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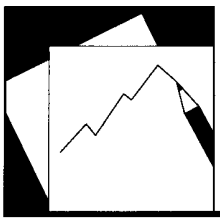

IMF Working Paper

Unemployment and Productivity in the Long Run: the Role of Macroeconomic Volatility

Pierpaolo Benigno, Luca Antonio Ricci, and Paolo Surico

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Unemployment and Productivity in the Long Run: the Role of Macroeconomic Volatility*

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Abstract

We propose a theory of low-frequency movements in unemployment based on asymmetric real wage rigidities. The theory generates two main predictions: long-run unemployment increases with (i) a fall in long-run productivity growth and (ii) a rise in the variance of productivity growth. Evidence based on U.S. time series and on an international panel strongly supports these predictions. The empirical specifications featuring the variance of productivity growth can account for two U.S. episodes which a linear model based only on long-run productivity growth cannot fully explain. These are the decline in long-run unemployment over the 1980s and its rise during the late 2000s.

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1 Introduction

This paper proposes a theory in which the low-frequency movements in unemployment are explained by the low-frequency movements and the volatility of productivity growth.[1] On the one hand, an increase in long-run productivity growth lowers long-run unemployment. On the other hand, a fall in the variance of productivity growth leads to a fall in long-run unemployment even when long-run productivity growth remains flat. The key mechanism that explains these relationships rests on the assumption that real wages, or more broadly real marginal costs, adjust more easily upward than downward.

A recent literature has highlighted the importance of real wage rigidities to explain labor-market dynamics at business cycle frequencies. Shimer (2005), Hall (2005), Gertler and Trigari (2009) and Blanchard and Gali (2010) show that real wage rigidities are important to account for a number of stylized facts including the high volatility of employment and vacancies as well as the low volatility of real wages.[2] This paper complements these studies by showing that real rigidities can also account for unemployment dynamics at low frequencies and therefore it offers a rationale for the empirical relationship between long-run unemployment, long-run productivity growth and its variance.

Our analysis is motivated by a number of empirical papers, including Bruno and Sachs (1985), Phelps (1994), Blanchard et al. (1995), Blanchard and Wolfers (2000), Staiger, Stock, and Watson (2001) and Pissarides and Vallanti (2007), which show time-series and cross-country evidence in favor of a negative relationship between unemployment and productivity growth at low frequencies. This literature is exemplified by Figure 1 which reports the trend in unemployment, the trend in productivity growth and the variance of productivity growth for a postwar sample of U.S. data. The time series plotted in the charts on the first row are obtained computing averages and variances over five-year rolling windows. The charts on the second row display similar objects obtained using the time-varying Vector AutoRegressive (VAR) model described in Section 3.

Two main features are evident. First, irrespective of the strategy used to look at the data over the long-run, the charts on the first column of Figure 1 confirm the negative relationship between long-run unemployment and long-run productivity growth documented in earlier contributions.[3] Second, a probably less known, yet very interesting, feature of the data is the strong positive association between long-run unemployment and the vari-

[1] The terms *long-run*, *trend*, *mean* and *low-frequency* are used interchangeably throughout the paper.

[2] Pissarides (2009) offers a critical appraisal of wage stickiness as a driver of the cyclical volatility of unemployment in search models.

[3] Results similar to Figure 1 are obtained using ten-year rolling windows, the Hodrick-Prescott and Christiano-Fitzgerald filters.

ance of productivity growth, which is uncovered in the charts on the second column. The Great Moderation in the variance of productivity growth, for instance, coincides with a sharp fall in the unemployment trend.

The contribution of this paper is twofold. On the theoretical side, we develop a simple model of the labor market based on the assumption of asymmetric real wage rigidities that can account for the two empirical findings summarized in Figure 1.[4] On the empirical side, we evaluate formally the predictions of the model by exploiting low-frequency movements in unemployment and productivity growth either over time or across countries.

In our model, wage setters face convex costs for adjusting real wages which can be either symmetric or asymmetric up to a limiting point that nests complete downward inflexibility. Asymmetric real-wage rigidities have two key implications. First, for a given volatility of productivity growth, a slowdown in long-run productivity growth generates a significant rise in long-run unemployment. This is the case because too high real wages make it more likely that real revenues will fall relative to costs, thereby forcing firms to reduce labor demand in order to protect profits. With symmetric rigidities, this trade-off is weaker. Second, for a given long-run productivity growth, a higher volatility raises the probability of an adverse shock and then leads to higher long-run unemployment. Conversely, even when the trend in productivity growth is low, a decline in its volatility reduces these risks and causes the unemployment trend to fall.

We present evidence consistent with the predictions of the theoretical model. Time series for the long-run mean and the variance of U.S. unemployment and productivity growth are obtained using an estimated VAR with drifting coefficients and stochastic volatility á la Cogley and Sargent (2005), and Primiceri (2005). Panel regressions are obtained using averages and variances over ten-year windows within a dataset of industrialized and emerging economies.

Our main results can be summarized as follows. First, the long-run mean and the variance of productivity growth are significant determinants of the long-run mean of U.S. unemployment. This is true even when we control for changes in the demographic composition of the labor force. Second, the empirical specifications that include a measure of productivity growth volatility (either linearly or non-linearly) are associated with a significant improvement in the goodness of fit relative to a linear specification in long-run productivity growth only. This is exemplified by two episodes that cannot be fully explained by movements of productivity growth at low frequencies: the fall in long-run

[4] The significance of downward real wage rigidity has been documented by a large number of empirical studies on micro-data, which are difficult to summarize in a few lines. Prominent examples include Dickens et al. (2008), Du Caju et al. (2009), Fagan and Messina (2009), Holden and Wulfsberg (2009) for the industrialized world and Calvo et al. (2006) for emerging markets.

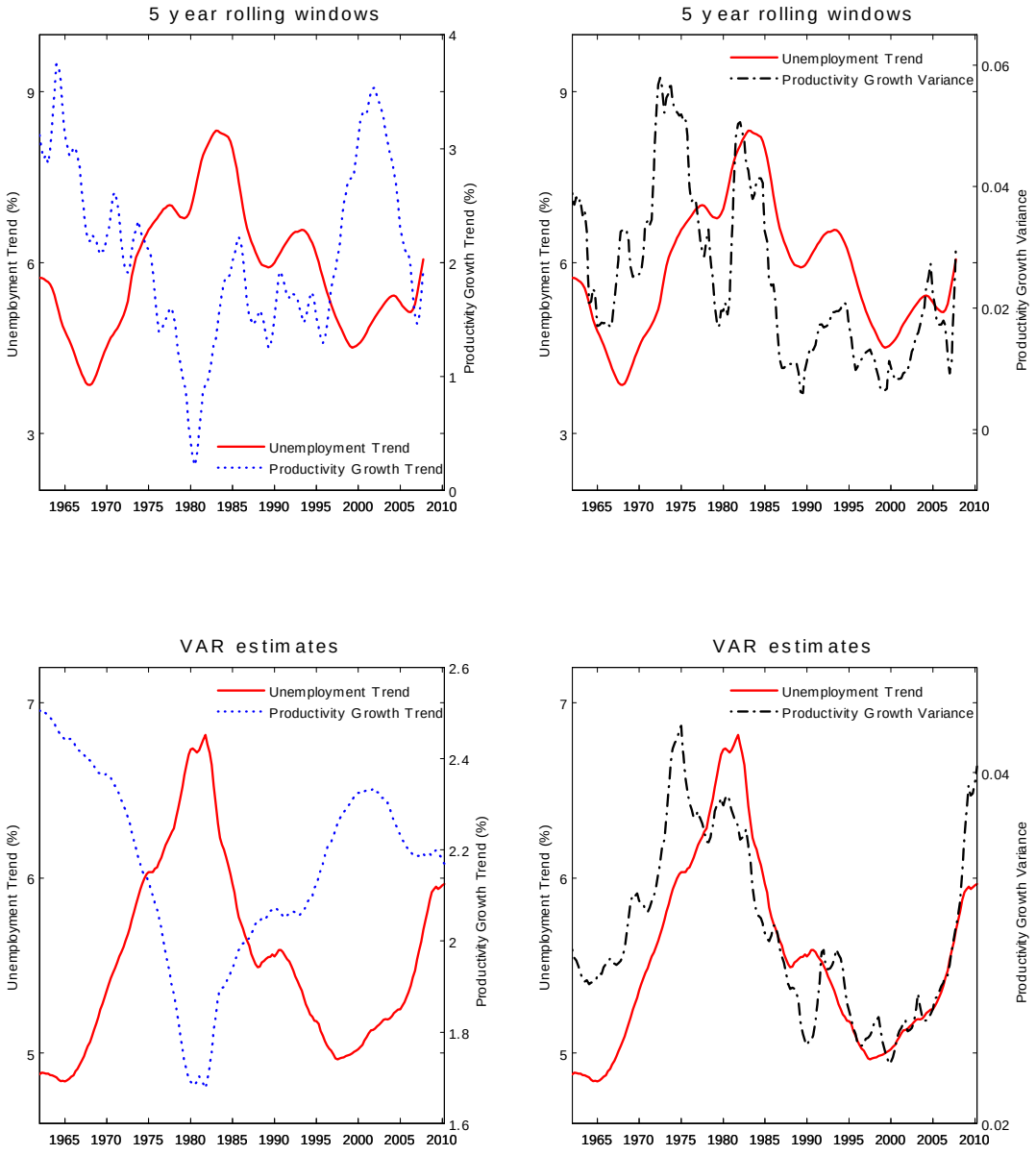

Figure 1: Long-run unemployment, long-run productivity growth and variance of productivity growth for the U.S., computed using five-year rolling windows for the charts on the first row and the time-varying VAR of section 3 for the charts on the second row.

unemployment over the 1980s and its rise during the late 2000s. Third, the panel regressions reveal that variation over time is more important than variation across countries for the mean and variance of productivity growth to account for fluctuations in the mean of unemployment.

Few theoretical papers have studied the implications for the long-run relationship between unemployment and productivity growth but, to the best of our knowledge, none has emphasized the importance of time variation in macroeconomic volatility for the unemployment trend. In traditional labor search models, the relationship between productivity and unemployment is generally uncertain, as it depends mostly on the extent to which jobs can be upgraded or need to be eliminated when new technology arises (Mortensen and Pissarides, 1998). If firms cannot embody the new technology into existing jobs, higher productivity would lead to job destruction and higher unemployment (Aghion and Howitt, 1994). If productivity increases for all existing jobs, demand for labor would increase and unemployment would decline (Pissarides, 2000, Pissarides and Vallanti, 2007). In line with our assumption of real wage rigidities, Ball and Mankiw (2002) suggest a possible rationale for a negative relationship between unemployment and productivity "resting on the idea that 'wage aspirations' adjust slowly to shifts in productivity growth", as "workers come to view the rate of real wage increase that they receive as normal and fair and to expect it to continue".

Our work complements an important literature which has built the case for demographic changes in labor force participation to explain low-frequency movements in unemployment (see Shimer, 1998, and Francis and Ramey, 2009, among others). We show that the finding of a significant role for the trend and the variance of productivity growth to account for the trend in unemployment is robust to controlling for movements in the share of young workers in the labor force as well as to using the measure of "genuine" unemployment that Shimer (1998) argues to be unaffected by demographics influences.

The paper is organized as follows. Section 2 presents the model and shows the mechanism through which asymmetric real wage rigidity generates a long-run relationship between unemployment, productivity growth, and its volatility. Section 3 confronts the predictions of the model to the time series properties of U.S. data while Section 4 provides evidence for an international panel of developed and developing economies. Section 5 concludes. The appendices provide details of the theoretical and empirical models.

2 The model

We describe a closed-economy model in which there is a continuum of infinitely lived households and firms (both in a [0,1] interval). Each household derives utility from the consumption of a continuum of goods aggregated using a Dixit-Stiglitz consumption index, and disutility from supplying one of the varieties of labor to firms in a monopolistic-competitive market. Each firm hires all varieties of labor to produce one of the continuum of consumption goods and operates in a monopolistic-competitive market. The economy is subject to an aggregate productivity shock. This is denoted by A_t, whose logarithmic a_t is distributed as a Brownian motion with drift g and variance σ^2

$$da_t = gdt + \sigma dB_t \tag{1}$$

where B_t denotes a standard Brownian motion with zero drift and unit variance.

Household j has preferences over time given by

$$E_{t_0}\left[\int_{t_0}^{\infty} e^{-\rho(t-t_0)}\left(\ln C_t^j - \frac{l_t^{1+\eta}(j)}{1+\eta}\right)dt\right] \tag{2}$$

where the expectation operator $E_{t_0}(\cdot)$ is defined by the shock processes (1) and $\rho > 0$ is the rate of time preference. Current utility depends on the Dixit-Stiglitz consumption aggregate of the continuum of goods produced by the firms operating in the economy

$$C_t^j \equiv \left[\int_0^1 c_t^j(i)^{\frac{\theta_p}{\theta_p - 1}} di\right]^{\frac{\theta_p - 1}{\theta_p}}$$

where $\theta_p > 0$ is the elasticity of substitution among consumption goods and $c_t^j(i)$ is household j's consumption of the variety produced by firm i. An appropriate consumption-based price index is defined as

$$P_t \equiv \left[\int_0^1 p_t(i)^{1-\theta_p} di\right]^{\frac{1}{1-\theta_p}},$$

where $p_t(i)$ is the price of the single good i.

