WP/06/19

IMF Working Paper

Equilibrium Wage Dispersion: An Example

Damien Gaumont, Martin Schindler, and Randall Wright

INTERNATIONAL MONETARY FUND

 WP/06/19

IMF Working Paper

Research Department

Equilibrium Wage Dispersion: An Example

Prepared by Damien Gaumont, Martin Schindler, and Randall Wright[1]

Authorized for distribution by Eswar Prasad

January 2006

Abstract

Search models with posting and match-specific heterogeneity generate wage dispersion. Given K values for the match-specific variable, it is known that there are K reservation wages that could be posted, but generically never more than two actually are posted in equilibrium. What is unknown is *when* we get two wages, and *which* wages are actually posted. For an example with $K = 3$, we show equilibrium is unique; may have one wage or two; and when there are two, the equilibrium can display any combination of posted reservation wages, depending on parameters. We also show how wages, profits, and unemployment depend on productivity.

JEL Classification Numbers: D83, E24, J31, J63

Keywords: Search equilibrium, wage posting, wage dispersion, labor theory

Author's E-Mail Address: damien.gaumont@free.fr; mschindler@imf.org; rwright@econ.upenn.edu

---

[1] Damien Gaumont is a Professor of Economics at the Université Panthéon-Assas (Paris II); Martin Schindler is an Economist in the Financial Studies Division of the IMF Research Department; and Randall Wright is a Professor of Economics at the University of Pennsylvania. The authors thank the Federal Reserve Bank of Cleveland, Paris II, and the National Science Foundation for research support, as well as seminar participants at the University of Pennsylvania, Université du Québec à Montréal, Georgetown University, and the IMF Institute for comments.

Contents   Page

I. Introduction

Wage dispersion—a deviation from the law of one price in the labor market—is a subject of long-standing theoretical and empirical interest in economics.[2] In Gaumont, Schindler, and Wright (2005), hereinafter referred to as GSW, we discuss several models of wage dispersion in search equilibrium with wage posting. Models based on ex ante homogeneous agents but ex post heterogeneous matches are shown to have advantages over earlier specifications based on ex ante heterogeneity (e.g., Albrecht and Axell, 1984, or Diamond, 1987). In particular, they do not "unravel" with the introduction of small but positive search costs. Although they do admit deviations from the law of one price, however, models with ex post heterogeneity are bound by the law of two prices (Curtis and Wright, 2004).

To explain this, suppose there are K possible realizations of the match-specific random variable. Then there are K distinct reservation wages, say w_k, $k = 1, ...K$, and no profit-maximizing firm would post anything but one of these w_k. One can show that in a given equilibrium, generically, no more than two of them actually are posted. What is unknown is *when* we get two wages, as opposed to a single wage; and when we get two, *which* reservation wages are posted—the two highest, the highest and the lowest, two consecutive wages, or another combination. Here we present an example with $K = 3$ and characterize the outcome. We show there is always a unique equilibrium, which will have one wage or two, depending on parameters. Then we show that when the equilibrium has two wages, again depending on parameters, these can be either w_1 and w_2, w_2 and w_3, or w_3 and w_1.

This example is instructive because it helps us understand how and when wage dispersion happens, and exactly what kinds of wage dispersion can arise. It is appealing that we have a unique equilibrium, and the economic structure of this equilibrium is intuitively reasonable and simple. It does take some effort, however, to solve the example. In this paper, we show how to do it.

II. The Model

There is a $[0, 1]$ continuum of firms and a $[0, L]$ continuum of workers. Time is continuous. All agents live forever, are risk neutral, and discount at rate r. Each firm has a constant returns technology with labor as the only input and productivity y. Firms with vacancies contact workers at rate γ, and unemployed workers contact firms at rate α; there is no on-the-job search. For our purposes, it makes sense to set $L = 1$, so that the arrival rates α and

[2] See Mortensen (2003) or Rogerson, Shimer, and Wright (2005) for extended discussions and references.

γ are effectively pinned down exogenously, helping to keep the analysis simple.[3] Matches end at exogenous rate δ. Firms post wages to maximize expected profit, given other firms' wages and worker behavior.

Workers are ex ante homogeneous but matches are ex post heterogeneous. Thus, when a worker contacts a firm he draws at random $c \in \{c_1, ..., c_K\}$, where c is the per period cost to taking the job, with $c_1 < c_2 < ... < c_K < y$, and the probability of $c = c_j$ is λ_j. For example, c could be the cost of commuting. Generally there is also an opportunity cost b to taking a job, incorporating leisure, home production, etc. To reduce notation, normalize $b = 0$. Also, we assume that c is permanent for the duration of the match.[4]

Let $W_j(w)$ be the value to having a job with wage w and $c = c_j$, and U the value of unemployed search. Clearly, conditional reservation wage strategies are optimal: given $c = c_j$, accept a job iff $w \geq w_j$, where $W_j(w_j) = U$. Notice $w_{j+1} > w_j$. Hence there can be at most K wages posted since, as is completely standard, no firm would post anything other than one of the reservation wages: a firm posting $w \in (w_j, w_{j+1})$ could reduce w to w_j and make more profit per worker without changing the set of workers who accept. Let θ_j denote the fraction of firms posting w_j, $\sum_j \theta_j = 1$.

