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# Deriving Market Expectations for the Euro-Dollar Exchange Rate from Option Prices

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**Deriving Market Expectations for the Euro-Dollar Exchange Rate from Option Prices**

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**Abstract**

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Option prices provide valuable information on market expectations. This paper attempts to extract market expectations, as conveyed by an implied risk-neutral probability distribution, from option prices for the dollar-euro exchange rate. Returns' volatilities are inferred from observed and interpolated option prices. To address robustness, two distributions, one from actual data and the other from interpolated data, were computed. The main conclusion of the paper is that traders have wide-ranging expectations, and large movements in either direction would not occur as a surprise. The main implication for monetary policy is that should markets become too volatile, then intervention may be required.

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# I. Introduction

Traders' expectations influence the volatility of asset prices and the hedging strategies in both assets and derivatives' markets. Obtaining information about these expectations is therefore relevant for both market participants and policy-makers. Indicators of uncertainty and market sentiment are closely watched by policy-makers and investors. Value-at-risk analysis relies on these indicators. Higher uncertainty will cause risk managers to hedge their portfolios against sharp depreciation in the value of their assets. In the same vein, many central banks do infer the risk-neutral probability distribution (RND) of asset prices[2] from assets and derivatives prices and use this information in deciding the timing and scope of monetary policy and interventions in foreign exchange markets.

Asset prices are known to follow a stochastic process whose parameters and transition densities are key ingredients in measuring risk and designing hedging strategies. In deciding on hedging strategies, traders assess relevant prevailing market indicators for measuring risk. That is, observing today's asset price, traders predict where the price might be at a given future date based on the dynamics of an inherent stochastic process. The different values the price may take in the future are called state values. Among the indicators used are the forward prices which contain information about the expected mean of future states. A more complete characterization of uncertainty would require knowledge of the volatility of the asset price which is the second moment of the probability distribution. Considering that financial time series data are generally not log-normally distributed, and often exhibit skewness and kurtosis,[3] investors need to estimate the full distribution of the states, or equivalently undertake a density forecast. A distribution with fatter tails than the lognormal implies higher probabilities for sharp moves, or market crash, in asset prices. Similarly, a

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[2] The risk-neutral probability distribution for asset prices is equivalently known as the state price density, Arrow-Debreu prices of primitive securities, or stochastic discount factors.

[3] Skewness is a measure of asymmetry of the distribution of the series around its mean. Skewness is computed as: $S = \frac{1}{N}\sum_{i=1}^{N}(\frac{y_i - \bar{y}}{\hat{\sigma}})^3$ ,where $\hat{\sigma}$ is an estimator for the standard deviation that is based on the biased estimator for the variance . The skewness of a symmetric distribution, such as the normal distribution, is zero. Positive skewness means that the distribution has a long right tail and negative skewness implies that the distribution has a long left tail. Kurtosis measures the peakedness or flatness of the distribution of the series, especially the concentration of values near the mean. Kurtosis is computed as $K = \frac{1}{N}\sum_{i=1}^{N}(\frac{y_i - \bar{y}}{\hat{\sigma}})^4$ . The kurtosis of the normal distribution is 3. If the kurtosis exceeds 3, the distribution is peaked (leptokurtic) relative to the normal; if the kurtosis is less than 3, the distribution is flat (platykurtic) relative to the normal.

distribution which is skewed to the right, i.e., with a longer right tail than the lognormal, implies higher probabilities for right tail events, and vice-versa for left skewed distribution with a longer left tail.

Describing the market implied probability distribution of asset prices has now become a focal area in finance. Such implied probability distribution would reflect investors' expectations or market sentiment about the future movements of asset prices. The method typically used to derive market expectations relies on option prices since options are hedging instruments. The method is referred to as the inverse problem, or the calibration model, in option pricing theory; it involves estimating the RND, or more precisely the parameters of the stochastic process of the underlying asset price that are consistent with observed option prices.

