace}(\Gamma' W \Gamma FF') \,.$$

L depends on the policy coefficients ϕ_m, ϕ_x, and ϕ_π because Γ depends on them. Finally, we compute the derivatives of L with respect to the policy coefficients using Matlab's symbolic toolbox.

Modeling discount factor shocks

We model the co-movement among the IS shock g_t, the velocity shock ε_t^m, and the productivity shock a_t by augmenting the impact matrix F of the shock process $s_t = Ds_{t-1} + Fv_t$ by the two off-diagonal elements $F_{13} = -b\sigma_g/(1+\omega)$ and $F_{43} = b\eta_y\sigma_g$. The new elements imply that any movement in the residual v_{gt} to the g_t shock also affects the productivity shock and the velocity shock. The scaling of the new elements implies that g_t affects all terms in the period utility function uniformly. For the case $\rho = 0$, the covariance matrix $E(s_t s_t') = FF'$ of the shocks s_t is

$$E(s_t s_t') = \begin{bmatrix} \frac{b^2}{(1+\omega)^2}\sigma_g^2 + \sigma_a^2 & 0 & -\frac{b}{1+\omega}\sigma_g^2 & -\frac{\eta_y b^2}{1+\omega}\sigma_g^2 & 0 \\ 0 & \sigma_u^2 & 0 & 0 & 0 \\ -\frac{b}{1+\omega}\sigma_g^2 & 0 & \sigma_g^2 & \eta_y b\sigma_g^2 & 0 \\ -\frac{\eta_y b^2}{1+\omega}\sigma_g^2 & 0 & \eta_y b\sigma_g^2 & \eta_y^2 b^2\sigma_g^2 + \sigma_m^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\xi^2 \end{bmatrix} .$$

We derive the OLS interpretation of the money gap coefficient by computing the covariance between the money gap and the natural interest rate and by computing the variance of the money gap in the model with the DF shock. The covariance is

$$\begin{aligned} \text{cov}(m_t^g, r_t^n) &= E[m_t^g r_t^n] = E[(\eta_y g_t - \varepsilon_t^m)(\tfrac{1}{1+\omega}g_t - a_t)] \\ &= E[\tfrac{\eta_y}{1+\omega}g_t^2 - \tfrac{1}{1+\omega}g_t\varepsilon_t^m - \eta_y g_t a_t + a_t\varepsilon_t^m] = \tfrac{\eta_y}{1+\omega}(1-b)(1+b)\sigma_g^2 . \end{aligned}$$

In the last setp we use the fact that all second moments in the second to last step are in the covariance matrix $E(s_t s_t')$. The variance of the money gap is equal to

$$\text{var}(m_t^g) = E[(\eta_y g_t - \varepsilon_t^m)^2] = E[\eta_y^2 g_t^2 - 2\eta_y g_t\varepsilon_t^m + (\varepsilon_t^m)^2] = \eta_y^2(1-b)^2\sigma_g^2 + \sigma_m^2 .$$

In the last step we again use the fact that all second moments are in $E(s_t s_t')$. Dividing $\text{cov}(m_t^g, r_t^n)$ by $\text{var}(m_t^g)$ yields $\phi_m^\star$.

APPENDIX C. THE MONEY GAP WITH A DYNAMIC MONEY DEMAND FUNCTION

To derive the money gap in equation (23) in the main text, we iterate the dynamic money demand function (20) forward and obtain

$$m_t - \phi m_{t-1} - \varepsilon_t^m = E_t \sum_{s=0}^{\infty} (\phi\beta)^s [\theta_y I(Y_{t+s}, g_{t+s}) - \theta_i i_{t+s}] .$$

We plug in $I(Y_{t+s}, g_{t+s}) = E_t(1 - \eta\beta L^{-1})[(1 - \eta L)Y_{t+s} - g_{t+s}]/((1 - \eta\beta)(1 - \eta))$ and re-arrange,

$$m_t - \phi m_{t-1} - \varepsilon_t^m = E_t \sum_{s=0}^{\infty} (\phi\beta)^s \left[\theta_y \frac{(1 - \eta\beta L^{-1})(1 - \eta L)}{(1 - \eta\beta)(1 - \eta)} Y_{t+s} - \theta_i i_{t+s} - \theta_y \left(\frac{1 - \eta\beta L^{-1}}{(1 - \eta\beta)(1 - \eta)} \right) g_{t+s} \right].$$