A special case of this is the well-known result of Diamond (1971) that arises when $K = 1$: with homogeneous matches, all firms post $w_1 = c_1$. A problem with that model is that when there is any cost to search, no matter how small, the market will shut down since workers get no surplus from employment at $w = c_1$. The same is true when there are ex ante heterogeneous workers, say K distinct types with different (but fixed) values of c. The highest c workers get no surplus from employment, so they drop out, and so on, and so the market "unravels" and shuts down. This is why we study models with ex post heterogeneity; in these models, as long as $\theta_1 < 1$, workers get gains from search (e.g. he may get offer $w > w_1$ and draw $c = c_1$).

Bellman's equations for a worker are

$$rU = \alpha \sum_{j=1}^{K} \lambda_j \sum_{i=j}^{K} \theta_i \left[W_j(w_i) - U \right] \tag{1}$$

and

$$rW_j(w) = w - c_j + \delta \left[U - W_j(w) \right]. \tag{2}$$

In words, (1) says that he contacts firms at rate α, draws $c = c_j$ with probability λ_j, and accepts if the posted wage is $w_i \geq w_j$, which occurs with probability θ_i. Given w is

[3] This is because, with $L = 1$, the ratio of unemployed workers to vacancies is always 1. See GSW for details concerning the arrival rates, and how to solve for them in equilibrium, in generalized versions of the model.

[4] In GSW we also consider the case where employed workers draw a new c each period.

acceptable, (2) says that an employed worker gets $w - c_j$ until the match ends, which occurs at rate δ. Using $W_j(w_j) = U$, we have

$$w_j = c_j + rU. \tag{3}$$

Expected profit for a firm posting a vacancy at w_j is

$$\Pi_j = \frac{\gamma \rho_j (y - w_j)}{r + \delta} = \frac{\gamma \rho_j (y - c_j - rU)}{r + \delta}, \tag{4}$$

where γ is the arrival rate of workers, $\rho_j = \sum_{h=1}^{j} \lambda_h$ is the probability a random worker accepts, and we use (3) to substitute for w_j in terms of U. As we said, no firm posts anything other than one of the K reservation wages. Following Curtis and Wright (2004), one can strengthen this to show that generically there are no more than two wages posted.

Proposition 1 *For generic parameter values, we can have $\theta_h > 0$ for at most two values of h.*

Proof. Suppose $\theta_i > 0$, $\theta_j > 0$, $\theta_k > 0$ for distinct i, j, and k. Then $\Pi_i = \Pi_j = \Pi_k = \max\{\Pi_1, ..., \Pi_K\}$. Hence, $g_i(U) = g_j(U) = g_k(U)$, where from (4)

$$g_h(U) \equiv \rho_h(y - c_h) - r\rho_h U.$$

For generic parameter values, there does not exist a solution U to $g_i(U) = g_j(U) = g_k(U)$. ■

In GSW we studied the case $K = 2$. We showed there always exists a unique equilibrium, which may or may not entail wage dispersion. If y is small, all firms post $w_1 = c_1$; if y is big all firms post $w_2 \in (c_2, y)$; and if y is intermediate, a fraction post $w_1 \in (c_1, w_2)$ while the rest post $w_2 \in (c_2, y)$. For other values of K, although we know there can be no more than two wages posted, we do not know when there are two, as opposed to one. And when there are two, we also do not know which of the two reservation wages they will be.

III. The Example

Consider $K = 3$. As the only wages posted are in $\{w_1, w_2, w_3\}$, we write $W_{ij} = W_i(w_j)$ for the value of employment at reservation wage w_j when a worker draws c_i. Then (1) and (2) reduce to

$$\begin{aligned} rU &= \alpha\theta_2\lambda_1 (W_{12} - U) + \alpha\theta_3 [\lambda_1 (W_{13} - U) + \lambda_2 (W_{23} - U)] \\ rW_{ij} &= w_j - c_i + \delta (U - W_{ij}). \end{aligned}$$

Here we use the result that a worker who draws $w = w_j$ and $c = c_j$ gets no surplus from the match (in equilibrium he still accepts). Using $w_j = c_j + rU$,

$$rU = \eta\theta_2\lambda_1 (c_2 - c_1) + \eta\theta_3\lambda_1 (c_3 - c_1) + \eta\theta_3\lambda_2 (c_3 - c_2), \tag{5}$$

where $\eta = \alpha/(r+\delta)$. Also, (4) reduces to $\Pi_j = \frac{\gamma}{r+\delta}\sum_{i=1}^{j} \lambda_i (y - w_j)$.