The objective of this paper is to estimate the market implied RND for the dollar-euro rate from option prices. Based on some key elements of option pricing theory explained in the Appendix, Section II discusses the inverse problem in option pricing theory. Section III applies the inverse problem methods to exchange-traded option data in order to derive the RND for the dollar-euro rate. Section IV concludes. A main finding of the paper is that traders have wide-ranging expectations and large movements in either direction would not occur as a surprise. The main implication for monetary policy is that should markets become volatile and pressure builds in one direction, then intervention may be required. Shifts in the RND, however, determine the effectiveness of the intervention and the ability of such intervention to influence investors' sentiment

## II. The Inverse Problem in Option Pricing Theory

Option pricing theory[4] shows how to go from a stochastic process of the underlying asset to the price of a contingent claim. The objective of this section is to address the inverse problem and go from observed market prices for options to infer the stochastic process of the underlying asset in a risk-neutral world. In such a world, what matters is the knowledge of the volatility, since the drift is always given by the risk-free interest rate. The inverse problem therefore amounts to estimating from option prices a volatility function, $\sigma(K,T)$, relating for a given maturity $T$ volatility to each exercise price. The knowledge of $\sigma(K,T)$ would enable to compute the transition probabilities of the process, as a function of $\sigma(K,T)$, i.e., $p(K,T) = p(\sigma(K,T))$, and therefore the RND of the state variable. The inverse problem would have been easily solved if the asset price is log normally distributed and volatility is constant. Volatility being the only unknown in the Black-Scholes (BS) formula, implied volatility would follow from equating the BS formula to the observed market price of the option. However, it is well known that there is a smile effect, whereby the implied volatility varies with the strike  and out-of-the-money (OTM) options have higher volatility than near-the-money options. The RND, thus, may not be lognormal and volatility may be state and time dependent.

[4] See Appendix.

Noting the simplicity of the payoff structure for a European call option, and given the formula for computing the value of this option, namely:

$$C(S,t;K,T)=e^{-r(T-t)}\int_{0}^{\infty}Max(0,S_T-K)p(S_T)dS_T \quad (1)$$

Breeden and Litzenberger (1978) showed that market prices of European call options and the RND are related as follows:

$$\frac{\partial^2 C}{\partial K^2}=e^{-r(T-t)}p(K,T) \quad (2)$$

Thus, if for a given maturity $T$ a continuous twice differentiable call pricing function $C(K,T)$ can be found, then the computation of the RND is simple, and is obtained by twice differentiating the call pricing function.

Several approaches for the calibration of option pricing models via the estimation of the volatility function $\sigma(K,T)$ from observed market data were proposed. Shimko (1993) and Maltz (1997) used the BS implied volatility to construct a BS call pricing function $C(K,T)$. Rubinstein (1994) and Derman and Kani (1994), however, used a tree method to infer volatility. Dupire's forward approach (1994) for estimating $\sigma(K,T)$ used the dual Fokker-Planck parabolic PDE for the value of options, coupled with initial and boundary conditions. Dupire showed that the Breeden-Litzenberger (1978) formula can lead to a direct relation between $\sigma(K,T)$ and the partial derivatives of the call pricing function.

Implementation of the Breeden-Litzenberger formula requires a continuum of European options with the same time-to-maturity on a single underlying asset spanning strike prices from zero to infinity. Unfortunately, since option contracts are only traded at discretely spaced strike price levels, and for a limited range either side of the at-the-money (ATM) strike, there are many RND functions that can fit their market prices. Hence, all the procedures for estimating RND essentially amount to interpolating between observed strike prices and extrapolating outside their range to model the tail probability.

### A. Black-Scholes Implied Volatility: Market Quotations and the Smile Effect

Market pricing convention uses volatility for quoting option prices and delta ($\Delta$) for referring to the strike price, namely options are quoted at 25-delta, 10-delta, and ATM option corresponds approximately to 50-delta. Given the known market values $\Delta$ and $\sigma(\Delta,T)$, the passage to strike $K$ and currency units $C(K,T)$ is made via the BS formula and the delta formula, namely:

$$C(S_t,t,K,T,r,\gamma,\sigma)=e^{-\gamma\tau}S_tN(d_1)-e^{-r\tau}KN(d_2) \quad (3)$$

$$\Delta=\frac{\partial C}{\partial S_t}=e^{-\gamma\tau}N(d_1)=e^{-\gamma\tau}N\left(\frac{Ln(\frac{S_t}{K})+(r-\gamma+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \quad (4)$$

where: $d_1 = \dfrac{Ln(\frac{S_t}{K}) + (r - \gamma + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$, $d_2 = \dfrac{Ln(\frac{S_t}{K}) + (r - \gamma - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$

the variables are: $S_t$ = the current underlying asset price, $K$ = the option strike price, $r$ = the domestic interest rate, $\gamma$ = the foreign interest rate, and $\tau = T - t$, time to expiration, and $N(.)$ = the standard cumulative normal distribution function. The probability distribution implied by the BS formula is the lognormal distribution. For known $\Delta$ and $\sigma$, equations (3) and (4) solve for $K$ and $C$. Inversely, given an observed call price $C(K,T)$ and the observed values of $S_t, K, \tau, r, \gamma$, equation $C(K,T) = C(S_t, t, K, T, r, \gamma, \sigma)$ can be solved for the unique implied volatility $\sigma(K,T)$ corresponding to $C(K,T)$.