The utility flow is logarithmic in the consumption aggregate. In (2), labor disutility is assumed to be isoelastic with respect to the labor supplied $l_t(j)$, with $\eta \geq 0$ measuring the inverse of the Frisch elasticity of labor supply.[5] Household j's intertemporal budget

[5] These preferences are consistent with a balanced-growth path as we assume a drift in technology.

constraint is given by

$$E_{t_0}\left\{\int_{t_0}^{\infty} Q_t P_t C_t^j dt\right\} \leq E_{t_0}\left\{\int_{t_0}^{\infty} Q_t \left[W_t(j) l_t(j) + \Pi_t^j\right] dt\right\} \tag{3}$$

where Q_t is the stochastic nominal discount factor in capital markets where claims to monetary units are traded; $W_t(j)$ is the nominal wage for labor of variety j, and Π_t^j is the profit income of household j.

Starting with the consumption decisions, household j chooses goods demand, $\{c_t^j(i)\}$, to maximize (2) under the intertemporal budget constraint (3), taking prices as given. The first-order conditions for consumption choices imply

$$e^{-\rho(t-t_0)} C_t^{-1} = \xi Q_t P_t \tag{4}$$

$$\frac{c_t(i)}{C_t} = \left(\frac{p_t(i)}{P_t}\right)^{-\theta_p} \tag{5}$$

where the multiplier ξ does not vary over time. The index j is omitted from the consumption's first-order conditions, because we are assuming perfect consumption risk-sharing through a set of state-contingent claims to monetary units.

Before we turn to the labor supply decision, we analyze the firms' problem. We assume that the labor used to produce each good i is a CES aggregate, $L(i)$, of the continuum of individual types of labor j defined by

$$L_t(i) \equiv \left[\int_0^1 l_{i,t}(j)^{\frac{\theta_w - 1}{\theta_w}} dj\right]^{\frac{\theta_w}{\theta_w - 1}}$$

with an elasticity of substitution $\theta_w > 1$. Here $l_{i,t}(j)$ is the demand of firm i for labor of type j. Given that each differentiated type of labor is supplied in a monopolistic-competitive market, the demand for labor of type j on the part of a wage-taking firm of type i is given by

$$l_{i,t}(j) = \left(\frac{W_t(j)}{W_t}\right)^{-\theta_w} L_t(i), \tag{6}$$

where W_t is the Dixit-Stiglitz aggregate wage index

$$W_t \equiv \left[\int_0^1 W_t(j)^{1-\theta_w} dj\right]^{\frac{1}{1-\theta_w}}; \tag{7}$$

whereas the aggregate demand of labor of type j is given by

$$l_t^d(j) = \left(\frac{W_t(j)}{W_t}\right)^{-\theta_w} L_t \tag{8}$$

and aggregate labor L_t is defined as

$$L_t \equiv \int_0^1 L_t(i)di.$$

We assume a common linear technology for the production of all goods

$$y_t(i) = A_t L_t(i)^{\alpha}, \tag{9}$$

for a parameter α with $0 < \alpha < 1$ measuring decreasing return to scale. Profits of the generic firm i, $\Pi_t(i)$, are given by

$$\Pi_t(i) = p_t(i)y_t(i) - W_t L_t(i).$$

In a monopolistic-competitive market, given (5), each firm faces the demand

$$y_t(i) = \left(\frac{p_t(i)}{P_t}\right)^{-\theta_p} Y_t$$

where total output is equal in equilibrium to aggregate consumption ($Y_t = C_t$). We assume that firms can freely adjust their prices. Standard optimality conditions under monopolistic competition imply that all firms set the same price given by

$$p_t(i) = P_t = \mu_p \frac{W_t L_t(i)}{Y_t} = \mu_p \frac{W_t L_t}{Y_t} \tag{10}$$

where $\mu_p \equiv \theta_p/[(\theta_p - 1)\alpha] > 1$ denotes the mark-up of prices over marginal costs. An implication of (10) is that labor income is a constant fraction of total income

$$P_t Y_t = \mu_p W_t L_t. \tag{11}$$

Using the production function (9) into (11), aggregate demand of labor

$$L_t = \left(\mu_p \frac{W_t}{P_t}\frac{1}{A_t}\right)^{\frac{1}{\alpha-1}} \tag{12}$$

depends negatively on the real wage and positively on productivity. Demand of labor is

critical to understand the main intuition behind our results. When productivity falls and real wage remains too high, firms have to cut on labor to protect their profits.

In what follows, we define $w_t(j) = W_t(j)/P_t$ as the real wage for worker of type j and $w_t = W_t/P_t$ as the aggregate real wage.[6] The choice of real wages is modelled in a similar way to the monopoly-union model of Dunlop (1944). Given firms' demand (8), a household of type j (or a union) chooses real wages in a monopolistic-competitive market to maximize (2) under the intertemporal budget constraint (3) taking as given prices $\{Q_t\}$ and the other relevant aggregate variables. An equivalent formulation of this problem is the maximization of the following objective

$$E_{t_0}\left[\int_{t_0}^{\infty} e^{-\rho(t-t_0)}\pi(w_t(j), w_t, A_t)dt\right] \tag{13}$$

by choosing $\{w_t(j)\}_{t=t_0}^{\infty}$, where

$$\pi(w_t(j), w_t, A_t) \equiv \frac{1}{\mu_p}\left(\frac{w_t(j)}{w_t}\right)^{1-\theta_w} - \frac{1}{1+\eta}\left(\frac{1}{\mu_p}\right)^{\frac{1+\eta}{1-\alpha}}\left(\frac{w_t(j)}{w_t}\right)^{-(1+\eta)\theta_w}\left(\frac{A_t}{w_t}\right)^{\frac{1+\eta}{1-\alpha}}.$$

Households would then supply as much labor as demanded by firms in (8) at the chosen real wages. In deriving $\pi(\cdot)$ we have used (4), (8) and (11).

2.1 Flexible wages

We first analyze the case in which wages are set without any friction, so that they can be moved freely. With flexible wages, maximization of (13) corresponds to per-period maximization and implies the following optimality condition

$$\pi_{w_j}(w_t(j), w_t, A_t) = 0 \tag{14}$$

where $\pi_{w_j}(\cdot)$ is the derivative of $\pi(\cdot)$ with respect to the first argument. Since equation (14) holds for each j, there is a unique equilibrium where $w_t(j) = w_t = w_t^f$ and in which w_t^f denotes the equilibrium level of real wages under flexible wages. Equation (14) defines the equilibrium level of labor under flexible wages, which is a constant given by

$$L^f = (\mu_p\mu_w)^{-\frac{1}{1+\eta}},$$

[6] Notice that equation (11) holds because of the assumption of flexible prices which is necessary for analytical tractability.

where the wage mark-up is defined by $\mu_w \equiv \theta_w/(\theta_w - 1)$. Real wages are proportional to the aggregate productivity shock

$$w_t^f = \frac{1}{\mu_p}(L^f)^{\alpha-1}A_t. \tag{15}$$

2.2 Definition of unemployment rate

Following Galì (2010), we define the unemployment rate as the difference between the "notional" amount of labor that workers would be willing to supply in a competitive and frictionless market at the current real wage and the amount of labor currently employed. Given our preference specifications, "notional" labor supply, L_t^s, is defined as the amount of labor that equates the marginal rate of substitution between labor and (current) consumption to the current real wage

$$(L_t^s)^\eta C_t = \frac{W_t}{P_t}. \tag{16}$$

Accordingly, the unemployment rate u_t is given by $u_t = \ln L_t^s - \ln L_t$. Combining (16) with (11) and using $Y_t = C_t$ we can write

$$u_t = u^f - \frac{1+\eta}{\eta}x_t \tag{17}$$

where u^f denotes the unemployment rate in the flexible-wage model given by $u^f = \ln \mu_w/\eta$ and where the employment gap x_t, equal to the output gap, is defined as the log difference between actual labor and the flexible-wage level

$$x_t = \ln L_t - \ln L_t^f. \tag{18}$$

With flexible wages, unions set too high real wages and at these real wages workers would be willing to supply more labor than currently demanded by firms. Unemployment is given by u^f and indeed captures the unions' monopoly power. With real wage rigidities, unemployment depends also on the output gap and can vary over time inversely proportional to the variation of the output gap. This second component will be the most relevant in our model to explain the dynamics of unemployment at low frequencies.

2.3 Sticky real wages

In this section, we investigate a general model in which real wages are allowed to adjust either upward or downward but with some cost. In particular, we allow for both symmetric and asymmetric adjustment costs through a linex function of the form

$$h(\pi_{R,t}(j)) = \frac{e^{\chi\lambda\pi_{R,t}(j)} - \chi\lambda\pi_{R,t}(j) - 1}{\lambda^2}$$

for some parameters χ, λ, where we have defined the rate of real wage changes as $\pi_{R,t}(j)dt \equiv dw_t(j)/w_t(j)$. In particular χ is a measure of the costs of adjustment, while λ measures the asymmetries in the cost function.[7] When $\lambda \to 0$, we retrieve the standard symmetric quadratic cost function

$$h(\pi_{R,t}(j)) = \chi^2 \frac{(\pi_{R,t}(j))^2}{2},$$

while when $\lambda < 0$ it is more costly to adjust real wages downward than upward and viceversa for $\lambda > 0$. When λ goes to minus infinity, we nest the case in which real wages are inflexible downward and fully flexible upward. In the next section, we discuss this case more extensively as it allows us to derive a closed form solution for the long-run mean of unemployment.

In this setting, we assume that wage setters maximize (13) taking into account the present discounted value of the costs of changing real wages[8]

$$V(w_t(j), w_t, A_t) = \max_{\pi_{R,t}(j)} E_{t_0}\left[\int_{t_0}^{\infty} e^{-\rho(t-t_0)}[\pi(w_t(j), w_t, A_t) - h(\pi_{R,t}(j))]dt\right]. \tag{19}$$

The value function associated with the objective function (19) can be written as

$$\rho V dt = \max_{\pi_{R,t}(j)} [\pi(w_t(j), w_t, A_t) - h(\pi_{R,t}(j))]dt + E_t dV_t \tag{20}$$

where

$$E_t dV_t = V_{w_j} w_t(j)\pi_{R,t}(j)dt + V_w E dw_t + V_a g' dt + \frac{1}{2}V_{aa}\sigma^2 dt, \tag{21}$$

and in which we have used the results that $(dw_t(j))^2 = (dw_t)^2 = dw_t(j)dA_t = dw_t dA_t = 0$ and defined $g' \equiv g + (1/2)\sigma^2$.[9]

[7] Varian (1974) has first introduced this specification. Kim and Murcia (2009) have recently used it to model asymmetric nominal wage rigidities.

[8] With similar tools, Abel and Eberly (1994) have analyzed costly investment decisions.

[9] The fact that dw_t has the same properties of $dw_t(j)$ follows from the symmetry of the equilibrium.

Using the expression for the value function given by (20) and (21), we obtain the optimal value of $\pi_{R,t}(j)$ as implicitly defined by the following condition

$$h_\pi(\pi_{R,t}(j)) = V_{w_j} w_t(j), \tag{22}$$

where

$$h_\pi(\pi_{R,t}(j)) = \chi \frac{e^{\chi\lambda\pi_{R,t}(j)} - 1}{\lambda}. \tag{23}$$

In the symmetric case, i.e. when $\lambda \to 0$, the rate of real wage changes is proportional to its marginal cost

$$\pi_{R,t}(j) = \frac{1}{\chi^2} h_\pi(\pi_{R,t}(j)).$$

Using (20), (21) and (22), we show, in Appendix D, that the marginal costs of changing real wages follow a stochastic differential equations of the form

$$\rho h_\pi(\pi_{R,t})dt = \frac{\theta_w - 1}{\mu_p} \left[\left(\frac{L_t}{L^f} \right)^{1+\eta} - 1 \right] dt + E_t dh_\pi(\pi_{R,t}) \tag{24}$$

and therefore

$$h_\pi(\pi_{R,t}) = \frac{\theta_w - 1}{\mu_p} E_t \int_t^\infty e^{-\rho(s-t)} \left[\left(\frac{L_s}{L^f} \right)^{1+\eta} - 1 \right] ds. \tag{25}$$

Under a quadratic cost function, we can simplify equation (24) to

$$\rho \pi_{R,t} dt = k \left[\left(\frac{L_t}{L^f} \right)^{1+\eta} - 1 \right] dt + E_t d\pi_{R,t}$$

which is the continuos-time non-linear version of the Rotemberg's (1982) cost of adjustment model where the stickiness is applied to real wages rather than to nominal wages and where we have defined $k \equiv (\theta_w - 1)/(\mu_p \chi^2)$.

Using the definition of the employment gap (18), equation (12) implies that

$$x_t = -\ln L_t^f + \frac{1}{1-\alpha}(a_t - \ln w_t - \ln \mu_p), \tag{26}$$

and therefore a diffusion process for x_t of the form

$$dx_t = \frac{1}{1-\alpha} \left(g - \pi_R(x_t) \right) dt + \frac{1}{1-\alpha} \sigma dB_t, \tag{27}$$

which can be used to derive the long-run distribution and in particular the long-run mean

of the employment gap, x. To this end, we need to solve for the unknown functional $\pi_R(x_t)$. By defining $p(x_t) \equiv h_\pi(\pi_R(x_t))/\chi^2$, the optimality condition (23) implies

$$\pi_R(x_t) = \frac{\ln[1 + \lambda\chi p(x_t)]}{\chi\lambda}. \tag{28}$$

In particular, using Ito's Lemma in equation (24) and the diffusion process (27), we obtain that the functional $p(x_t)$ satisfies the following differential equation

$$\rho p(x_t) = k\left[e^{(1+\eta)x_t} - 1\right] + \frac{1}{1-\alpha}p_x(x_t)\left(g - \pi_R(x_t)\right) + \frac{1}{2}\frac{1}{(1-\alpha)^2}p_{xx}(x_t)\sigma^2. \tag{29}$$

Notice again that with quadratic adjustment costs $\pi_R(x_t) = p(x_t)$. We use (28) and (29) to solve for the functional $\pi_R(x_t)$ and $p(x_t)$ and then (27) to solve for the long-run distribution of x_t, if it exists.