By Proposition 1, at least one $\theta_j = 0$, so there are exactly 6 possible equilibria as listed in Table 1. We now give conditions determining when each equilibrium exists. We give these conditions in two ways: as restrictions on y, which are relatively easy and facilitate comparison with earlier work (e.g. the results reported in the last paragraph of Section 2); and as restrictions on $\lambda = (\lambda_1, \lambda_2)$, which provide a nice graphical representation of the equilibrium set. To begin, it will be useful to define the following:

$$\begin{aligned}
\underline{y}_1 &= \eta\lambda_1 (c_2 - c_1) + \frac{c_3 - (\lambda_1 + \lambda_2) c_2}{1 - \lambda_1 - \lambda_2} \text{ and } \overline{y}_1 = \underline{y}_1 + \eta(\lambda_1 + \lambda_2)(c_3 - c_2) \\
\underline{y}_2 &= \frac{c_3 - \lambda_1 c_1}{1 - \lambda_1} \text{ and } \overline{y}_2 = \underline{y}_2 + \eta \left[\lambda_1 (c_3 - c_1) + \lambda_2 (c_3 - c_2)\right] \\
\underline{y}_3 &= \frac{(\lambda_1 + \lambda_2) c_2 - \lambda_1 c_1}{\lambda_2} \text{ and } \overline{y}_3 = \underline{y}_3 + \eta\lambda_1 (c_2 - c_1)
\end{aligned}$$

Equilibrium 1: $\theta_1 = 1$. This case implies $rU = 0$ by (5); hence $w_j = c_j$ and equilibrium profit is

$$\Pi_1 = \frac{\gamma}{r+\delta}\lambda_1(y - c_1).$$

Given all firms post $w = w_1 = c_1$, no firm wants to deviate and post w_2 iff $\Pi_2 \leq \Pi_1$ and no firm wants to post w_3 iff $\Pi_3 \leq \Pi_1$, where

$$\begin{aligned}
\Pi_2 &= \frac{\gamma}{r+\delta}(\lambda_1 + \lambda_2)(y - c_2) \\
\Pi_3 &= \frac{\gamma}{r+\delta}(y - c_3).
\end{aligned}$$

Algebra implies $\Pi_2 \leq \Pi_1$ iff $y \leq \underline{y}_3$ and $\Pi_3 \leq \Pi_1$ iff $y \leq \underline{y}_2$. The corresponding conditions in λ-space are given by

$$\begin{aligned}
\lambda_1 \geq \tilde{\lambda}_1 &\equiv \frac{y - c_3}{y - c_1} \\
\lambda_2 \leq \ell_1 (\lambda_1) &\equiv \frac{c_2 - c_1}{y - c_2}\lambda_1.
\end{aligned} \tag{6}$$

This gives necessary and sufficient conditions for equilibrium 1.

Table 1. Possible Equilibria

Equilibrium 1	Equilibrium 2	Equilibrium 3	Equilibrium 4	Equilibrium 5	Equilibrium 6
$\theta_1 = 1$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_1\theta_2 > 0$	$\theta_1\theta_3 > 0$	$\theta_2\theta_3 > 0$

Equilibrium 2: $\theta_2 = 1$. Given $\theta_2 = 1$, $rU = \eta\lambda_1(c_2 - c_1)$ by (5), and hence $w_j = c_j + rU = c_j + \eta\lambda_1(c_2 - c_1)$. Using this,

$$\begin{aligned}\Pi_1 &= \frac{\gamma}{r+\delta}\lambda_1(y - c_1 - rU)\\ \Pi_2 &= \frac{\gamma}{r+\delta}(\lambda_1+\lambda_2)(y - c_2 - rU)\\ \Pi_3 &= \frac{\gamma}{r+\delta}(y - c_3 - rU).\end{aligned}$$

No firm wants to deviate and post w_1 iff $\Pi_1 \le \Pi_2$, which holds iff $y \ge \overline{y}_3$, and no firm will deviate and post w_3 iff $\Pi_3 \le \Pi_2$, which holds iff $y \le \underline{y}_1$. In λ-space, these conditions on y can be expressed as

$$\begin{aligned}\lambda_1 &< \hat{\lambda}_1 \equiv \frac{y - c_2}{\eta\left(c_2 - c_1\right)}\\ \lambda_2 &\ge \ell_2\left(\lambda_1\right) \equiv \frac{\left(1-\lambda_1\right)\left[y - \eta\lambda_1(c_2 - c_1)\right] - c_3 + \lambda_1 c_2}{y - \eta\lambda_1\left(c_2 - c_1\right) - c_2}\\ \lambda_2 &> \ell_3\left(\lambda_1\right) \equiv \frac{\lambda_1\left(c_2 - c_1\right)}{y - c_2 - \eta\lambda_1\left(c_2 - c_1\right)}.\end{aligned} \tag{7}$$

The properties of the ℓ functions are given below, but we need some properties of ℓ_3 now to conclude the following: although $\overline{y}_3 \le y \le \underline{y}_1$ is also satisfied if the above three inequalities are all reversed, this case can be ignored because $\lambda_1 > \hat{\lambda}_1$ implies $\ell_3\left(\lambda_1\right) < 0$. Thus, (7) gives necessary and sufficient conditions for equilibrium 2.