Dealers often sell options in combinations. The straddle is a combination of a put and a call with identical strike prices, usually at the money forward (ATM). Two combinations, the strangle (STR) and the risk reversal (RR), involve two OTM options with the same delta and contain most of the information about the volatility smile and skewness. The strangle is a position consisting of a long OTM put and call; this strategy is implemented when traders anticipate increased volatility in the underlying security's price. The RR consists of a long OTM call and a short OTM put. The prices of these option combinations are expressed in volatility rather than in currency units. Straddle quotes are given by the ATM forward volatility. In a $\Delta$-delta strangle, the dealer sells or buys both OTM options from the counterparty. Dealers generally record strangle prices as the spread of the strangle volatility over the ATM forward volatility:

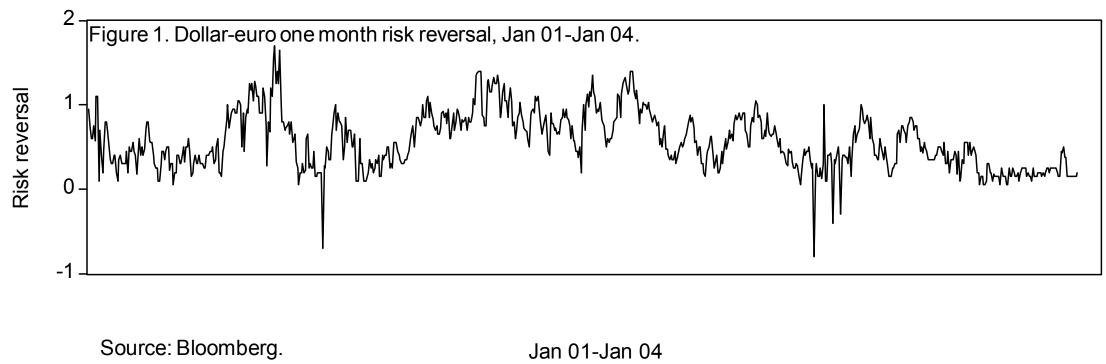
$$STR(\Delta,T) = \frac{1}{2}(\sigma^{Call}_{implied}(\Delta,T) + \sigma^{Put}_{implied}(\Delta,T)) - \sigma_{implied}(\Delta neutral, T) \quad (5)$$

If traders were convinced that exchange rates are moving log normally, the OTM options would have the same implied volatility as the ATM options, and strangle spreads would be zero. Strangles, then, indicate the degree of curvature of the volatility smile.

In a risk reversal, the dealer exchanges one of the options for the other with the counterparty. Because the put and the call are generally not of equal value, the dealer pays or receives a premium for exchanging the options, quoted as implied volatility differential for exchanging a $\Delta$-delta call for a $\Delta$-delta put:

$$RR(\Delta,T) = \sigma^{Call}_{implied}(\Delta,T) - \sigma^{Put}_{implied}(\Delta,T) \quad (6)$$

For example, if the dollar-euro rate is strongly expected to rise (dollar depreciation), an option dealer might quote RRs as "one month 25-delta RRs are 0.8 at 1.2 euro calls over". This means that the dealer stands ready to pay a net premium (0.8) to buy a 25-delta euro call and sell a 25-delta euro put against the dollar, and charges a net premium (1.2) to sell a 25-delta euro call and buy a 25-delta euro put. A risk reversal is a skewness premium. Under the BS log normality assumption the RR price is zero; the probability of OTM call being ATM at maturity is the same as that of an equally OTM put being ATM at maturity. In practice, positive RRs exist when market expectations are skewed relative to the lognormal distribution. Thus a RR is a measure of the skewness of an implied RND function. RRs can have a positive or negative sign, depending on the direction of skewness in expected exchange rate changes. In the dollar-yen market, if 25-delta OTM dollar calls are trading at a volatility 0.5 percentage points higher than 25-delta OTM dollar put, then 25-delta RR might be quoted as "0.5, dollar calls over", indicating a skewness biased towards dollar appreciation. When RRs are positive, the market sentiment favors the underlying currency on which the call is written. Figure 1 shows one month RRs for the dollar-euro rate over January 2001-June 2004. Markets were consistently expecting a depreciation of the dollar.