2.3.1 The productivity growth-unemployment trade-off

The differential equation (29) is solvable using approximation methods. In particular, an educated guess would be to approximate the solution $p(x_t)$ with a finite-order polynomial.[10] An interesting case, which can be helpful to discuss first, is that of a first-order polynomial. Consider the symmetric quadratic adjustment cost model, with $\lambda \to 0$, and consider small deviations of x_t from zero. In particular, approximate the term $e^{(1+\eta)x_t}$ in (29) as $e^{(1+\eta)x_t} \approx 1 + (1+\eta)x_t$. In this case, the solution for $p(x_t)$, which is equal to $\pi_R(x_t)$, is linear and of the form $p(x_t) = \pi_R(x_t) = a_0 + a_1 x_t$ where a_1 is the positive root of the following quadratic equation

$$\frac{a_1^2}{1-\alpha} + \rho a_1 - k(1+\eta) = 0$$

and

$$a_0 = \frac{a_1}{\rho(1-\alpha) + a_1}g.$$

From the stochastic differential equation (27), it can be seen that the employment gap, x_t, follows an Ornstein-Uhlenbeck process which in the long run converges to a normal distribution with mean given by

$$E(x_\infty) = \frac{\rho(1-\alpha)}{\rho(1-\alpha) + a_1}g$$

[10] This is an educated guess since both the exponential in (29) and the logarithmic in (28) can be represented with infinite-order polynomials, althought the latter only when $|p(x_t)| < 1$.

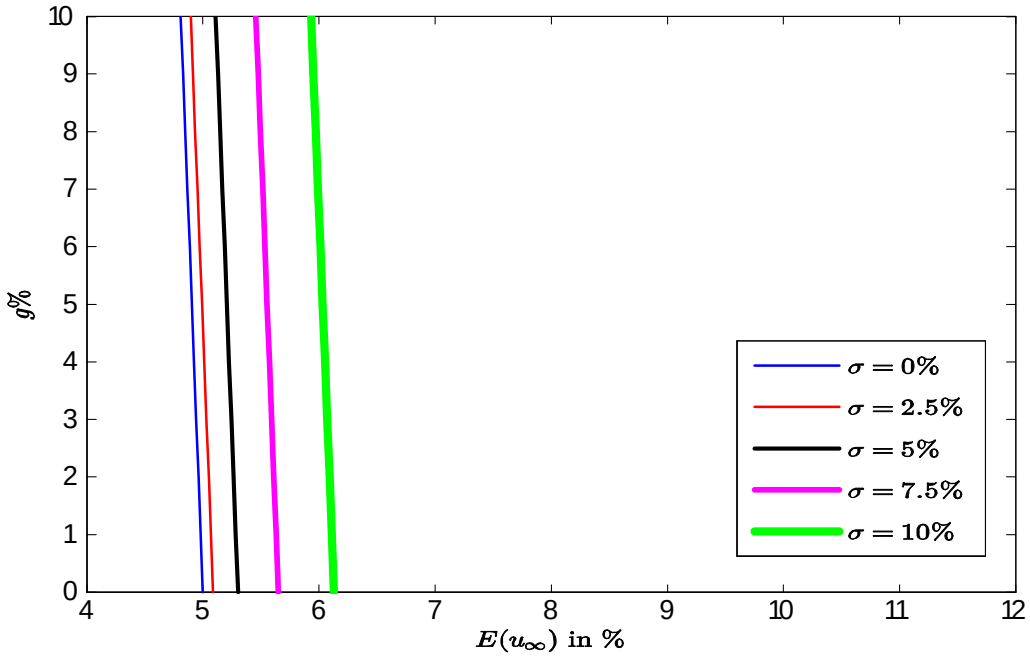

Figure 2: Model with symmetric real-wage rigidities: long-run relationships between the mean of unemployment, $E(u_\infty)$, and the mean of productivity growth, g, for different values of the standard deviation of productivity growth, σ. All variables in % and at annual rates.

where x_∞ denotes the long-run level of the employment gap. The above equation displays a positive relationship between the employment gap and productivity growth and therefore a negative linear relationship between unemployment and productivity growth

$$E(u_\infty) \approx u_f - \frac{1+\eta}{\eta}\frac{\rho(1-\alpha)}{\rho(1-\alpha)+a_1}g,$$

where we have used (17). Notice that at lower levels of real-wage stickiness (lower χ) the link between unemployment and productivity growth is weakened and unemployment becomes close to the frictional level. Furthermore, in this linear solution, there is no relationship between unemployment and the volatility of productivity growth.

In order to find a role for volatility, we need to take at least a second-order polynomial approximation for $p(x_t)$ and $\pi_R(x_t)$.[11] However, as shown in Figure 2, when we assume a symmetric adjustment-cost function, $\lambda \to 0$, we find that the trade-off between unemployment and productivity growth is negligible and the curve is almost vertical. Moreover

[11] The approximations are accurate as long as x_t, $p(x_t)$ and $\pi_R(x_t)$ remain appropriately bounded within the unit circle. In particular, a larger λ in absolute value requires stricter bounds for x_t.

the variance has a small role in accounting for significant shifts in such a trade-off.[12]

A stronger trade-off and a more important role for volatility emerge when there are asymmetries in real wage rigidities, as shown in Figure 3 where we let the parameter λ take negative values. As λ decreases the trade-off becomes more pronounced in a way that it also depends on the level of productivity growth. Moreover, the lower λ the higher the impact of volatility on unemployment. This channel is larger the closer the trend in productivity growth is to zero.

When there are asymmetric rigidities on the downward side, lower levels of productivity growth are associated with higher unemployment because bad productivity shocks are more likely to be absorbed by lower employment demand on the side of firms, as in (12). Firms cut on labor to protect their profits since real wages cannot fall much. At these too high real wages workers would like to supply more labor than what firms demand. When the volatility of productivity growth is high, these bad draws on productivity are even more likely requiring a larger adjustment on labor.

The mechanisms underlined by our model would be absent in a simple framework of symmetric real wage rigidities unless there is a substantial and persistent misalignment between real wages growth and productivity growth. Not only would a model with symmetric real rigidities imply a weak relationship between productivity growth and unemployment but also no role for the volatility of productivity growth in explaining unemployment.

In the next section, we discuss more extensively the results in the limiting case of complete downward real wage inflexibility.

2.4 Downward real wage rigidity

In this section, we assume that real wages are completely rigid on the downward side and flexible on the upward side. This model can be solved in closed-form and its derivation of its solution is helpful to illustrate the influence of volatility on unemployment.[13] With complete downward wage inflexibility, the wage setters maximize (13) under

$$dw_t(j) \geq 0, \tag{30}$$

with $w_{t_0} > 0$. In other words, agents choose a non-decreasing positive real wage path to maximize (13). In appendix E, we show that this optimization problem leads to a

[12] In the Figure, we use the following calibration: $\eta = 2.5$, $\rho = 0.01$, $\alpha = 0.66$, $\theta = 6$, $\mu_p = 1.15$, $\mu_f = 0.05$, $\chi = 9.14$. In particular, the assumption on χ would translate in a Calvo's model into an average duration of contracts on real wages equal to five quarters.

[13] Benigno and Ricci (2010) study the implications of a model with downward nominal wage rigidities.

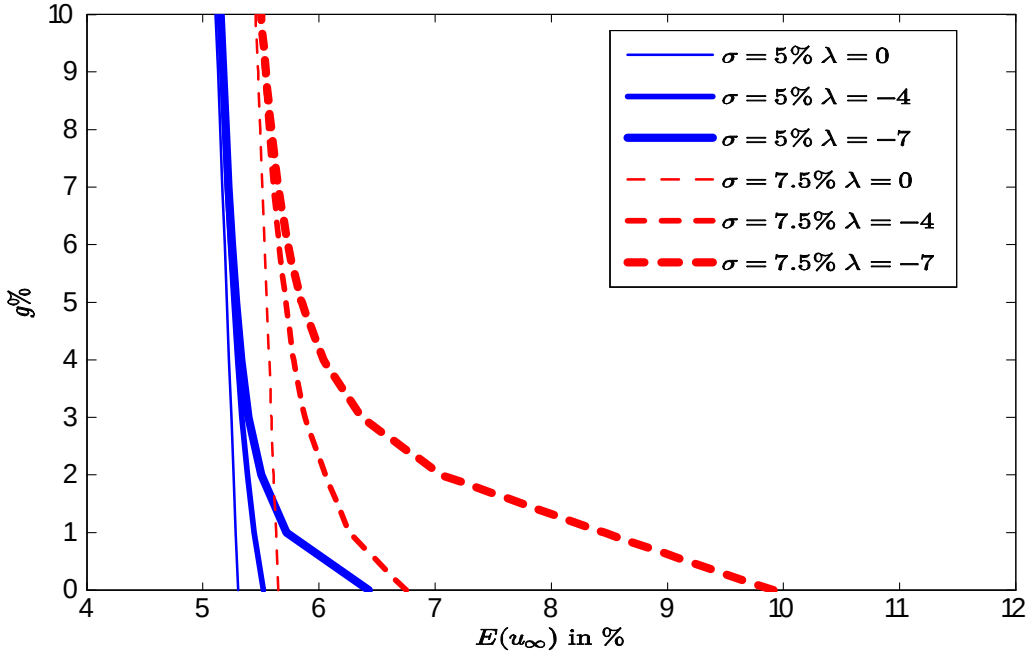

Figure 3: Model with asymmetric real-wage rigidities: long-run relationships between the mean of unemployment, $E(u_\infty)$, and the mean of productivity growth, g, for different values of the standard deviation of productivity growth, σ, and different levels of asymmetries, λ. All variables in % and at annual rates.

simple decision rule. Wage setters compare their past choice on real wages to a current desired real wage. Whenever the past real wage is higher than the desired one, they are constrained by the past decisions and cannot move their real wage. Otherwise, whenever the current desired real wage is higher than the past real wage, they adjust upward to that desired real wage, w_t^d, which is a fraction of the flexible-wage level and given by

$$\begin{aligned} w_t^d &= c(g,\sigma^2,\eta,\rho,\alpha)^{1-\alpha}\cdot\frac{1}{\mu_p}(L^f)^{\alpha-1}A_t \\ &= c(g,\sigma^2,\eta,\rho,\alpha)^{1-\alpha}\cdot w_t^f \end{aligned} \tag{31}$$

where $c(\cdot)$ is a non-negative function of the model parameters:

$$c(g,\sigma^2,\eta,\rho,\alpha)\equiv\left(\frac{g+\frac{1}{2}\gamma(g,\sigma^2,\rho)\cdot\sigma^2}{g+\frac{1}{2}\left(\gamma(g,\sigma^2,\rho)+\frac{\eta+1}{1-\alpha}\right)\cdot\sigma^2}\right)^{\frac{1}{1+\eta}}\leq 1 \tag{32}$$

and $\gamma(\cdot)$ is the following non-negative function

$$\gamma(g,\sigma^2,\rho)=\frac{-g+\sqrt{g^2+2\rho\sigma^2}}{\sigma^2},$$

which is derived in Appendix E.

Agents' optimizing behavior in the presence of exogenous downward real wage rigidities implies an endogenous tendency for limiting the upward revisions in real wages. When wages adjust upward, they adjust to the *desired* level w_t^d, which is always below the flexible-wage level by a factor $c(\cdot)$. Indeed, optimizing wage setters choose an adjustment rule that tries to minimize the inefficiencies of downward real wage inflexibility. Wage setters are worried to get locked with an excessively high real wage were future unfavorable shocks require a real wage decline (as downward real wage rigidities would imply a fall in employment). As a consequence, optimizing agents refrain from excessive real wage increases when favorable shocks require upward adjustment, pushing current employment above the flexible-case level.

The above optimizing decision rule nests also a myopic rule in which agents do not take into account the consequences of the current real wage choice for future decisions and simply adjust real wages to a flexible-wage level whenever this level is above their previous choice. In this case $w_t = w_t^f$, whenever $dw_t > 0$. This myopic rule, which will be of particular interest for the empirical section that follows, corresponds to the limiting case in which agents do not discount the future at all, i.e. when $\rho \to \infty$ implying $c \to 1$.

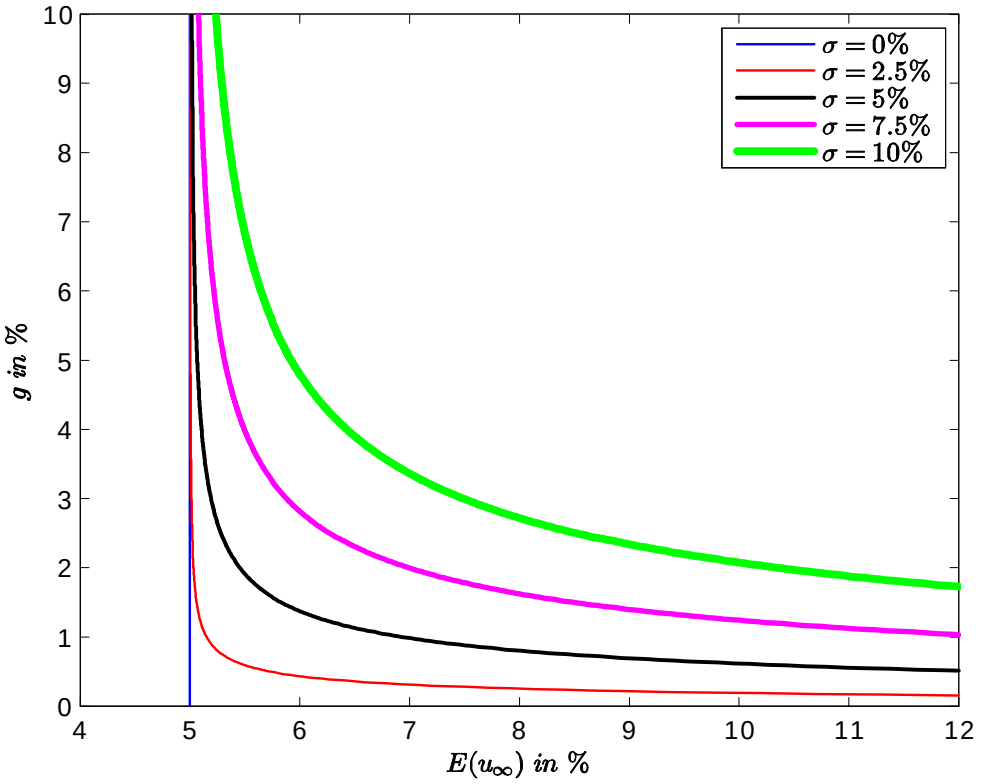

Figure 4: Model with downward real-wage rigidities: long-run relationships between the mean of unemployment, $E(u_\infty)$, and the mean of productivity growth, g, for different values of the standard deviation of productivity growth, σ. All variables in % at annual rates.