Lemma 1 $\ell_3(\lambda_1)$ *goes through the origin, is strictly increasing, strictly convex and positive if* $\lambda_1 < \hat{\lambda}_1$*, and strictly concave and negative if* $\lambda_1 > \hat{\lambda}_1$*, with a discontinuity at* $\lambda_1 = \hat{\lambda}_1$*.*

Proof. $\ell_3\left(0\right) = 0$ is obvious. The first derivative is $\ell_3' = \frac{(y-c_2)(c_2-c_1)}{[y-c_2-\eta\lambda_1(c_2-c_1)]^2} > 0$. The second derivative is $\ell_3'' = \frac{2\eta(c_2-c_1)}{[y-c_2-\eta\lambda_1(c_2-c_1)]^3}$ which is positive if $\lambda_1 < \hat{\lambda}_1$ and negative otherwise. ■

Equilibrium 3: $\theta_3 = 1$. Given $\theta_3 = 1$ we can solve for rU, w_j, and Π_j, and check that no firm will deviate iff $y \ge \overline{y}_1$ and $y \ge \overline{y}_2$. The first condition $y \ge \overline{y}_1$ can be written as a quadratic in λ_2 for a given λ_1, say $Q(\lambda_2) = A\lambda_2^2 + B\lambda_2 + C \geq 0$, where

$$\begin{aligned} A &= \eta\,(c_3 - c_2) \\ B &= -y + c_2 + \eta\lambda_1\,(c_3 - c_1) - \eta(1-\lambda_1)(c_3 - c_2) \\ C &= (1-\lambda_1)\,y - c_3 + \lambda_1 c_2 - \eta\,(1-\lambda_1)\,\lambda_1\,(c_3 - c_1)\,. \end{aligned}$$

It is easy to see that $Q(\lambda_2)$ is convex and $Q(\lambda_2) < 0$ at $\lambda_2 = 1 - \lambda_1$. Hence, $Q(\lambda_2)$ has two real roots that depend on λ_1, say $\ell^-(\lambda_1)$ and $\ell^+(\lambda_1)$, one on each side of $1 - \lambda_1$. Since only $\lambda_2 \leq 1 - \lambda_1$ is relevant, we conclude that $Q(\lambda_2) \geq 0$ iff

$$\lambda_2 \leq \ell^-(\lambda_1) = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

The second condition $y \geq \overline{y}_2$ is equivalent to

$$\lambda_2 \leq \ell_4(\lambda_1) = \frac{(1-\lambda_1)\,y - c_3 + \lambda_1 c_1 - \eta\,(1-\lambda_1)\,\lambda_1\,(c_3 - c_1)}{\eta\,(1-\lambda_1)\,(c_3 - c_2)}.$$

Hence, equilibrium 3 exists iff

$$\lambda_2 \leq \min\left\{\ell^-(\lambda_1)\,, \ell_4(\lambda_1)\right\}. \tag{8}$$

The description above exhausts the single-wage equilibria. By inspection of the y-cutoffs, these cases are mutually exclusive, so there cannot be multiple single-wage equilibria. We now consider two-wage equilibria.

Equilibrium 4: $\theta_1, \theta_2 > \theta_3 = 0$. This equilibrium requires $\Pi_2 = \Pi_1$, an equality that can be solved for

$$\theta_2 = \frac{\lambda_2 y - (\lambda_1 + \lambda_2)\,c_2 + \lambda_1 c_1}{\eta\lambda_1\lambda_2\,(c_2 - c_1)}.$$

Notice $\theta_2 \in (0, 1)$ iff $y \in \left(\underline{y}_3, \overline{y}_3\right)$, which is equivalent to $\lambda_2 > \ell_1(\lambda_1)$ and

$$\lambda_2 < \ell_3(\lambda_1) \text{ if } \lambda_1 < \hat{\lambda}_1;\ \lambda_2 > \ell_3(\lambda_1) \text{ if } \lambda_1 > \hat{\lambda}_1.$$

No firm wants to deviate and post w_3, rather than either w_1 or w_2, iff[5]

$$\lambda_2 \geq \ell_5(\lambda_1) = \frac{\lambda_1(1-\lambda_1)(c_2 - c_1)}{c_3 - \lambda_1 c_1 - (1-\lambda_1)c_2}.$$

Hence, equilibrium 4 exists iff

$$\lambda_2 < \ell_3(\lambda_1) \text{ if } \lambda_1 < \hat{\lambda}_1; \text{ and } \lambda_2 > \max\{\ell_5(\lambda_1)\,, \ell_1(\lambda_1)\}\,. \tag{9}$$

[5] There is no corresponding condition in terms of y: for equilibria 4-6, the no deviation conditions depend only on λ_2 versus $\ell_5(\lambda_1)$.