Source: Bloomberg.

Denoting the 25-delta and 75-delta call quotes by $\sigma^{(0.25)}$ and $\sigma^{(0.75)}$, respectively, the quotes for a strangle and a risk reversal would be: $STR = 0.5(\sigma^{(0.25)} + \sigma^{(0.75)}) - ATM$,
and $RR = \sigma^{(0.25)} - \sigma^{(0.75)}$, respectively. Knowing the market quotes for ATM, STR, and RR, it is easy to derive $\sigma^{(0.25)}$ and $\sigma^{(0.75)}$ as:

$$\sigma^{(0.25)} = ATM + STR + 0.5RR$$
$$\sigma^{(0.75)} = ATM + STR - 0.5RR.$$

Similar to Shimko (1993), Maltz (1997) proposed an approximation of the volatility function $\sigma(\Delta, T)$ as:

$$\sigma(\Delta, T) = ATM - 2RR(\Delta - 0.5) + 16STR(\Delta - 0.5)^2 \quad (7)$$

This functional form is the simplest one that captures the information about the smile that the three option prices express. The ATM volatility gives the general level of implied volatility; it is a measure of location for the volatility smile. The RR price indicates the skewness in the smile, and the strangle price indicates the degree of curvature of the smile. This approximation is made in the delta-volatility space. To retrieve approximation in a strike-volatility space, delta is replaced by its formula (4), yielding thus:

$$\sigma(K,T) = ATM - 2RR(e^{-\gamma\tau}N(d_1) - 0.5) + 16STR(e^{-\gamma\tau}N(d_1) - 0.5)^2 \quad (8)$$

Equation (8) solves for $\sigma(K,T)$ as a function of $K$. Next, the BS formula is used to obtain a continuous call function: $C(K,T) = C(S_t, t, K, T, r, \gamma, \sigma(K,T))$. The cumulative distribution, denoted by $P(K,T)$, which estimates the probability that the future exchange rate will be less or equal to $K$, and the probability density function denoted by $p(K,T)$ can be easily calculated from $C(K,T)$. Indeed, for each $K$, the estimated cumulative distribution function at that point is given by: $\frac{\partial C(K,T)}{\partial K} = e^{-r\tau}[1 - P(K,T)]$; and the risk neutral probability density is given by: $\frac{\partial^2 C(K,T)}{\partial K^2} = e^{-r\tau} p(K,T)$.

### B. The Implied Tree Method

The tree method to infer the risk-neutral transition probabilities, or equivalently the volatility smile $\sigma(K,T)$, from option prices was explored by Rubinstein (1994), and Derman and Kani (1994). Rubinstein's approach uses a binomial tree as the discrete approximation of the implied process. First, probabilities are assigned to the terminal nodes of the tree by minimizing the deviation from prior estimates, subject to the constraint that option prices must fall within a bid-ask spread of observed prices. The rest of the tree is constructed using a backward recursive procedure on the ad hoc assumption that all paths leading to the same terminal node have the same risk-neutral probability. This assumption permits the local volatility to be determined uniquely by indirectly supplying the missing information.

Derman and Kani (1994) showed that $\sigma(K,T)$ can be completely determined from the smile by requiring that option prices calculated from the tree model fit the smile. Option prices for all strikes and expirations, obtained by interpolation from known option prices, will determine the position and the probability of reaching each node in the implied tree. The transition of the tree from time $t_n$ to time $t_{n+1}$ is constructed by induction. At time $t_{n+1}$, the $(n+1)th$ level of the tree has $(n+1)$ nodes, corresponding to $(n+1)$ unknown underlying asset prices $S^i_{n+1}$ and $n$ unknown transition probabilities $p^i$. Thus, the construction of the $(n+1)th$ level of the tree requires the solution for $(2n+1)$ unknowns. These can be determined using the smile, implied by $n$ forward prices, $F^i_{n+1}$, where $F^i_{n+1} = e^{r\Delta t} S^i_n$, and $n$ options prices, $C(S^i_n, t_{n+1})$, obtained from interpolating observed market data, all expiring at time $t_{n+1}$. The forward and options' prices provide $2n$ equations for $(2n+1)$ unknowns. The one remaining degree of freedom is used to make the center of the tree coincide with the center of the standard Cox-Ross-Rubinstein binomial tree. The computation of the theoretical option prices, to be matched with the interpolated option prices, assumes given the Arrow-Debreu (AD) prices at the $(n)th$ level. These are computed by forward induction as the sum over all paths, from the root of the tree to the node $(n,i)$, of the product of the riskless-discounted transition probabilities at each node in each path leading to node $(n,i)$. The local volatility at each node is computed, as in any binomial tree, as follows:

$$p^i (S^{i+1}_{n+1} - F^i_{n+1})^2 + (1 - p^i)(S^i_{n+1} - F^i_{n+1})^2 = (F^i_{n+1})^2 \sigma^2 (S^i_n) \Delta t \quad (9)$$

Besides the tree methods proposed by Rubinstein (1994), and Derman and Kani (1994), an implied tree can also be constructed using a finite difference method. The partial differential equation (PDE) for a European call given by equation (A.6) becomes after a change of variable $x = \ln S$ and applying Ito's lemma:

$$\frac{\partial C}{\partial t} + (r - \frac{1}{2}\sigma^2)\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial S^2} - rC = 0 \quad (10)$$

with boundary condition: $C(x_T, T; \ln K, T) = \max[0, e^{x_T} - e^{\ln K}]$. Chriss and Tsiveriotis (1995) showed that this PDE can be approximated on a finite grid $t_i = t_0 + i\Delta t$, $x_j = x_0 + j\Delta x$, $i = 1,..,M$, and $j = 1,...,N$, by a finite difference equation:

$$-\vec{C}^m + \Sigma^{m+1}B\vec{C}^{m+1} + D\vec{C}^{m+1} = 0,\ m = 1,...,M \quad (11)$$

Where $\vec{C}^m = (N,1)$ is a vector of $N$ option values; an element $\vec{C}^m(j)$ of $\vec{C}^m$ gives the value of the option at nodes $(t_m, x_j)$ of the grid, $j = 1,...,N$. The matrix $\Sigma^{m+1} = diag(\frac{(\sigma_j^{m+1})^2}{2})$ gives the volatility of the underlying asset prices at nodes $(t_{m+1}, x_j)$, $j = 1,...,N$. The matrices $B = (N,N)$ and $D = (N,N)$ are independent of the volatilities and depend on the remaining coefficients of the PDE as well as the mesh size of the grid. The PDE can be written as:

$$\vec{C}^m = (\Sigma^{m+1}B + D)\vec{C}^{m+1} = \mathrm{A}^{m+1}\vec{C}^{m+1} \quad (12)$$

where $\mathrm{A}^{m+1} = \Sigma^{m+1}B + D$. Iteration of the PDE to present time $t_0$ yields:

$$\vec{C}^0 = (\mathrm{A}^1\mathrm{A}^2......\mathrm{A}^m)\mathrm{A}^{m+1}\vec{C}^{m+1} \quad (13)$$

This iterated equation shows how to compute today's option prices for all $x$-values on the finite difference grid, given the option prices on the grid at a later time $t_{m+1}$. Denoting $G^m = \mathrm{A}^1\mathrm{A}^2......\mathrm{A}^m$, $G^m$ is a Green's function that could be seen as the AD state prices at the level $t_m$ obtained by forward induction, and $\mathrm{A}^{m+1}$ could be seen as the backward risk-neutral discounted transition probabilities that relate option values at level $t_m$ to option values at level $t_{m+1}$.The matrix formula of the iterated PDE equation provides a simple method for backing out the local volatilities $\sigma_j^m$ from a set of option prices by solving the transition step from level $t_m$ to level $t_{m+1}$ of the grid assuming $G^m$ is already known and computed by forward induction from $t_0$ to $t_m$.