2.4.1 The productivity growth-unemployment trade-off

We can now solve for the equilibrium level of employment and characterize the productivity-unemployment trade-off in the presence of downward real wage rigidities. Since we have shown that $w_t \geq c(\cdot)^{1-\alpha} w_t^f$, equation (26) implies that $-\infty \leq x_t \leq -\ln c(\cdot)$. The existence of downward real wage rigidities endogenously adds an upward barrier on the employment gap. Since a_t follows a Brownian motion with drift g and standard deviation σ, also x_t is going to follow a Brownian motion with mean $g/(1-\alpha)$ and variance $(\sigma/(1-\alpha))^2$ but with a regulating barrier at $-\ln c(\cdot)$. The probability distribution function for such process can be computed at each point in time.[14] We are interested in studying whether this probability distribution converges to an equilibrium distribution when $t \to \infty$, in order to characterize the long-run probability distribution for employ-

[14] See Cox and Miller (1990, pp. 223-225) for a detailed derivation.

ment, and thus unemployment. Standard results assure that this is the case when the drift of the Brownian motion of x_t is positive, which requires $g > 0$. In this case, it can be shown that the long-run cumulative distribution of x_t, denoted with $P(\cdot)$, is given by

$$P(x_\infty \leq z) = e^{\frac{2g}{\sigma^2}(1-\alpha)(z+\ln c)}$$

for $0 \leq z \leq -\ln c(\cdot)$ where x_∞ denotes the long-run equilibrium level of the employment gap. We can compute the long-run mean of the employment gap,

$$E[x_\infty] = -\frac{1}{2}\frac{1}{1-\alpha}\frac{\sigma^2}{g} - \ln c(g, \sigma^2, \eta, \rho, \alpha). \tag{33}$$

and therefore the long-run mean of unemployment

$$E[u_\infty] = u^f + \frac{1}{2}\frac{1+\eta}{\eta(1-\alpha)}\frac{\sigma^2}{g} + \frac{1+\eta}{\eta}\ln c(g, \sigma^2, \eta, \rho, \alpha). \tag{34}$$

In this model the average growth rate of real wages converges in the long run to the productivity trend, g for any positive g.[15] In the presence of downward real wage rigidities, we find a strong negative relationship between the unemployment rate and the rate of productivity growth, which is shifted by the volatility of productivity. The shift is quantitatively important as shown in Figure 4. For given growth of productivity, a higher volatility implies a higher unemployment rate. For given volatility, a lower productivity growth implies a higher unemployment rate. Notice that under the myopic adjustment rule, in which $\rho \longrightarrow \infty$, the mean of unemployment rate is simply given by

$$E[u_\infty] = u^f + \frac{1}{2}\frac{1+\eta}{\eta(1-\alpha)}\frac{\sigma^2}{g}, \tag{35}$$

as the function $c(\cdot)$ in (34) is now equal to 1. Indeed, a value of $c(\cdot)$ below one is capturing the benefits in terms of lower unemployment due to the intertemporal optimizing behavior of wage setters who are taking into account the future consequences of their current real wage choices and therefore set lower real wages when adjusting upward. Absent this channel, unemployment would simply reflect the structural level of unemployment, u^f, and the costs of the downward real wage rigidity constraint given by the ratio between the variance and the mean of productivity growth. The relevance of this ratio to explain long-run unemployment will be investigated in the empirical analysis below.

[15] This is an appealing feature of the limiting case in contrast with the model of symmetric rigidities.

3 Evidence for the United States

A key prediction of the theoretical model is that the variance of productivity growth has explanatory power for the mean of the unemployment rate over and above the mean of productivity growth. There are two ways we can take this prediction to the data. First, focusing on a single country, we can construct time-varying measures of mean and volatility, and then ask whether periods of higher variance in productivity growth are associated with a higher mean of unemployment, for a given mean of productivity growth. Second, we can investigate this relationship within a panel of countries. This section describes the strategy and the results for the first avenue. Section 4 presents evidence based on the second avenue.

As for exploiting the time variation within a single country, the U.S. Great Moderation appears a natural candidate for assessing the empirical merits of our theory. During the first half of the 1980s, the volatility of several measures of real activity, including real GDP growth, residential investment and unemployment fell sharply in the U.S.. To the extent that productivity growth also showed a pronounced decline in volatility, our model predicts that this should have been accompanied by a pronounced fall in the mean of unemployment. Figure 1 provides prima facie evidence in support of this prediction. In this section, we first spell out the way the estimates in Figures 1 have been constructed and we then use the time-varying measures of mean and volatility for productivity growth to assess the ability of the model to account for the low-frequency variation in the unemployment rate.

3.1 Measuring unemployment and productivity trends

The econometric literature offers several ways to model time-variation in the variance of the stochastic disturbances as well as in the autoregressive coefficients of stochastic processes. Some of the best-known examples in macroeconomics include models of AutoRegressive Conditional Heteroskedasticity (ARCH), Regime-Switching volatility models (RS) and Vector AutoRegressions with stochastic volatility (VAR). It is worth emphasizing that our theoretical model has predictions for the rate of unemployment in the long-run. The focus on the long-run makes the ARCH specification less attractive than the RS and the VAR. Furthermore, the notion of real rigidities in the labor market hinges upon the presumption that changes in productivity diffuse gradually, rather than abruptly, to the rest of the economy, thereby making the RS model less attractive than the time-varying VAR for our purposes.

Following the literature pioneered by Cogley and Sargent (2001 and 2005), and followed

among others by Primiceri (2005) and Sargent and Surico (2010), we model the evolution of productivity growth, g_t, real wage growth, Δw_t, and the rate of unemployment, u_t, using a VAR with drifting coefficients and stochastic volatility. The drifting coefficients enable us to construct a time-varying measure for the mean of the endogenous variables. Both the drifting coefficients and the stochastic volatility allow us to construct a time-varying measure of volatility.

The statistical model is a VAR(p) of the following form:

$$Y_t = B_{0,t} + B_{1,t}Y_{t-1} + ... + B_{p,t}Y_{t-p} + \epsilon_t \equiv X_t^{'}\theta_t + \epsilon_t \tag{36}$$

where $X_t^{'}$ collects the first p lags of Y_t, θ_t is a matrix of time-varying parameters, ϵ_t are reduced-form errors, Y_t is defined as $Y_t \equiv [g_t,\ \Delta w_t, u_t]'$, and p is set equal to 2. The parameters of the error covariance matrix, $Var(\epsilon_t) \equiv \Omega_t$, are assumed to evolve as geometric random walks while the parameters of the matrix of autoregressive coefficients are assumed to evolve as random walks.

The time-series for long-run unemployment and long-run productivity growth are computed as local-to-date t approximations to the mean of the endogenous variables of the VAR, evaluated at the posterior mean $E(\theta_{t|T})$. Let us rewrite equation (36) in companion form:

$$z_t = C_{t|T} + D_{t|T}z_{t-1} + \varsigma_t$$

where z_t contains current and lagged values of Y_t, $C_{t|T}$ is the vector of intercepts, $D_{t|T}$ is the vector of stacked time-varying parameters and ς_t is a conformable vector containing ϵ_t and zeros. Following Cogley and Sargent (2005), the long-run mean for the vector z_t can then be computed as:

$$\tilde{z}_t = \left(I - D_{t|T}\right)^{-1} C_{t|T} \tag{37}$$

where, given the order of the variables in the VAR, the first and third elements of $\tilde{z}_t$ correspond to the mean of productivity growth, $\tilde{g}_t$, and the mean of unemployment, $\tilde{u}_t$, at time t.

The time-series for the variance of the variables in the VAR can be estimated using the integral of the spectral density over all frequencies, $\int_{\varpi} f_{t|T}(\omega)$, where $f_{t|T}$ is defined as:

$$f_{t|T}(\omega) = (I - D_{t|T}e^{-i\omega})^{-1}\frac{\Omega_{t|T}}{2\pi}\left[(I - D_{t|T}e^{-i\omega})^{-1}\right]' \tag{38}$$

The element $(1,1)$ of the matrix $f_{t|T}(\omega)$ represents the variance of productivity growth, $\tilde{\sigma}_t^2$, at time t. Details of the model specification and estimation method are provided in Appendix B.

The data were collected in September 2010 from the Fred database available at the Federal Reserve bank of St. Louis. Productivity is the non-farm business sector output per hour of all persons (acronym 'OPHNFB'), wage is the non-farm business sector real compensation per hour (acronym 'COMPRNFB'), and unemployment is the rate of civilian unemployment for persons with 16 years of age or older (acronym 'UNRATE').[16] All variables are seasonally adjusted at the source. As we are not interested to explain quarter on quarter changes, we compute annual growth rates for productivity and real wage to smooth out the high frequency components in the data. Growth rates are approximated by log differences. Results are robust to using quarterly changes. To calibrate the priors for the VAR coefficients, we use a training sample of thirteen years, from 1949Q1-1961Q4. The results hereafter, then, refer to the period 1962Q1 to 2010Q2.

We can therefore compute the estimates of long run unemployment ($\tilde{u}_t$), long run productivity ($\tilde{g}_t$), and the variance of productivity ($\tilde{\sigma}_t^2$) from the estimates of the VAR (36) together with the formulas (37) and (38). These series are shown in Figure 1.

3.2 The fit of the linear model

This section assesses empirically the main predictions of the model: the mean of unemployment depends negatively from the mean of productivity growth and positively from the variance of productivity growth. More formally, we can write:

$$E[u_\infty] = f(g, \sigma^2, \vartheta)$$

where the vector $\vartheta \equiv (\eta, \rho, \alpha, \lambda, u^f)$ contains the relevant parameters of the model and $f(\cdot)$ is a generic non-linear function which in the limiting case of downward real wage inflexibility corresponds to (34).

A natural benchmark of comparison for this exercise is the linear specification employed in earlier contributions (see for instance Pissarides and Vallanti, 2007), which relates long-run unemployment to long-run productivity growth:

$$\tilde{u}_t = a - b\tilde{g}_t + \varepsilon_t \tag{39}$$

where a and b are parameters and ε_t is a well-behaved stochastic disturbance. Using

[16] To make our empirical results comparable with earlier contributions (see for instance Staiger, Stock and Watson, 2001), we measure productivity as the ratio of output to total hours in the non-farm business sector, Y/L. This measure is computed and released by the Bureau of Labour Statistics. In our model, productivity is defined as Y/L^α and the first difference of its logarithm is denoted by g. It should be noted, however, that assuming a standard labour to capital ratio of 2/3 the correlation between g and the first difference of the logarithm of Y/L is 0.91 over our sample period.

the estimates of the VAR derived in the previous Section, we obtain the following OLS estimates for equation (39):

$$\tilde{u}_t = \underset{(0.002)}{0.10} - \underset{(0.088)}{2.24} \cdot \tilde{g}_t + \hat{\varepsilon}_t \tag{40}$$

where standard errors are reported in parentheses. The R^2 of the regression is 0.77. The estimates of this simple model show that there is a tight negative relationship between productivity growth and unemployment in the long-run. In particular, a 1% fall in long-run productivity growth corresponds to an increase in long-run unemployment of 2.24 percentage points. Alternatively, an increase of one standard deviation (0.002) in long-run productivity growth would lower long-run unemployment by 0.47 percentage points.

Figure 5 confronts long-run unemployment, depicted as red continuous line, with the fitted values from equation (40), depicted as blue dotted line. The linear model does a good job in tracking qualitatively the movements in the unemployment rate. However, a closer inspection of the figure reveals that neither the decline in trend unemployment between 1984 and 1992 nor the rise since the late 1990s can be adequately explained by the linear model, whose fit seems particularly inadequate to explain the developments in long-run unemployment since 2007.

The theoretical model of section 2 suggests two departures from the linear specification (39). First, it highlights the relevance of the variance of productivity growth. Consistent with Figure 1, movements in the variance of productivity growth coincide with movements in long-run unemployment, especially during the periods where the mean of productivity growth was flat. Second, under the limiting case of downward real wage inflexibility, the model allows us to derive a nonlinear relationship between unemployment and productivity growth in closed form. To appreciate the relative importance of these modifications, we proceed in two steps. First we augment the linear specification in (39) with a variance term. Then, we estimate the relationship between unemployment and productivity growth nonlinearly.