Equilibrium 5: $\theta_1, \theta_3 > \theta_2 = 0$. This requires $\Pi_3 = \Pi_1$, which can be solved for

$$\theta_3 = \frac{(1-\lambda_1)\,y - c_3 + \lambda_1 c_1}{\lambda_1\,(c_3 - c_1) + \lambda_2\,(c_3 - c_2)} \frac{1}{(1-\lambda_1)\,\eta}.$$

Hence $\theta_3 \in (0,1)$ iff $y \in \left(\underline{y}_2, \overline{y}_2\right)$, which is equivalent to $\lambda_1 < \tilde{\lambda}_1$ and $\lambda_2 > \ell_4(\lambda_1)$. No firm wants to deviate and post w_2 iff $\lambda_2 \le \ell_5(\lambda_1)$. Hence equilibrium 5 exists iff

$$\lambda_1 < \tilde{\lambda}_1,\ \lambda_2 > \ell_4(\lambda_1),\ \text{and}\ \lambda_2 \le \ell_5(\lambda_1). \tag{10}$$

Equilibrium 6: $\theta_2, \theta_3 > \theta_1 = 0$. This requires $\Pi_3 = \Pi_2$, which can be solved for

$$\theta_2 = -\frac{(1-\lambda_1-\lambda_2)\Psi - c_3 + (\lambda_1+\lambda_2)\,c_2}{\eta(1-\lambda_1-\lambda_2)\,(\lambda_1+\lambda_2)\,(c_3-c_2)},$$

where $\Psi = y - \eta\lambda_1\,(c_3 - c_1) - \eta\lambda_2\,(c_3 - c_2)$. Observe that the denominator in this expression is the quadratic $Q\,(\lambda_2)$ defined in the discussion of equilibrium 3. Hence, we can write $\theta_2 \in (0,1)$ iff $y \in \left(\underline{y}_1, \overline{y}_1\right)$, which is equivalent to $\lambda_1 < \hat{\lambda}_1$, $\lambda_2 > \ell^-(\lambda_1)$, and $\lambda_2 < \ell_2(\lambda_1)$. Actually, the condition $\theta < 1$ is also satisfied if $\lambda_1 > \hat{\lambda}_1$ and $\lambda_2 > \ell_2\,(\lambda_1)$, but the following lemma shows that this can never be satisfied in the relevant region.

Lemma 2 *$\ell_2(0) = \frac{y-c_3}{y-c_2} \in (0,1)$; $\ell_2(\lambda_1) \to -\infty$ as $\lambda_1 \to \hat{\lambda}_1$ from below; $\ell_2(\lambda_1) \to \infty$ as $\lambda_1 \to \hat{\lambda}_1$ from above; $\ell_2(\lambda_1) > 1-\lambda_1$ iff $\lambda_1 > \hat{\lambda}_1$; $\ell_2(\lambda_1) \to 1-\lambda_1$ from above as $\lambda_1 \to \infty$; $\ell_2(\lambda_1) \to 1-\lambda_1$ from below as $\lambda_1 \to -\infty$; $\ell_2(\lambda_1) > \ell^-(\lambda_1)$.*

Proof. The first parts involve straightforward analysis. Proving $\ell_2(\lambda_1) \to 1-\lambda_1$ is equivalent to proving $\ell_2(\lambda_1)/\,(1-\lambda_1) \to 1$, which follows from l'Hôpital's rule. For the last part, observe that $\partial\theta_2/\partial\lambda_1 > 0$; hence, as we increase λ_1 for any given λ_2, we hit the threshold at which $\theta_2 = 0$ before we hit the threshold at which $\theta_2 = 1$. This means the $\ell_2(\lambda_1)$ curve lies above the $\ell^-(\lambda_1)$ curve in (λ_1, λ_2) space. ■

Finally, no firm wants to deviate and post w_1 iff $\lambda_2 \ge \ell_5(\lambda_1)$. Hence equilibrium 6 exists iff

$$\lambda_1 < \hat{\lambda}_1,\ \lambda_2 > \ell^-(\lambda_1),\ \lambda_2 < \ell_2(\lambda_1)\ \text{and}\ \lambda_2 \ge \ell_5(\lambda_1). \tag{11}$$

The completes our analysis of every case in Table 1. For each candidate equilibrium 1-6 we provide necessary and sufficient conditions for existence in terms of y and also λ. We can summarize the results as follows:

Proposition 2 *Generically equilibrium exists and is unique. If $\lambda_2 < \ell_5\,(\lambda_1)$ then: equilibrium 1 exists iff $y \le \underline{y}_2$; equilibrium 5 exists iff $y \in (\underline{y}_2, \overline{y}_2)$; and equilibrium 3 exists iff $y \ge \overline{y}_2$.*

If $\lambda_2 > \ell_5(\lambda_1)$ *then: equilibrium 1 exists iff* $y \leq \underline{y}_3$*; equilibrium 4 exists iff* $y \in (\underline{y}_3, \overline{y}_3)$*; equilibrium 2 exists iff* $y \in (\overline{y}_3, \underline{y}_1)$*; equilibrium 6 exists iff* $y \in (\underline{y}_1, \overline{y}_1)$*; and equilibrium 3 exists iff* $y \geq \overline{y}_1$*.*

Proof. There are two generic cases, $\lambda_2 < \ell_5(\lambda_1)$ and $\lambda_2 > \ell_5(\lambda_1)$. In the former case it is clear that for all y there is a unique equilibrium. The same is true in the latter case once one recognizes that $\overline{y}_3 < \underline{y}_1$ in the case where $\lambda_2 > \ell_5(\lambda_1)$. ■

In order to present the results graphically, we move to λ-space, and make use of conditions (6)-(11). To do so, we first describe some more properties of the ℓ_j functions used in these conditions. Proofs are omitted where obvious.