To value an option with time to expiry $t_{m+1}$, the boundary condition becomes: $\vec{C}^{m+1} = \max(e^{x_j} - K, 0)$, $j = 1,..,N$. To find local volatilities $\sigma(x,t)$ using the iterated PDE, observed market option prices are required; namely as many option prices are needed as there are nodes in the finite difference grid. Therefore, all $C(i,j)$, corresponding to nodes $(i,j)$ are assumed known through interpolation of market data. The backward and forward induction can be summarized as: $\vec{C}^0 = G^m(\Sigma^{m+1}B + D)\vec{C}^{m+1}$ and $G^{m+1} = G^m(\Sigma^{m+1}B + D)$, respectively. In the backward equation, all matrices are known, except $\Sigma^{m+1}$. The latter is therefore computed by solving the backward induction. The computed $\Sigma^{m+1}$ is then used to update the forward induction and provide $G^{m+1}$. The iteration is initiated at time $t_0$ with $G^0 = I$ (the identity matrix). The backward transition between $t_1$ and $t_0$ is given by: $\vec{C}^0 = G^0(\Sigma^1 B + D)\vec{C}^1$. The computation of $\Sigma^1$ is used in the forward induction to give $G^1 = G^0(\Sigma^1 B + D)$. The transition between time $t_1$ and $t_2$ is given by $\vec{C}^0 = G^1(\Sigma^2 B + D)\vec{C}^2$, and enables to derive $\Sigma^2$. This iteration is carried out until the expiry date of the option. Therefore, the iteration exploits two types of relations: a forward relation, which relates AD price of a node to the AD prices of its predecessors; a standard backward relation, which links the value of a claim at a node to its value at the immediate successors.

### C. The Forward Approach to the Inverse Problem: The Fokker-Planck Equation

Dupire (1994) suggested the use of the adjoint PDE to extract local volatilities from observed option market prices. Assuming a complete model where the asset price is driven by the following risk-neutral diffusion process:

$$dS_t = (r-\gamma)S_t dt + \sigma(S_t,t)S_t d\hat{Z}_t$$

a call premium, $C = C(S_t,t;K,T,\sigma(S_t,t))$, satisfies the PDE:

$$\frac{\partial C}{\partial t} + (r-\gamma)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} - rC = 0,$$

with boundary conditions: $C(0,t;K) = 0$, and $C(S_T,T,K) = \max(0, S_T - K)$.Under regularity conditions on $r$, $\gamma$, and $\sigma$, the Feynman-Kac theorem shows that the solution to this PDE can be written as a conditional expectation:

$$C(S,t;K,T) = E[g(S_T)\exp\{-\int_t^T rdu\} \mid S_t = S] = e^{-r(T-t)}\int_0^\infty g(S_T)p(S_t,t;S_T,T)dS_T\ ,\ 0 \le t \le T\ .$$

where $p(S_t,t;S_T,T)$ is the Green's function or the fundamental PDE solution. This probability density function determines completely the behavior of the asset price Markov process as described by the SDE; it satisfies both the backward and forward (Fokker-Planck) Kolmogorov equations. The forward equation for $p(S_t,t;S_T,T)$ is:

$$\frac{\partial p}{\partial T} + \frac{\partial((r-\gamma)S_T p)}{\partial S_T} - \frac{1}{2}\frac{\partial^2(\sigma^2(S_T,T)S_T^2 p)}{\partial S_T^2} = 0\text{, for fixed } (S_t,t) \quad (14)$$

The initial condition is given by the Dirac delta: $p(S_t,t;S,t) = \delta(S_t - S)$. For a European call option, the payoff function is given by: $g(S_T) = C(S_T,T) = \max(S_T - K, 0), K > 0$ , yielding a solution:

$$C(S,t;K,T) = e^{-r(T-t)}\int_K^\infty (S_T - K)p(S_t,t;S_T,T)dS_T \quad (15)$$

Using Leibniz' rule to differentiate twice with respect to $K$ yields:

$$p(S_t, t; K, T) = e^{r(T-t)} \frac{\partial^2 C(S_t, t; K, T)}{\partial K^2} \quad (16)$$ [5]

Thus, given a continuum of traded European calls with different strikes and maturities, the transition densities of the asset price $S$ can be recovered directly from market prices. By replacing $p(S_t, t; K, T)$ in the Fokker-Planck equation and integrating twice with respect to $K$, Dupire obtained the adjoint partial differential equation:

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2(K,T)K^2 \frac{\partial^2 C}{\partial K^2} - (r-\gamma)K\frac{\partial C}{\partial K} - \gamma C \quad (17)$$

with boundary conditions: $C|_{K=0} = e^{-(r-\gamma)(T-t)}S$, $C|_{T=t} = \max(0, S_T - K)$. This forward equation describes the evolution of a call option price as a function of maturity and strike; it enables to express the unknown volatility function directly in terms of known option prices. Therefore, letting $S$ follow a continuous-time one-factor diffusion, and assuming given observed arbitrage-free market prices of European calls for all strikes $K \in [0, \infty)$, and maturities $T \in (t, \tau]$, the implied local volatility function of $S$ that is consistent with the market is given uniquely by:

$$\sigma^2(K,T) = 2\frac{\partial C/\partial T + \gamma C + K(r-\gamma)\partial C/\partial K}{K^2(\partial^2 C/\partial K^2)} \quad (18)$$

Customarily, the PDE is amenable to some simplification by a change in variables. Let $y = \ln\frac{K}{S_t}$, $\tau = T - t$, $V(y,\tau) = e^{\gamma\tau}\frac{C(;K,T)}{S_t}$, $a(y) = \frac{1}{2}\sigma^2(K)$, then $V(y,\tau)$ satisfies the following inverse Cauchy parabolic problem:

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[5] The Feynman-Kac solution can also be stated in terms of the AD prices denoted by $G(S_t, t; K, T)$. Let $C(S,t;K,T) = \int_0^\infty g(S_T)G(S_t,t;S_T,T)dS_T$, $0 \le t \le T$, with $G(S_t,t;K,T) = \frac{\partial^2 C(S_t,t;K,T)}{\partial K^2}$. The function $G(S_t,t;K,T)$ satisfies: $\frac{\partial G}{\partial t} + \frac{1}{2}S^2\sigma^2(S)\frac{\partial^2 G}{\partial S^2} + (r-\gamma)S\frac{\partial G}{\partial S} - rG = 0$ with the terminal data: $G(S_t,t;K,T) = \delta(S_T - K)$. From the well-known property of the Green's function, $G(S_t,t;K,T)$ also satisfies the adjoint equation: $\frac{\partial G}{\partial T} = \frac{1}{2}\frac{\partial^2(K^2\sigma^2(K)G)}{\partial K^2} - (r-\gamma)\frac{\partial(KG)}{\partial K} - rG$. Integrating this equation twice with respect to $K$, and using integration by parts, yields: $\frac{\partial C}{\partial \tau} = \frac{1}{2}K^2\sigma^2(K)\frac{\partial^2 C}{\partial K^2} + (r-\gamma)K\frac{\partial C}{\partial K} - \gamma C$ were $\tau = T - t$, and the boundary conditions for $C = C(K,\tau)$ are: $C(K,\tau)|_{K=0} = e^{-\gamma\tau}S_t$, and $C(K,0) = Max(0, S_t - K)$.

$$V_\tau - a(y)(V_{yy} - V_y) + (r-\gamma)V_y = 0 \qquad (19)$$

with terminal observation $V(y,0) = Max(0, 1-e^y)$, and $V(y,\tau) = V_0(y)$, where $V_0(y) = e^{\gamma\tau}\frac{C_0(K,T)}{S_t}$ is the current market data. The inverse problem is, therefore, to find the unknown coefficient $a(y)$ from the known current market values $V_0(y)$. It is assumed that $V(y)$ satisfies the following conditions: $0 \le V_0(y) \le 1$, $\lim_{y\to-\infty} V_0(y) = 1$, $\lim_{y\to\infty} V_0(y) = 0$. The Fokker-Planck equation as a function of $(K,T)$ is, for fixed $(S,t)$, $T \in (t,\tau]$:

$$\frac{\partial p}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial K^2}(K^2\sigma^2(K)p) - (r-\gamma)\frac{\partial}{\partial K}(Kp) - rp \qquad (20)$$

the initial condition is $p(K,T;S,t)_{T=t} = \delta(K-S)$, and the boundary conditions are $p|_{K=0} = 0$, and $p|_{K\to\infty} = 0$. After the change in variables and using Ito's lemma, (20) becomes:

$$p_\tau = (ap)_{yy} - (ap)_y - (r-\gamma)p_y \quad (21)$$

with the initial condition $p(K,T;S,t)_{T=t} = \delta(K-S)$.

Dupire's approach consists therefore of estimating a local-volatility function $a(y)$ from the dual PDE (19); next the smile is used in the Fokker-Planck equation for the transition density function (21) to solve numerically for the implied dispersion of expected returns.