More specifically, we estimate the following linear specification in both the mean and the variance of productivity growth:

$$\tilde{u}_t = \underset{(0.001)}{0.08} - \underset{(0.047)}{1.68} \cdot \tilde{g}_t + \underset{(1.974)}{50.89} \cdot \tilde{\sigma}_t^2 + \hat{\varepsilon}_t \tag{41}$$

The variance term is highly significant and the R^2 is now 0.95, a significant increase relative to the estimates in (40) which are based on a linear specification in long-run

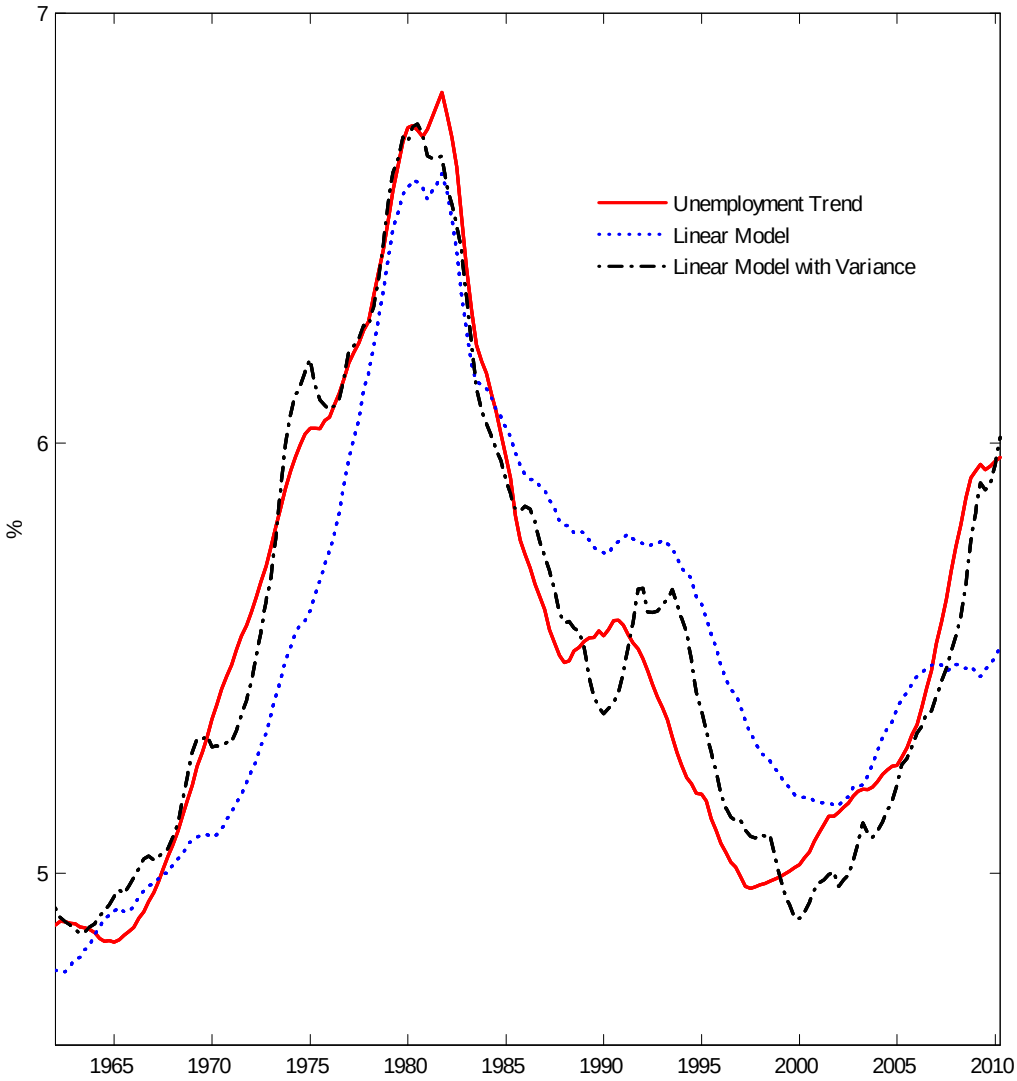

Figure 5: Trend in the unemployment rate implied by the estimates of the time-varying VAR (36) using formula (37), and fitted values of the Linear Model of equation (40) and of the Linear Model with Variance of equation (41). Percent rates.

productivity growth only.[17] The improvement is evident from Figure 5. The fitted values from equation (41) track unemployment trend far better than the linear model (40), and in particular they allow the model to account fully for the decline in long-run unemployment of the 1980 and the rise of the late 2000s. The coefficient on the productivity mean is somewhat lower than in the bivariate case.

The effect of the variance is also economically significant: an increase of one standard deviation (0.00005) would imply a rise in long-run unemployment of about 0.25 percent. The estimates in Figure 1 reveal that the variance of productivity growth declined from 0.0003 to about 0.0002 during the first half of the 1980s when long-run unemployment fell from about 6.5% to 5.5%. Together with the estimates in (41), this implies that the decline in the variance of productivity growth can account for about 50% of the fall in long-run unemployment during this episode. Between 2000 and 2009, the variance of productivity growth has increased from 0.00024 to 0.00038 against the backdrop of a rise in long-run unemployment from 5% to 6%. These numbers imply a 70% contribution of the variance of productivity growth to long-run unemployment during the 2000s.

3.3 Controlling for demographics

An important strand of the literature has convincingly argued that changes in the demographic composition of the labour force affects the low-frequency movements in unemployment (Shimer, 1998), the low-frequency movements in productivity (Francis and Ramey, 2009) and the variance of real output growth (Jaimovich and Siu, 2009).

In this section, we want to assess the extent to which the estimates of the linear models above may vary once we control for demographics. To this end, we construct time series for the share of workers in the labor force with age *(i)* between 16 and 21 (as in Francis and Ramey, 2009), *(ii)* between 16 and 34 (as in Shimer, 1998), and *(i)* the sum of the shares of workers in the 16-29 and the 60-64 windows of age (as in Jaimovich and Siu, 2009). Furthermore, we run a regression of the unemployment rate on a constant and the unemployment rate of workers in prime age (defined as those between 35 and 64 years), and then use the fitted values from this regression in place of the unemployment rate in the VAR to construct the trend of what Shimer (1998) refers to as a measure of genuine unemployment which is not affected by demographics.[18]

The labor force series were collected in September 2010 from the Bureau of Labor

[17] Similar results are obtained using averages and variances of unemployment and productivity growth computed over either five- or ten-year rolling windows.

[18] The estimates of this regression are: 0.0075 (.0014) for the intercept and 1.2716 (.0340) for the slope. Standard errors in parenthesis. $R^2 = 0.851$. Sample: 1948Q1:2010Q2.

Statistics using data gathered in the Current Population Survey. These data can also be used to compute the unemployment rate for prime-age workers. The series used in this section are reported in Appendix A. The results of these sensitivity analyses are collected in Table 1, which presents estimates for the linear model using the trend of productivity growth and the measures of labor force share in columns (1) to (3), and then adding the variance of productivity growth in columns (5) to (7). The estimates for the specifications using Shimer's measure of genuine unemployment are displayed in columns (4) and (8), without and with the variance of productivity growth respectively.

Two main results emerge from Table 1. First, controlling for demographics does not overturn our finding of a significant role for both the long-run mean and the variance of productivity growth to explain low-frequency movements in unemployment. In particular, the estimated coefficient on $\tilde{\sigma}_t^2$ in columns (5) to (8) is never statistically different from the estimates in (41), which omits any demographic measures. Similar results are obtained for the estimated coefficient on $\tilde{g}_t$, although in column (4) this is statistically lower than the estimates in (40). Second, in line with Shimer (1998), Francis and Ramey (2009) and Jaimovich and Siu (2009), the composition of the labor force has a significant influence on the low-frequency movements in unemployment, although its statistical and economic significance appear muted once the variance of productivity growth is added as additional regressor in the columns (5) to (7). The finding of an important role for the variance of productivity growth is robust to using Shimer's measure of genuine unemployment in column (8), although the coefficient on the productivity growth trend is statistically smaller than in (41).

In summary, we conclude that the long-run mean and the variance of productivity growth are significant determinants of U.S. long-run unemployment over and above changes in the demographic composition of the labor force in the post-WWII period.

Table 1: Controlling for demographics

specifications:	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*
workers age	*16-21*	*16-34*	*16-29_60-64*	*prime*	*16-21*	*16-34*	*16-29_60-64*	*prime*
dependent variable:								
$\tilde{u}_t$	✓	✓	✓		✓	✓	✓	
$trend\ in\ fitted\ u_t$				✓				✓
regressors:								
constant	0.092***	0.095***	0.076***	0.085***	0.076***	0.082***	0.076***	0.055***
	(0.002)	(0.005)	(0.004)	(0.003)	(0.001)	(0.003)	(0.002)	(0.045)
$\tilde{g}_t$	-2.014***	-2.023***	-1.717***	-1.252***	-1.679***	-1.824***	-1.685***	-0.680***
	(0.080)	(0.150)	(0.094)	(0.127)	(0.047)	(0.071)	(0.053)	(0.128)
$labor\ force\ share_t$	0.060***	0.011*	0.044***		-0.005	-0.008***	-0.001	
	(0.007)	(0.006)	(0.005)		(0.005)	(0.003)	(0.003)	
$\tilde{\sigma}_t^2$					52.37***	52.20***	51.41***	57.21***
					(2.488)	(1.997)	(2.512)	(6.714)
R^2	0.832	0.776	0.838	0.334	0.949	0.950	0.949	0.515

Estimation sample: 1962Q1-2010Q2. *** p<0.01, ** p<0.05, * p<0.1. $\tilde{u}_t$ is the unemployment trend, $\tilde{g}_t$ is the productivity growth trend and $\tilde{\sigma}_t^2$ is the productivity growth variance computed from the time-varying VAR. Labor force shares and prime-age unemployment rate are from the Current Population Survey as computed by the BLS. Prime-age refers to workers aged between 35 and 64. In columns (4) and (8), the trends and variances are computed from a time-varying VAR in which the unemployment rate is replaced by the fitted values of a regression of unemployment rate on a constant and the prime-age unemployment rate.

3.4 The fit of the nonlinear model

The results above point toward a significant role for asymmetries in real wage rigidities. To investigate this channel further, we estimate the non-linear equation implied by the model under the limiting case of complete downward real wage rigidities:

$$\tilde{u}_t = u^f + \frac{1}{2}\frac{1+\eta}{\eta(1-\alpha)}\frac{\tilde{\sigma}_t^2}{\tilde{g}_t} + \frac{1+\eta}{\eta}\ln c(\tilde{g}_t, \tilde{\sigma}_t^2, \eta, \rho, \alpha) + \varepsilon_t. \tag{42}$$

Unfortunately, the parameters α and η are not separately identified. Nevertheless, we can still estimate a reduced-form version of (42), which we refer to as the 'unrestricted model'.

The estimates of the unrestricted model yield so high estimates for ρ as to imply values of the function $c(\cdot)$ very close to one, which correspond to the case of myopic agents. We therefore estimate also a simplified version of the theoretical model in (42) where we impose $c = 1$ prior to estimation:

$$\tilde{u}_t = u^f + \frac{1}{2}\frac{1+\eta}{\eta(1-\alpha)} \cdot \frac{\tilde{\sigma}_t^2}{\tilde{g}_t} + \varepsilon_t \tag{43}$$

The simplified version (43), which is linear in the variance-to-mean ratio of productivity growth, is referred to as the 'restricted model'.

The fitted values associated with the non-linear unrestricted model and with the variance-to-mean restricted model are presented in Figure 6. Both specifications track long-run unemployment remarkably well and they clearly outperform the linear specification of Figure 5 which is based on long-run productivity growth only. In particular, the specifications in (42) and (43) capture well the fall in long-run unemployment during the 1984-1992 period and its increase during the (late) 2000s.

The non-linear model has a R^2 of 0.92 and a point estimate (standard error) for the flexible-wage unemployment rate, μ^f, of 3.88% (0.29). The restriction implied by the variance-to-mean ratio implies only a modest deterioration in the goodness of fit with a R^2 of 0.90 and a coefficient μ^f of 3.41% (0.05). This suggests that the simplified expression (43) provides a reasonable approximation to the unrestricted specification (42). Notice again that in the simplified model (43) with myopic agents, downward real wage rigidities play a crucial role through the influence of the variance-to-mean ratio of productivity growth in affecting unemployment in the long-run.

In summary, versions of the theoretical model that feature strong *asymmetries* in real rigidities appear to account for the low-frequency movements in the U.S. unemployment rate which a model with *symmetric* real rigidity has hard time to explain. Similar results,

available upon request, are obtained using Shimer's measure of genuine unemployment, which controls for demographic changes.

4 International evidence

In this section, we explore the empirical implications of the model in Section 2 within a panel of international data. In particular, we are interested in whether the variance of productivity growth has predictive power for the mean of unemployment across different countries over a sufficiently long period of time. Our international dataset is an unbalanced panel of quarterly observations for developed and developing economies over the post-WWII period.[19]

For each country i, we compute over a window of ten years: (i) the mean of unemployment, $\tilde{u}_{it}$, (ii) the mean of productivity growth, $\tilde{g}_{it}$, (iii) the variance of productivity growth, $\tilde{\sigma}^2_{it}$, and (iv) the ratio between the variance of productivity growth and the mean of productivity growth, $V\text{-}to\text{-}M\ ratio_{it}$.

Unemployment is taken from various data sources (World Development Indicators, IFS, WEO, OECD, and Datastream, via splicing in the respective order); the sample spans the years between 1960 and 2008. For productivity, we use real GDP per worker where real GDP is taken from employment World Development Indicators, IFS, and WEO (via splicing in the respective order) and employment is taken from the same sources as unemployment. Prior to estimation, we drop observations for which there are less than eight periods in a ten-year window.

The estimates are displayed in Table 2 and each column refers to a different specification and estimation method. The estimates in the first eight columns are based on the Fixed-effect Estimator (FE) with time dummies included only in the columns (5) to (8). The last two columns refer to the Between Estimator (BE) and will allow us to assess the extent to which cross-country variation in the mean of unemployment is due to cross-country variation in the mean and variance of productivity growth. In all specifications, standard errors are adjusted for heteroskedasticity. In the FE columns, standard errors are also adjusted for intra-group correlation.