Lemma 3 $\ell^-(0) \in (0,1)$*. There is a unique* $\lambda_1^0 \in (0,1)$ *such that* $\ell^-(\lambda_1^0) = 0$*.*

Proof. For the first part, note that $\lambda_1 = 0$ implies $Q(0) > 0$ and $Q(1) < 0$, and since $Q(\lambda_2)$ has two real roots $\ell^-(0)$ and $\ell^+(0)$, one on each side of $1 - \lambda_1$, we conclude $\ell^-(0) \in (0,1)$. The second part is shown by noting that $\lambda_1 = 1$ implies $Q(0) < 0$, which in turn implies $\ell^-(1) < 0$. Convexity of Q then implies the existence of a unique $\lambda_1^0 \in (0,1)$ such that $\ell^-(\lambda_1^0) = 0$. ■

Lemma 4 $\ell_5(\lambda_1)$ *is concave, lies below* $\lambda_2 = 1 - \lambda_1$*, and goes through* $(0,0)$ *and* $(1,0)$*.*

Lemma 5 $\ell_4(\lambda_1)$ *is monotonically decreasing with* $\ell_4(0) = (y - c_3)/\eta(c_3 - c_2) > 0$*.*

Lemma 6 $\ell_1(\lambda_1)$ *is a linearly increasing function with* $\ell_1(0) = 0$ *and slope* $\ell_1'(\lambda_1) = \frac{c_2 - c_1}{y - c_2} > 0$*.*

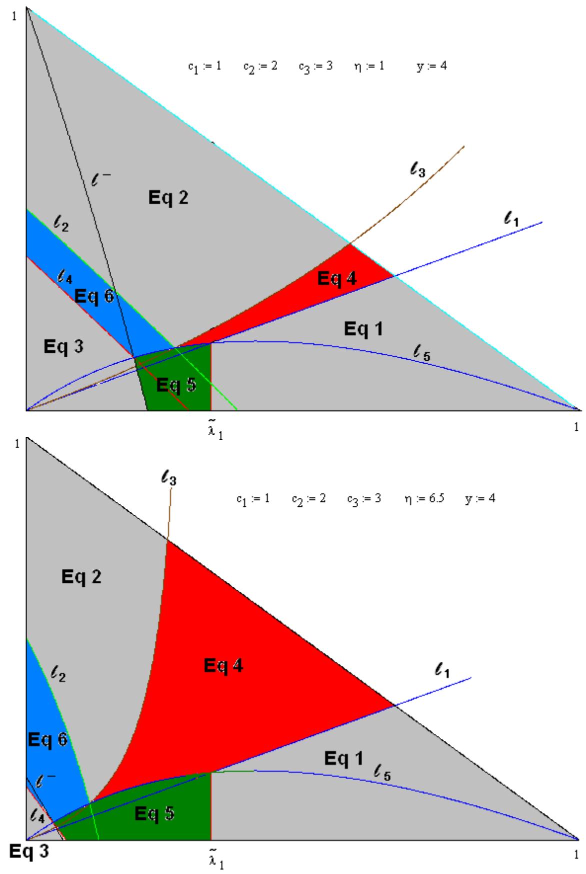
Lemma 7 $\ell^-(\lambda_1)$*,* $\ell_5(\lambda_1)$ *and* $\ell_4(\lambda_1)$ *intersect at the same point* λ_1^a*.*

Proof. This follows from considering the corresponding functions in y-space, $\overline{y}_1$ and $\overline{y}_2$, where it is easy to show that $\overline{y}_1 = \overline{y}_2$ iff $\lambda_2 = \ell_5$. ■

Lemma 8 $\ell_5(\lambda_1)$*,* $\ell_2(\lambda_1)$ *and* $\ell_3(\lambda_1)$ *intersect at the same point* $\lambda_1^b < \hat{\lambda}_1$*.*

Proof. That the three functions intersect at the same point λ_1^b follows from simple algebra. Then $\lambda_1^b < \hat{\lambda}_1$ follows from the properties of $\ell_2(\lambda_1)$ which imply that if ℓ_2 intersects with another function within the simplex, it must be at some $\lambda_1 < \hat{\lambda}_1$. ■

Figure 1. Equilibrium Regions



Lemma 9 *$\ell_5(\lambda_1)$ and $\ell_1(\lambda_1)$ intersect at $\lambda_1^c = \tilde{\lambda}_1$.*