The first term in square brackets corresponds to the income measure $I(Y_{t+s}, 0)$, which does not refer to the IS shock, multiplied by θ_y. Replacing this measure and substituting for the money demand elasticities θ_y and θ_i, we obtain

$$m_t - \phi m_{t-1} - \varepsilon_t^m = (1 - \phi)(1 - \phi\beta)E_t \sum_{s=0}^{\infty} (\phi\beta)^s \left[\eta_y I(Y_{t+s}, 0) - \eta_i i_{t+s} - \eta_y \left(\frac{1 - \eta\beta L^{-1}}{(1 - \eta\beta)(1 - \eta)} \right) g_{t+s} \right].$$

Plugging in the definition (22) of m_t^p from the main text and rearranging, we then obtain

$$m_t = \phi m_{t-1} + (1 - \phi)m_t^p - \left[\eta_y (1 - \phi)(1 - \phi\beta)E_t \sum_{s=0}^{\infty} (\phi\beta)^s \left(\frac{1 - \eta\beta L^{-1}}{(1 - \eta\beta)(1 - \eta)} \right) g_{t+s} - \varepsilon_t^m \right].$$

Using the partial adjustment representation of the money demand function (21) shows that the term in square brackets corresponds to the money gap. Rewriting the infinite sum in square brackets as the inverted lead polynomial $(1 - \phi\beta L^{-1})^{-1}$ yields the money gap in equation (23) in the main text.

APPENDIX D.   CALIBRATION

This appendix refers to further literature to support our calibration and put it into a larger perspective. It also provides our calibration of the deep parameters required to compute the slope of the NKPC.

Degree of dynamics in the money demand ϕ

Andres, Lopez-Salido, and Nelson (2009) estimate coefficients of lagged money demand in their money demand function between 0.4 and 0.5. They consider a DSGE model with the MIU specification of money demand and use U.S. and euro area data. Others arrive at higher estimates. Nelson (2002), for example, calibrates a value of 0.7 in a DSGE model with the MIU specification of money demand, using time-series evidence. Ball (2002) estimates 0.8 using a time-series model for U.S. data for M1, while Stracca (2003) estimates 0.92 using a time-series model for euro area data for M1. Heller and Khan (1979) obtain estimates

between 0.7 and 0.8 using a time-series model for U.S. data for M2, and Coenen and Vega (2001) estimate 0.87 using a time-series model for euro area data for M3. Tin (1999) again arrives at much lower estimates between 0.25 and 0.45 using U.S. micro data for monetary assets.

Standard deviation of velocity shocks σ_m

Andres, Lopez-Salido, and Valles (2006) and Andres, Lopez-Salido, and Nelson (2009) estimate a relative standard deviation of velocity shocks $\sigma_m \approx \frac{1}{3}\sigma_g$ in their DSGE models with dynamic money demand and using euro are data. When they use U.S. data, however, Andres, Lopez-Salido, and Nelson (2009) obtain a much larger estimate, $\sigma_m \approx 2\sigma_g$. Arestis, Chortareas, and Tsoukalas (2010) use U.S. data and obtain $\sigma_m \approx 1.4\sigma_g$. McCallum and Nelson (1999) also use U.S. data and obtain $\sigma_m \approx 1.3\sigma_g$.[24] Ireland (2004) estimates a DSGE model with a static instead of a dynamic money demand function using U.S. data. However, he obtains $\sigma_m \approx \frac{1}{2}\sigma_g$, an estimate that is more in line with the euro area evidence than with the U.S. evidence.

Standard deviation of noise shocks σ_ξ

Orphanides (2003), Coenen, Levin, and Wieland (2005) and Scharnagl, Gerberding, and Seitz (2010) examine the amount of noise in the observed output gap for U.S., euro area, and German data, respectively. They report estimates for σ_ξ very close to but below 0.01. Furthermore, Ireland (2004), Andres, Lopez-Salido, and Valles (2006), and Andres, Lopez-Salido, and Nelson (2009) estimate σ_g as 0.019, 0.012, and 0.015 respectively. Based on these numbers, a reasonable estimate of the relative standard deviation of noise shocks is $\sigma_\xi \approx \frac{2}{3}\sigma_g$, and we center our interval around this value.

Slope of NKPC μ

As is evident from Appendix A, a number of deep parameters determine slope μ of the NKPC in the basic and quantitative model (see Appendix A for the basic model). We calibrate these parameters as follows. The mean duration of price contracts is four quarters using α equal to 0.75. The steady state markup is 20% using θ equal to 6, and the labor supply elasticity with

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[24]We disentangle their money demand residual, which comprises IS and velocity shocks, into its components.

respect to real wages ν is 0.2. Moreover, a Cobb–Douglas technology $f(.)$ with a labor co-efficient of 2/3 implies that ω_w is equal to 0.3 and ω_p is equal to 0.5. These two numbers are comparable to numbers in Giannoni and Woodford (2005). They imply that $\omega = \omega_w + \omega_p$ is equal to 0.8.