In line with the theory, the coefficient on average productivity growth is negative and the coefficient on the variance of productivity growth is positive. While the latter is

[19] The countries are: Argentina, Australia, Austria, Belgium, Bulgaria, Canada, Chile, China,P.R.:Hong Kong, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Israel, Italy, Japan, Korea, Latvia, Lithuania, Luxembourg, Malaysia, Malta, Morocco, Netherlands, New Zealand, Norway, Peru, Philippines, Portugal, Russia, Slovak Republic, Slovenia, Spain, Sweden, Switzerland, Taiwan Prov.of China, Thailand, United Kingdom, United States, Venezuela, Rep. Bol.

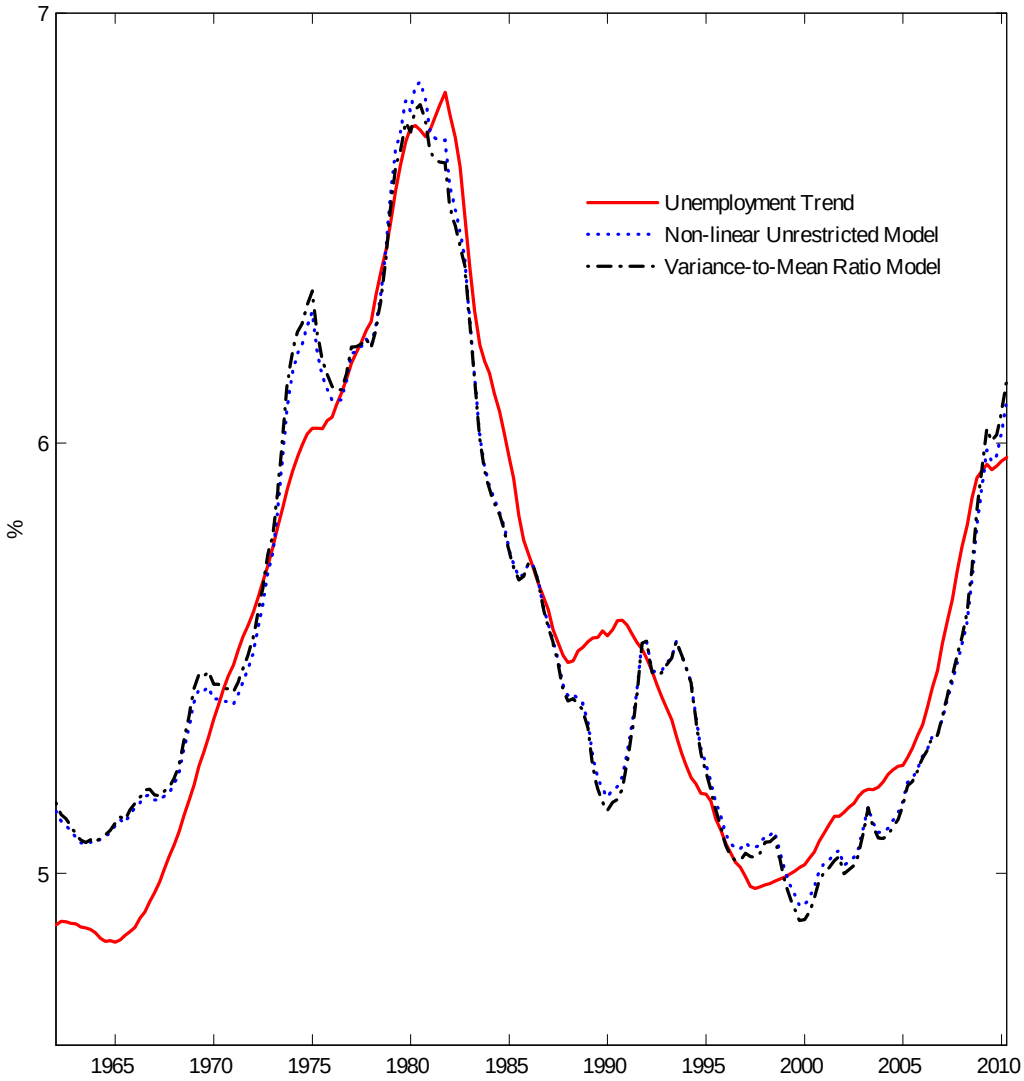

Figure 6: Trend in the unemployment rate implied by the estimates of the time-varying VAR (36) using formula (37), and fitted values for the Non-linear Unrestricted model of equation (42) and the Variance-to-Mean-Ratio model of equation (43). Percent rates.

always significant, the former is significant only in the specifications that do not include time dummies. A possible interpretation of this result is that the countries in our panel share a common trend in productivity growth which absorbs the negative correlation with national unemployment. The coefficients are somewhat lower than those estimated in the previous section, in part reflecting higher standard deviations for all variables in this international sample. Indeed, the coefficient in column 3 indicates that the effect of a one standard deviation increase in productivity growth (about 0.02) is to lower average unemployment by a full percentage point.

The significance of the variance of productivity growth is strongly supported. The estimated coefficient on the variance to mean ratio is positive and statistically different from zero in both specifications (4) and (8), and therefore it accords with the prediction of the model of section 2. As for the goodness of fit, the specifications that contain the variance term (either linearly or as a ratio) have the largest R^2. The coefficient reported in column 3 suggests that the effect of a one standard deviation increase in the variance of productivity growth (about 0.0005) is to lower average unemployment by more than one percentage point.

While the between-country effects of average productivity growth on average unemployment (columns 9 and 10) are not statistically different from the within-country effects (columns 1 to 8), the results of the last two columns reveal that the effects across countries are imprecisely estimated. The FE specifications are associated with a better fit than the BE specifications, thereby corroborating the view that the theory is more successful in explaining fluctuations in long-run unemployment over time. This finding is not surprising as the cross-sectional dimension of the long-run unemployment rates is more likely to reflect other factors such as labor market structures and institutions.

Table 2: Panel regressions

estimation method:	FE	FE	FE	FE	FE	FE	FE	FE	BE	BE
specifications:	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*
	mean	*variance*	*both*	*V-to-M ratio*	*mean*	*variance*	*both*	*V-to-M ratio*	*both*	*V-to-M ratio*
Dependent variable: $\tilde{u}_t$										
Regressors:										
$\tilde{g}_t$	-0.355*		-0.561***		-0.019		-0.200		-0.599	
	(0.190)		(0.190)		(0.258)		(0.258)		(0.410)	
$\tilde{\sigma}_t^2$		21.10*	26.70**			23.3**	24.4***		19.0	
		(11.4)	(10.7)			(8.80)	(8.50)		(14.46)	
$\tilde{\sigma}_t^2/\tilde{g}_t$				0.330***				0.280**		0.434
				(0.119)				(0.113)		(0.262)
time dummies	*no*	*no*	*no*	*no*	*yes*	*yes*	*yes*	*yes*		
observations	110	110	110	110	110	110	110	110	39	39
R^2	0.045	0.120	0.223	0.181	0.357	0.490	0.497	0.479	0.083	0.105

Robust standard errors in parentheses, *** p<0.01, ** p<0.05, * p<0.1. FE: Fixed-effects Estimator; BE: Between Estimator. Intercept and coefficients on time dummies not reported. $\tilde{u}_t$ is unemployment mean, $\tilde{g}_t$ is productivity growth mean and $\tilde{\sigma}_t^2$ is productivity growth variance, computed over ten year windows.

5 Conclusions

A simple model of the labor market with sticky real wages implies that unemployment and productivity growth are negatively related in the long-run. When coupled with the assumption of asymmetric wage rigidities, we show that the model generates a stronger trade-off and the additional prediction that long-run unemployment depends positively on the variance of productivity growth. We employ two alternative strategies to bring these predictions to the data. The first, based on U.S. data, extracts the trend component of unemployment and productivity growth and therefore exploits low-frequency variation over time. The second strategy, based on a panel of international data, evaluates the association between averages and variances of unemployment and productivity growth for windows of ten years and therefore exploits low-frequency variation both over time and across countries.

The empirical results show robust support for both predictions of the theoretical model: higher volatility of productivity growth and lower levels of long-run productivity growth are associated with higher levels of long-run unemployment. Moreover they are robust to controlling for demographic factors, which have been recently shown to influence long-run unemployment. The panel regressions reveal that variation over time is more important than variation across countries to explain this pattern. Movements in the variance of productivity growth, for instance, allows our model to account for two episodes in U.S. data which cannot be fully accounted for by a linear specification in the trend of productivity growth only. These are (i) the fall in long-run unemployment during the second half of the 1980s and early 1990s; (ii) the rise in long-run unemployment during the late 2000s.

Our paper has also important policy implications. To the extent that stabilization policies played a significant role in the Great Moderation, our theoretical and empirical findings highlight a new channel through which such policies may contribute to lower long-run unemployment.

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A The data

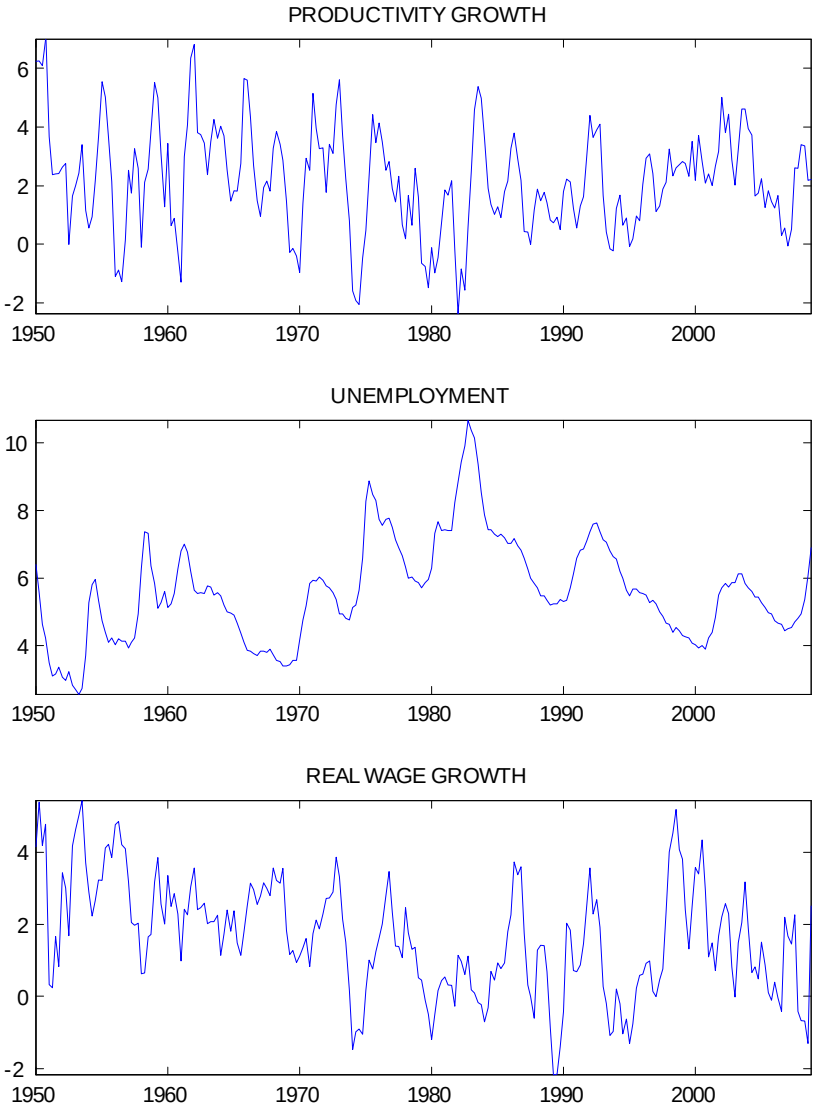

Figure 7: Productivity growth, unemployment and real wage growth, quarterly data on sample 1949Q1:2010Q2. All data are in percent. Productivity growth and real wage growth at annual rates.