Lemma 10 $\lambda_1^a < \lambda_1^b < \lambda_1^c$

Proof. Note first that $\ell_3(\lambda_1) > \ell_1(\lambda_1)$ for all $\lambda_1 > 0$, and $\ell_3(\lambda_1), \ell_1(\lambda_1) < \ell_5(\lambda_1)$ for small λ_1, which follows from noting that $\ell_3(0) = \ell_1(0) = 0$ and $\ell_3'(0) = \ell_1'(\lambda_1) = \frac{c_2-c_1}{y-c_2} < \ell_5'(0) = \frac{c_2-c_1}{c_3-c_2}$. This implies $\lambda_1^b < \lambda_1^c$. To show that $\lambda_1^a < \lambda_1^b$, we claim that $\ell^-(\lambda_1) < \ell_2(\lambda_1)$ for all λ_1 such that $\ell^-(\lambda_1) > \ell_5(\lambda_1)$. Suppose not. Then if $\ell^-(\lambda_1) > \ell_2(\lambda_1)$ for some λ_1 such that $\ell^-(\lambda_1) > \ell_5(\lambda_1)$, equilibria 2 and 3 coexist, and we would contradict the uniqueness part of Proposition 2; and if $\ell^-(\lambda_1) > \ell_2(\lambda_1)$ for all λ_1 such that $\ell^-(\lambda_1) > \ell_5(\lambda_1)$, we would contradict the existence part. Consequently, $\ell^-(\lambda_1)$ intersects $\ell_5(\lambda_1)$ at a smaller λ_1 than does $\ell_2(\lambda_1)$. ■

Lemma 11 *$\ell^-(\lambda_1) < \ell_4(\lambda_1)$ for $\lambda_2 > \ell_5(\lambda_1)$ and $\ell^-(\lambda_1) > \ell_4(\lambda_1)$ for $\lambda_2 < \ell_5(\lambda_1)$.*

Table 2. Wages

Equilibrium	Wages w_j for $j = 1, 2, 3$
1. $\theta_1 = 1$	$w_j = c_j$
2. $\theta_2 = 1$	$w_j = c_j + \eta\lambda_1(c_2 - c_1)$
3. $\theta_3 = 1$	$w_j = c_j + \eta\lambda_1(c_3 - c_1) + \eta\lambda_2(c_3 - c_2)$
4. $\theta_1\theta_2 > 0$	$w_j = c_j + y + \frac{\lambda_1 c_1 - (\lambda_1+\lambda_2)c_2}{\lambda_2}$
5. $\theta_1\theta_3 > 0$	$w_j = c_j + y + \frac{\lambda_1 c_1 - c_3}{1-\lambda_1}$
6. $\theta_2\theta_3 > 0$	$w_j = c_j + y + \frac{(\lambda_1+\lambda_2)c_2 - c_3}{1-\lambda_1-\lambda_2}$

Proof. It must be true that $\ell^-(\lambda_1) < \ell_4(\lambda_1)$ iff $\ell_4(\lambda_1) > \ell_5(\lambda_1)$, otherwise we violate the existence or uniqueness part of Proposition 2. ■

Given these properties we can draw the ℓ-functions, as shown in Figure 1 for two sets of parameter values. It is now a simple matter to fill in the different regions generated by the ℓ-functions with the equilibrium that exists in each case. In terms of economics, the results are quite intuitive. Consider, for example, the case of λ_1 close to 1. Then we get a single-wage equilibrium, all firms post the lowest reservation wage w_1, which maximizes profit per worker, and does not reduce the hiring probability too much as λ_1 is big. As λ_2 becomes big we get equilibrium where all firms post w_2, and as λ_3 becomes big we get wage equilibrium where all firms post w_3, because firms are willing to sacrifice profit per worker to keep the hiring probability from falling too much.

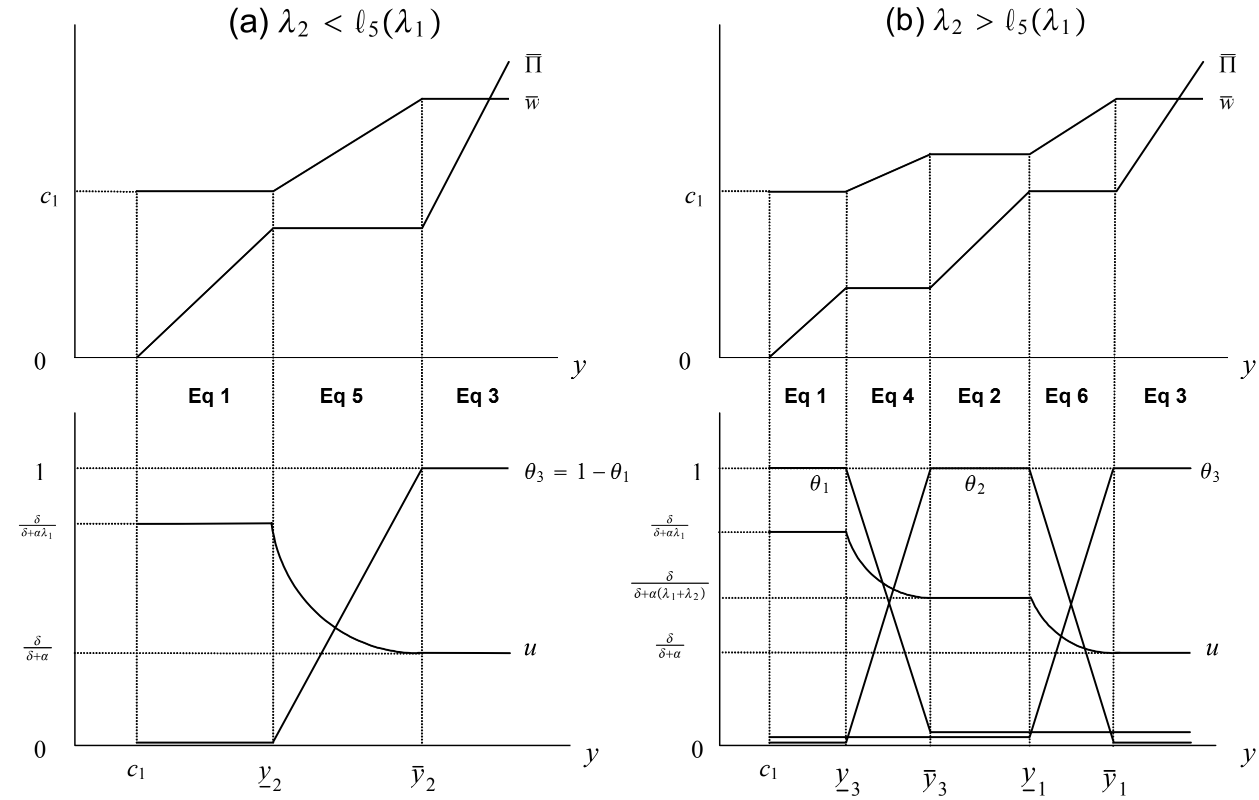
When we are in a region between those with a single-wage equilibrium, we get wage dispersion; for example, between the regions where all firms post w_1 and where all firms post w_2, some firms post w_1 and others w_2. The figure illustrates two key points. First, two-wage equilibria are not especially rare. Second, when two-wage equilibria exist, they may entail any combination of w_1, w_2 and w_3. Of course, these wages are themselves endogenous – they depend on the equilibrium as well as parameters. Table 2 lists wages in each equilibrium, including those that are not posted; note that in each case, consistent with (3), we have $w_j = c_j + rU$.