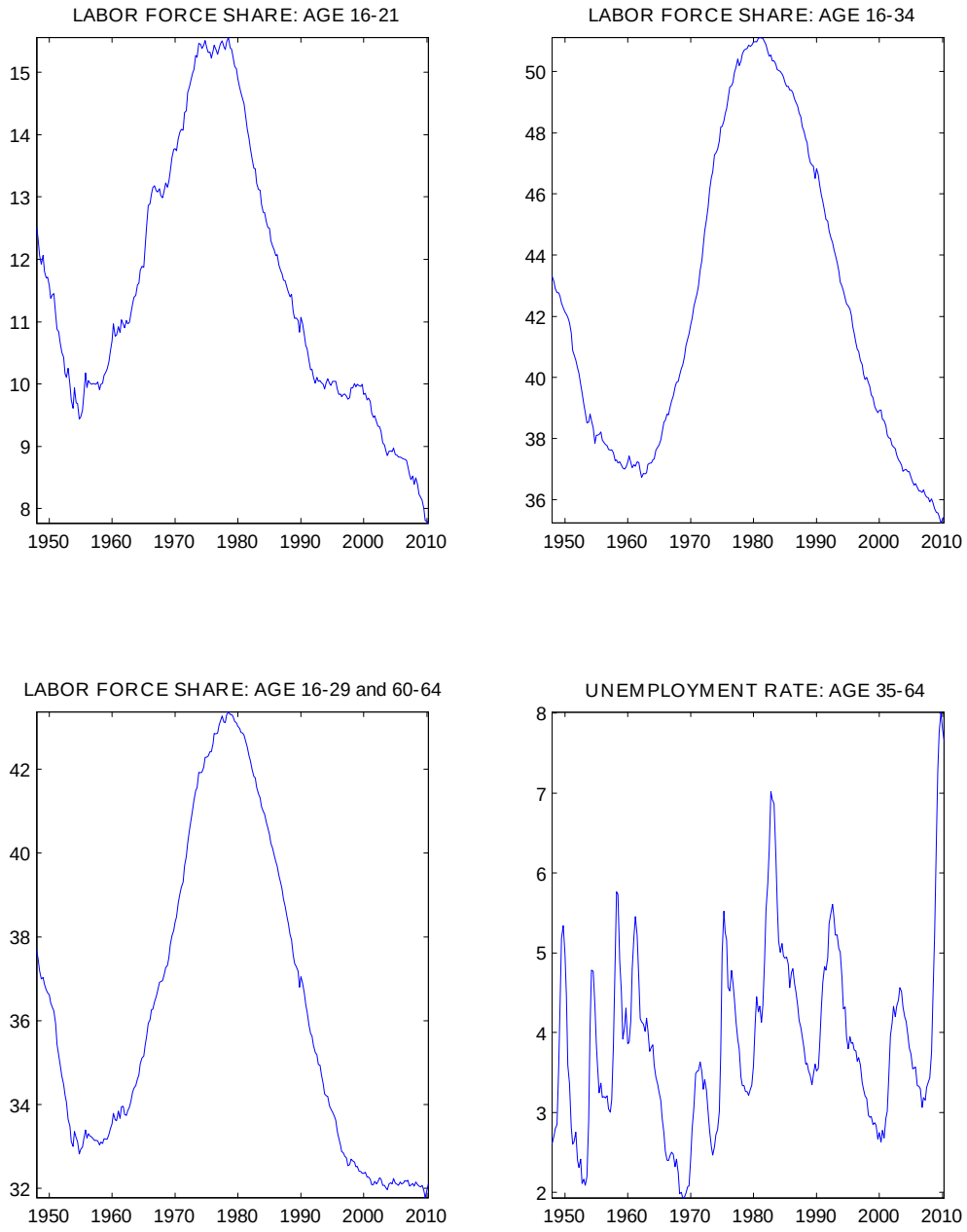

Figure 8: Labor force shares for workers with age between 16 and 21, between 16 and 34, between 16 and 29 plus between 60 and 64, unemployment rate for workers with age between 35 and 64, quarterly data on sample 1949Q1:2010Q2. Percent rates.

B A stochastic volatility model

The statistical model is a VAR(p) of the following form:

$$Y_t = B_{0,t} + B_{1,t}Y_{t-1} + ... + B_{p,t}Y_{t-p} + \epsilon_t \equiv X_t^{'}\theta_t + \epsilon_t \tag{B.1}$$

where $X_t^{'}$ collects the first p lags of Y_t, θ_t is a matrix of time-varying parameters, ϵ_t are reduced-form errors, Y_t is defined as $Y_t \equiv [g_t, \Delta w_t, u_t]'$, and p is set equal to 2. We stack the time-varying VAR parameters in the vector θ_t, which is assumed to evolve as:

$$p(\theta_t \mid \theta_{t-1}, Q) = I(\theta_t)\ f(\theta_t \mid \theta_{t-1}, Q) \tag{B.2}$$

where $I(\theta_t)$ is an indicator function that takes a value of 0 when the roots of the associated VAR polynomial are inside the unit circle and is equal to 1 otherwise. $f(\theta_t \mid \theta_{t-1}, Q)$ is given by

$$\theta_t = \theta_{t-1} + \eta_t \tag{B.3}$$

with $\eta_t \sim N(0, Q)$. The VAR reduced-form innovations in (36) are postulated to be zero-mean normally distributed, with time-varying covariance matrix Ω_t which is factored as

$$Var(\epsilon_t) \equiv \Omega_t = A_t^{-1}H_t(A_t^{-1})' \tag{B.4}$$

The time-varying matrices H_t and A_t are defined as:

$$H_t \equiv \begin{bmatrix} h_{1,t} & 0 & 0 \\ 0 & h_{2,t} & 0 \\ 0 & 0 & h_{3,t} \end{bmatrix} \qquad A_t \equiv \begin{bmatrix} 1 & 0 & 0 \\ \alpha_{21,t} & 1 & 0 \\ \alpha_{31,t} & \alpha_{32,t} & 1 \end{bmatrix} \tag{B.5}$$

with the elements $h_{i,t}$ evolving as geometric random walks:

$$\ln h_{i,t} = \ln h_{i,t-1} + \nu_{i,t} \tag{B.6}$$

Following Primiceri (2005), we postulate:

$$\alpha_t = \alpha_{t-1} + \tau_t \tag{B.7}$$

where $\alpha_t \equiv [\alpha_{21,t},\, \alpha_{31,t},\, \alpha_{32,t}]'$, and assume that the vector $[\varepsilon_t',\, \eta_t',\, \tau_t',\, \nu_t']'$ is distributed as

$$\begin{bmatrix} \varepsilon_t \\ \eta_t \\ \tau_t \\ \nu_t \end{bmatrix} \sim N\left(0, V\right), \text{ with } V = \begin{bmatrix} I_4 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & Z \end{bmatrix} \text{ and } Z = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix} \tag{B.8}$$

where ε_t is such that $\epsilon_t \equiv A_t^{-1} H_t^{\frac{1}{2}} \varepsilon_t$.

The model (B.1)-(B.8) is estimated using Bayesian methods (see Kim and Nelson (2000)). Full descriptions of the algorithm, including the Markov-Chain Monte Carlo (MCMC) used to simulate the posterior distribution of the hyperparameters and the states conditional on the data, are provided in a number of papers (see, for instance, Cogley and Sargent, 2005, and Primiceri, 2005) and will not be repeated here.

Even though one cannot characterize analytically the joint posterior distribution of the model parameters, it is possible to construct a Markov chain whose invariant distribution is the posterior. The MCMC procedure draws from the marginal density of a set of random variables j, conditional on some realizations for another set of random variables i, and then drawing from the marginal distribution of i conditional on the realizations of j in the previous step. Under some assumption, the chain converge to an invariant density that equals the desired posterior density.

The elements of S are assumed to follow an inverse-Wishart distribution centered at $2 * 10^{-3}$ times the prior mean(s) of the relevant element(s) of the vector α_t with the prior degrees of freedom equal to the minimum allowed The priors for all the other hyperparameters are borrowed from Cogley and Sargent (2005). We use 100000 Gibbs sampling replications, discard the first 80000 as burn-in.

C Convergence

In figure 9, we plot the posterior means of key model parameters. These statistics are computed recursively as the average for every 20^{th} draw of the retained repetitions of the Gibbs sampler. The figure reveals that the fluctuations in the posterior means are

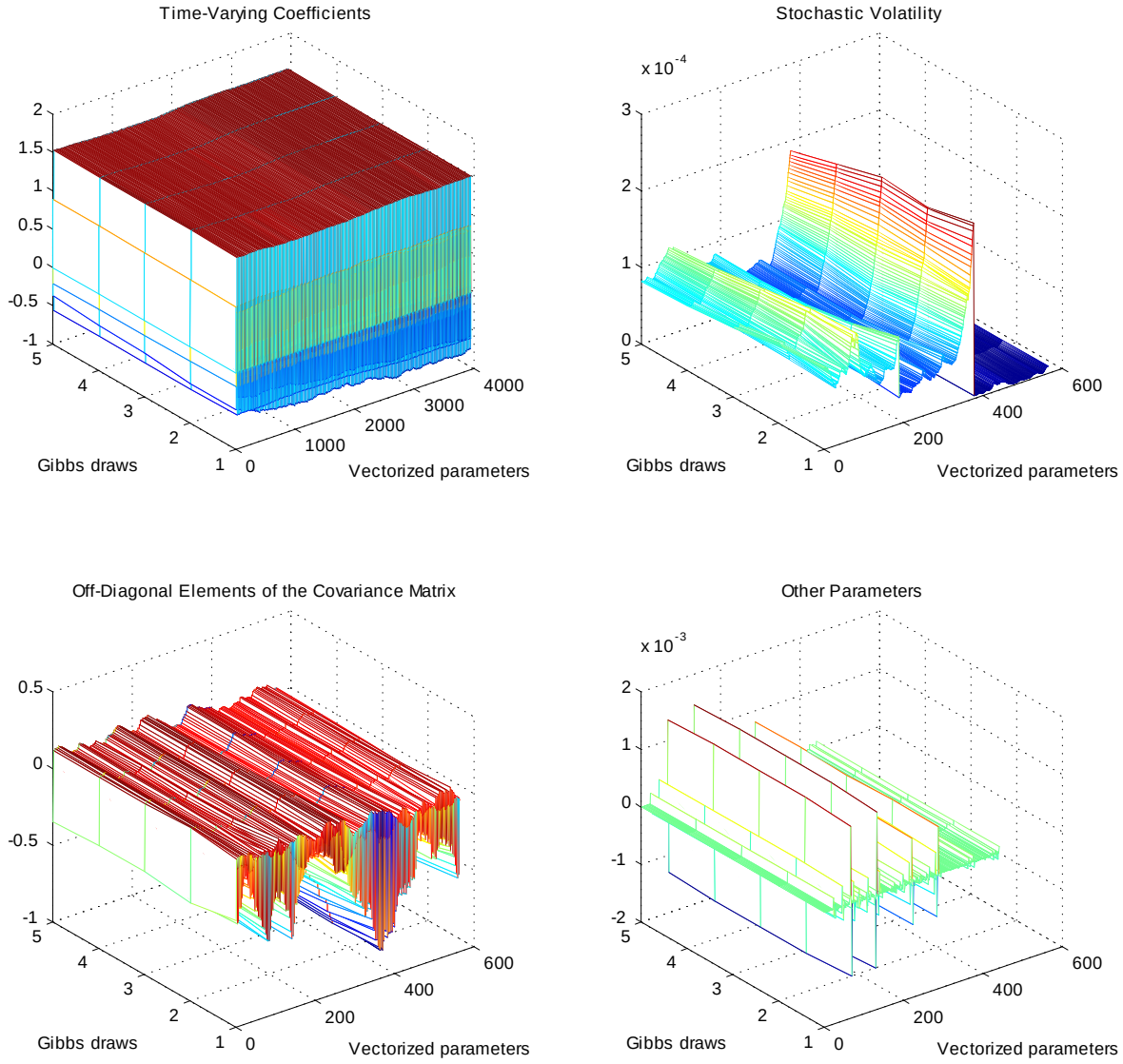

Figure 9: posterior means of key parameters of the time-varying VAR

D The asymmetric real-wage rigidity model

The value function associated with the objective function (19) can be written

$$\rho V dt = \max_{\pi_{R,t}(j)} [\pi(w_t(j), w_t, A_t) - h(\pi_{R,t}(j))]dt + E_t dV_t \tag{D.9}$$

where

$$E_t dV_t = V_{w_j} w_t(j)\pi_{R,t}(j)dt + V_w E_t dw_t + V_a g' dt + \frac{1}{2}V_{aa}\sigma^2 dt, \tag{D.10}$$

where we have used the results that $(dw_t(j))^2 = (dw_t)^2 = dw_t(j)dA_t = dw_t dA_t = 0$.[20] From (D.9) and (D.10), the optimal value of $\pi_{R,t}(j)$ is implicitly defined by the following condition

$$V_{w_j} w_t(j) = h_\pi(\pi_{R,t}(j)) = \chi \frac{e^{\chi\lambda\pi_{R,t}(j)} - 1}{\lambda}$$

from which it follows that

$$\pi_{R,t}(j) = f(V_{w_j} w_t(j)) \equiv \frac{\ln[1 + \lambda\chi^{-1} V_{w_j} w_t(j)]}{\chi\lambda}. \tag{D.11}$$

Substituting (D.11) into (D.9), we get

$$\begin{aligned} \rho V dt &= [\pi(w_t(j), w_t, A_t) - \tilde{h}(V_{w_j} \cdot w_t(j))]dt + \\ &\quad + V_{w_j} w_t(j) f(V_{w_j} \cdot w_t(j))dt + V_w E_t dw_t + V_a g' + \frac{1}{2}V_{aa}\sigma^2 dt \end{aligned} \tag{D.12}$$

where we have defined $\tilde{h}(\cdot) = h(f(\cdot))$ and $g' = g + 1/2\sigma^2$. Taking the derivative of (D.12) with respect to $w_t(j)$ we obtain

$$\begin{aligned} \rho V_{w_j} dt &= [\pi_{w_j}(w_t(j), w_t, A_t) - h_\pi(\pi_{R,t}(j)) f_1(\cdot)(V_{w_j w_j} w_t(j) + V_{w_j})dt + \\ &\quad + V_{w_j w_j} w_t(j) f(V_{w_j} \cdot w_t(j))dt + V_{w_j} f(V_{w_j} \cdot w_t(j))dt + \\ &\quad + V_{w_j} w_t(j) f_1(\cdot)(V_{w_j w_j} w_t(j) + V_{w_j})dt \\ &\quad + V_{w_j w} E_t dw_t + V_{w_j a} g' dt + \frac{1}{2}V_{w_j aa}\sigma^2 dt \end{aligned} \tag{D.13}$$

where $f_1(\cdot)$ is the derivative of $f(\cdot)$ with respect to its argument.

First note that

$$dh_\pi(\pi_{R,t}(j)) = dV_{w_j} \cdot w_t(j) + V_{w_j} \cdot dw_t(j)$$

[20] The fact that dw_t has the same properties of $dw_t(j)$ will follow from the symmetry of the equilibrium.

and therefore

$$\begin{aligned} E_t dh_\pi(\pi_{R,t}(j)) &= E_t dV_{w_j} \cdot w_t(j) + V_{w_j} \cdot E_t dw_t(j) \\ &= V_{w_j w_j} w_t(j) dw_t(j) + V_{w_j w} w_t(j) E_t dw_t + V_{w_j a} w_t(j) g' dt + \frac{1}{2} V_{w_j aa} w_t(j) \sigma^2 dt + \\ &\quad + V_{w_j} w_t(j) \cdot \pi_{R,t}(j) dt. \end{aligned}$$

Substituting into (D.13) we obtain

$$\rho h_\pi(\pi_{R,t}(j)) dt = w_t(j)[\pi_{w_j}(w_t(j), w_t, A_t)]dt + E_t dh_\pi(\pi_{R,t}(j)).$$

which in a symmetric equilibrium implies

$$\rho h_\pi(\pi_{R,t}) dt = w_t[\pi_{w_j}(w_t, w_t, A_t)]dt + E_t dh_\pi(\pi_{R,t}).$$

Note that

$$\begin{aligned} \pi_{w_j}(w_t, w_t, A_t) &= \frac{k_w}{w_t} \left[\frac{1}{\mu_p} - \mu_w \left(\frac{1}{\mu_p} \right)^{\frac{1+\eta}{1-\alpha}} \left(\frac{A_t}{w_t} \right)^{\frac{1+\eta}{1-\alpha}} \right] \\ &= \frac{\theta_w - 1}{\mu_p w_t} \left[\left(\frac{L_t}{L^f} \right)^{1+\eta} - 1 \right], \end{aligned}$$

where $k_w \equiv 1 - \theta_w$. It follows that

$$\rho h_\pi(\pi_{R,t}) dt = \frac{\theta_w - 1}{\mu_p} \left[\left(\frac{L_t}{L^f} \right)^{1+\eta} - 1 \right] dt + E_t dh_\pi(\pi_{R,t})$$

and finally

$$h_\pi(\pi_{R,t}) = E_t \int_t^\infty e^{-\rho(s-t)} \frac{\theta_w - 1}{\mu_p} \left[\left(\frac{L_s}{L^f} \right)^{1+\eta} - 1 \right] ds$$

which are equations (24) and (25) in the text.

E The downward real-wage rigidity model

Let $\mathcal{W}$ the space of non-decreasing non-negative stochastic processes $\{w_t(j)\}$. This is the space of processes that satisfy the constraint (30). First we show that the objective function is concave over a convex set. To show that the set is convex, note that if $\mathbf{x} \in \mathcal{W}$

and $\mathbf{y} \in \mathcal{W}$ then $\tau\mathbf{x} + (1-\tau)\mathbf{y} \in \mathcal{W}$ for each $\tau \in [0,1]$. Since the objective function is

$$E_{t_0}\left\{\int_{t_0}^{\infty} e^{-\rho(t-t_0)}\pi(w_t(j), w_t, A_t)dt\right\}$$

and $\pi(\cdot)$ is concave in the first-argument, the objective function is concave in $\{w_t(j)\}$ since it is the integral of concave functions.

Let $\{w_t^*(j)\}$ be a process belonging to $\mathcal{W}$ that maximizes (13) and $V(\cdot)$ the associated value function defined by

$$V(w_t(j), w_t, A_t) = \max_{\{w_\tau(j)\}\in\mathcal{W}} E_t\left\{\int_t^{\infty} e^{-\rho(\tau-t)}\pi(w_\tau(j), w_\tau, A_t)d\tau\right\}.$$

We now characterize the properties of the optimal process $\{w_t^*(j)\}$. The Bellman equation for the wage-setter problem can be written as

$$\rho V(w_t(j), w_t, A_t)dt = \max_{dw_t(j)} \pi(w_t(j), w_t, A_t)dt + E_t\{dV(w_t(j), w_t, A_t)\} \tag{E.14}$$

subject to

$$dw_t(j) \geq 0 \tag{E.15}$$

From Ito's Lemma we obtain that

$$\begin{aligned} E_t\{dV(w_t(j), w_t, A_t)\} &= E_t\{V_{w_j}(w_t(j), w_t, A_t)dw_t(j) + V_w(w_t(j), w_t, A_t)dw_t + \\ &\quad + V_a(w_t(j), w_t, A_t)dA_t + \frac{1}{2}V_{aa}(w_t(j), w_t, A_t)(dA_t)^2 + \\ &\quad + \frac{1}{2}V_{ww}(w_t(j), w_t, A_t)(dw_t)^2 + V_{aw}(w_t(j), w_t, A_t)dw_t dA_t\} \end{aligned}$$

$$\begin{aligned} E_t\{dV(w_t(j), w_t, A_t)\} &= V_{w_j}(w_t(j), w_t, A_t)dw_t(j) + V_w(w_t(j), w_t, A_t)E_t dw_t + \\ &\quad + V_a(w_t(j), w_t, A_t)A_t g' dt + \frac{1}{2}V_{aa}(w_t(j), w_t, A_t)A_t^2\sigma^2 dt \end{aligned} \tag{E.16}$$

since $dw_t(j)$, and therefore also dw_t, have finite variation implying $(dw_t(j))^2 = dw_t(j)dw_t = dw_t(j)dA_t = (dw_t)^2 = dw_t dA_t = 0$. We have defined $g' \equiv g + \frac{1}{2}\sigma^2$. Substituting (E.16) into (E.14) and maximizing over $dw_t(j)$ we obtain the complementary slackness condition:

$$V_{w_j}(w_t(j), w_t, A_t) \leq 0$$

for each t and

$$V_{w_j}(w_t(j), w_t, A_t) = 0$$

for each t when $dw_t(j) > 0$. We can write (E.14) as

$$\begin{aligned}\rho V(w_t(j), w_t, A_t)dt &= \pi(w_t(j), w_t, A_t)dt + V_w(w_t(j), w_t, A_t)E_t dw_t + \\ &\quad + V_a(w_t(j), w_t, A_t)A_t g' dt + \frac{1}{2}V_{aa}(w_t(j), w_t, A_t)A_t^2\sigma^2 dt,\end{aligned}$$

which can be differentiated with respect to $w_t(j)$ to obtain

$$\begin{aligned}\rho V_{w_j}(w_t(j), w_t, A_t)dt &= \pi_{w_j}(w_t(j), w_t, A_t)dt + V_{ww_j}(w_t(j), w_t, A_t)E_t dw_t + \\ &\quad + V_{aw_j}(w_t(j), w_t, A_t)A_t g' dt + \frac{1}{2}V_{aaw_j}(w_t(j), w_t, A_t)A_t^2\sigma^2 dt.\end{aligned} \tag{E.17}$$

Since the objective is concave and the set of constraints is convex, the optimal choice for $w_t(j)$ is unique. It follows that $w_t(j) = w_t$ for each j. Moreover, super-contact conditions require that when $dw_t > 0$

$$V_{w_j w_j}(w_t, w_t, A_t) = 0,$$

$$V_{w_j w}(w_t, w_t, A_t) = 0,$$

$$V_{w_j a}(w_t, w_t, A_t) = 0.$$

It follows that we can write (E.17) as

$$\rho v(w_t, A_t) = \tilde{\pi}_w(w_t, A_t) + v_a(w_t, A_t)A_t g' + \frac{1}{2}v_{aa}(w_t, A_t)A_t^2\sigma^2 \tag{E.18}$$

where we have defined $v(w_t, A_t) \equiv V_{w_j}(w_t, w_t, A_t)$

$$\tilde{\pi}_w(w_t, A_t) \equiv k_w \left[\frac{1}{w_t}\frac{1}{\mu_p} - \mu_w \left(\frac{1}{\mu_p}\right)^{\frac{1+\eta}{1-\alpha}} \left(\frac{A_t}{w_t}\right)^{\frac{1+\eta}{1-\alpha}} \frac{1}{w_t}\right],$$

with $k_w \equiv 1 - \theta_w$. In particular we can define the function $w(A_t)$ such that

$$v(w(A_t), A_t) = 0 \tag{E.19}$$

$$v_w(w(A_t), A_t) = 0, \tag{E.20}$$

$$v_a(w(A_t), A_t) = 0, \tag{E.21}$$

when $dw_t > 0$ while $v(w_t, A_t) \leq 0$ when $dw_t = 0$. We now solve for the functions $w(A_t)$ and $v(w_t, A_t)$. Thus we seek for functions $w(A_t)$ and $v(w_t, A_t)$ that satisfies (E.18) and

the boundary conditions (E.19)–(E.21). A particular solution to (E.18) is given by

$$v^p(w_t, A_t) = \frac{k_w}{\rho}\frac{1}{w_t}\frac{1}{\mu_p} - \frac{k_w}{v}\mu_w \left(\frac{1}{\mu_p}\right)^{\frac{1+\eta}{1-\alpha}} \left(\frac{A_t}{w_t}\right)^{\frac{1+\eta}{1-\alpha}} \frac{1}{w_t}$$

$$v \equiv \rho - g' \frac{1+\eta}{1-\alpha} - \frac{1}{2}\frac{1+\eta}{1-\alpha}\left[\frac{1+\eta}{1-\alpha} - 1\right]\sigma^2$$

while in this case the complementary solution has the form

$$v^c(W_t, A_t) = w_t^{-1-\gamma} A_t^{\gamma}$$

where γ is a root that satisfies the following characteristic equation

$$\frac{1}{2}\gamma^2\sigma^2 + \gamma g - \rho = 0 \tag{E.22}$$

i.e.

$$\gamma = \frac{-g + \sqrt{g^2 + 2\rho\sigma^2}}{\sigma^2}.$$

Since when $w_t \to \infty$ and/or $A_t \to 0$, the length of time until the next wage adjustment can be made arbitrarily long with probability arbitrarily close to one, then it should be the case that

$$\lim_{w_t \to \infty} [v(w_t, A_t) - v^P(w_t, A_t)] = 0$$

$$\lim_{A_t \to 0} [v(w_t, A_t) - v^P(w_t, A_t)] = 0$$

which both require that γ should be positive. The general solution is then given by the sum of the particular and the complementary solution, so that

$$v(w_t, A_t) = \frac{k_w}{\rho}\frac{1}{w_t}\frac{1}{\mu_p} - \frac{k_w}{v}\mu_w \left(\frac{1}{\mu_p}\right)^{\frac{1+\eta}{1-\alpha}} \left(\frac{A_t}{w_t}\right)^{\frac{1+\eta}{1-\alpha}} \frac{1}{w_t} + k w_t^{-1-\gamma} A_t^{\gamma} \tag{E.23}$$

for a constant k to be determined. Since

$$v_w(w_t, A_t) = -\frac{k_w}{\rho}\frac{1}{w_t^2}\frac{1}{\mu_p} + \frac{k_w}{v}\frac{2+\eta-\alpha}{1-\alpha}\mu_w \left(\frac{1}{\mu_p}\right)^{\frac{1+\eta}{1-\alpha}} \left(\frac{A_t}{w_t}\right)^{\frac{1+\eta}{1-\alpha}} \frac{1}{w_t^2} - (1+\gamma) k w_t^{-2-\gamma} A_t^{\gamma} \tag{E.24}$$

and

$$v_a(w_t, A_t) = -\frac{k_w(1+\eta)}{v(1-\alpha)}\mu_w \left(\frac{1}{\mu_p}\right)^{\frac{1+\eta}{1-\alpha}} \left(\frac{A_t}{w_t}\right)^{\frac{1+\eta}{1-\alpha}} \frac{1}{A_t w_t} + \gamma k w_t^{-1-\gamma} A_t^{\gamma-1}, \tag{E.25}$$

the boundary conditions (E.19)–(E.21) imply

$$\frac{k_w}{\rho}\frac{1}{\mu_p} - \frac{k_w}{v}\mu_w \left(\frac{1}{\mu_p}\right)^{\frac{1+\eta}{1-\alpha}} \left(\frac{A_t}{w_t(A_t)}\right)^{\frac{1+\eta}{1-\alpha}} + k\left(\frac{A_t}{w_t(A_t)}\right)^{\gamma} = 0, \tag{E.26}$$

$$-\frac{k_w}{\rho}\frac{1}{\mu_p} + \frac{k_w}{v}\frac{2+\eta-\alpha}{1-\alpha}\mu_w \left(\frac{1}{\mu_p}\right)^{\frac{1+\eta}{1-\alpha}} \left(\frac{A_t}{w_t(A_t)}\right)^{\frac{1+\eta}{1-\alpha}} - (1+\gamma)k\left(\frac{A_t}{w_t(A_t)}\right)^{\gamma} = 0, \tag{E.27}$$

$$-k_w\frac{(1+\eta)}{v(1-\alpha)}\mu_w \left(\frac{1}{\mu_p}\right)^{\frac{1+\eta}{1-\alpha}} \left(\frac{A_t}{w_t(A_t)}\right)^{\frac{1+\eta}{1-\alpha}} + \gamma k\left(\frac{A_t}{w_t(A_t)}\right)^{\gamma} = 0. \tag{E.28}$$

Note that this is a set of three equations, two of which are independent.[21] They determine k and the function $W_t(A_t)$. In particular, we obtain that

$$w_t(A_t) = c^{1-\alpha}\left(\frac{1}{\mu_w}\frac{1}{\mu_p}\right)^{\frac{\alpha-1}{1+\eta}}\frac{1}{\mu_p}A_t$$

where

$$c \equiv \left(\frac{\gamma - \frac{\eta+1}{1-\alpha}}{\gamma}\frac{\rho}{\rho - g'\frac{1+\eta}{1-\alpha} - \frac{1}{2}\frac{1+\eta}{1-\alpha}\left[\frac{1+\eta}{1-\alpha}-1\right]\sigma^2}\right)^{\frac{1}{1+\eta}}.$$

Using (E.22), we can write

$$c(g,\sigma^2,\eta,\rho,\alpha) = \left(\frac{g + \frac{1}{2}\gamma(g,\sigma^2,\rho)\sigma^2}{g + \frac{1}{2}\left[\gamma(g,\sigma^2,\rho) + \frac{1+\eta}{1-\alpha}\right]\sigma^2}\right)^{\frac{1}{1+\eta}}$$

which shows that $0 < c(g,\sigma^2,\eta,\rho,\alpha) \leq 1$.

[21] In fact, the homogenous function has been chosen appropriately for this purpose.