In Figure 2, we plot the average wage $\overline{w}$, profit $\overline{\Pi}$, unemployment u, and the fraction of firms posting each wage, as functions of productivity y, leaving explicit calculations as an exercise.[6] There are two panels, corresponding to the two cases in Proposition 2: $\lambda_2 < \ell_5(\lambda_1)$

[6] The only variable we have not defined is unemployment which, as is standard, evolves according to $\dot{u} = (1-u)\,\delta - u\alpha \sum_{i=1}^{3} \theta_i \sum_{j=1}^{i} \lambda_j$, so that in steady-state

$$u = \frac{\delta}{\delta + \alpha \sum_{i=1}^{3} \theta_i \sum_{j=1}^{i} \lambda_j}.$$

Figure 2. Selected Equilibrium Variables as a Function of y



and $\lambda_2 > \ell_5(\lambda_1)$. This figure shows the intuitive result that higher productivity must benefit either firms or workers in terms of wages or profit, but interestingly, never at the same time: $\overline{\Pi}$ is constant in single-wage equilibria and $\overline{w}$ is constant in two-wage equilibria. Also, u is constant in single-wage equilibria and decreasing in y in two-wage equilibria.[7]

IV. Conclusion

We analyzed in detail the case of $K = 3$ in a model with wage posting and ex post heterogeneity. We found that a unique equilibrium exists which may or may not exhibit wage dispersion. Also, any pair of reservation wages may be posted. We think the results teach us something interesting about endogenous wage dispersion and search theory more generally.

[7] The shapes in the figure are general with the exception of the relative position of $\overline{\Pi}$ and $\overline{w}$. In general, we can have $\overline{w} < \overline{\Pi}$ or vice versa, although for small y we must have $\overline{w} > \overline{\Pi}$ and for very large y we must have $\overline{w} < \overline{\Pi}$. Put differently, for very small (large) y workers extract a higher (lower) share of the surplus than firms.

References

Albrecht, James, and Bo Axell, 1984, "An Equilibrium Model of Search Unemployment," *Journal of Political Economy*, Vol. 92, pp. 824–40.

Curtis, Elisabeth, and Randall Wright, 2004, "Price Setting, Price Dispersion, and the Value of Money: or, the Law of Two Prices," *Journal of Monetary Economics*, Vol. 51, pp. 1599–1621.

Diamond, Peter A., 1971, "A Model of Price Adjustment," *Journal of Economic Theory*, Vol. 3, pp. 156–68.

________, 1987, "Consumer Differences and Prices in a Search Model," *Quarterly Journal of Economics*, Vol. 102, pp. 429–36.

Gaumont, Damien, Martin Schindler, and Randall Wright, 2005, "Alternative Theories of Wage Dispersion," IMF Working Paper 05/64 (Washington: International Monetary Fund); also forthcoming in *Contributions to Economic Analysis: Structural Models of Wage and Employment Dynamics*, ed. by Henning Bunzel and others (Amsterdam: Elsevier North-Holland).

Mortensen, Dale T., 2003, *Wage Dispersion* (Cambridge, United Kingdom: Zeuthen Lecture Book Series).

Rogerson, Richard, Robert Shimer, and Randall Wright, 2005, "Search Theoretic Models of the Labor Market: A Survey," forthcoming in the *Journal of Economic Literature*.