## III. Recovering the Dollar-Euro Implied Volatilities and RND

This section presents empirical estimates for the dollar-euro rate implied RND for June 2004 as expected by traders on May 5, 2004 using market data for the forward rate, call prices, put prices, and interest rates as observed on that day.[6] It computes the smile and then the implied RND and compares the RND to a lognormal distribution. As market data was available for a limited number of strikes and maturities, interpolation and extrapolation of data across strikes and maturities was undertaken. Cubic splining was used to generate data for all nodes on a finite difference grid, the number of generated data points changed according to the size of the grid mesh.

### A. Empirical Estimates of the Smile

The smile was estimated using Dupire's approach. The adjoint PDE (19):

$$V_\tau - a(y)(V_{yy} - V_y) + (r-\gamma)V_y = 0$$

[6] Source of the data is the Chicago Mercantile Exchange.

can be approximated as a finite difference equation in an explicit form, implicit form, or a Cranck-Nicholson form.[7] Let $\Delta y = h$, and let $D_t$, $D_y$, and $D_{yy}$ be, respectively, the time difference, the first central difference, and the second difference operators defined as: $\frac{V_n^{m+1} - V_n^m}{\Delta\tau} = D_t V^{m+1}$, $\frac{V_{n+1}^m - V_{n-1}^m}{2h} = D_y V^m$, and $\frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{h^2} = D_{yy} V^m$ for $n = 0,1,....,N,N+1$, and $m = 0,1,...,M$. The operators $D_y$ and $D_{yy}$ are $(N+2, N+2)$ tri-diagonal matrices given by:

$$D_y = \begin{bmatrix} 0.................................0 \\ -\frac{1}{2h} 0 \frac{1}{2h} 0.................0 \\ 0 - \frac{1}{2h} 0 \frac{1}{2h} 0..............0 \\ ..................................... \\ 0.................. -\frac{1}{2h} 0 \frac{1}{2h} \\ 0.................................0 \end{bmatrix} \quad \text{and} \quad D_{yy} = \begin{bmatrix} 0.................................0 \\ \frac{1}{h^2} - \frac{2}{h^2} \frac{1}{h^2} 0...............0 \\ 0 \ \frac{1}{h^2} - \frac{2}{h^2} \frac{1}{h^2} 0............0 \\ ..................................... \\ 0................. \frac{1}{h^2} - \frac{2}{h^2} \frac{1}{h^2} \\ 0.................................0 \end{bmatrix}$$

The boundary values are given by: $V_0^m = \bar{V}_0^m$, and $V_{N+1}^m = \bar{V}_{N+1}^m$ for $m = 0,1,...,M$.

(i) **An explicit scheme** for the PDE can be written as:

$$\frac{V_n^{m+1} - V_n^m}{\Delta\tau} = a(y)(\frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{(\Delta y)^2} - \frac{V_{n+1}^m - V_{n-1}^m}{2\Delta y}) - (r-\gamma)\frac{V_{n+1}^m - V_{n-1}^m}{2\Delta y}$$

Or, equivalently: $D_t V^{m+1} = a(y)(D_{yy} - D_y)V^m - (r-\gamma)D_y V^m$. After rearranging:

$$V^{m+1} = \Delta\tau a(y)(D_{yy} - D_y)V^m + (I - \Delta\tau(r-\gamma)D_y)V^m + \bar{V}^{m+1} \qquad (22)$$

where $\bar{V}^{m+1}$ is an $(N+2,1)$ vector given by the boundary values as the transpose of $(\bar{V}_{N+1}^{m+1}, 0.....,0,\bar{V}_0^{m+1})$. Let the known matrices be written as $B_1 = \Delta\tau(D_{yy} - D_y)$, and $B_2 = I - \Delta\tau(r-\gamma)D_y$, then the explicit finite difference scheme can be written as:

$$V^{m+1} = (A(y)B_1 + B_2)V^m + \bar{V}^{m+1} \qquad (23)$$

---

[7] Approximation techniques based on Green's theorem, variational, and finite element methods are also used for solving numerically partial differential equations. These techniques transform the PDE into an ordinary differential equation. Other techniques use inverse Fourier or Laplace transforms whereby the Green's function is obtained from a numerical inversion of the Fourier or Laplace transform.

where the only unknown is the diagonal matrix: $A(y) = Diag(a_n^m(y))$, representing local volatilities at nodes $(n,m)$, for $n = 0,1,....,N,N+1$.

(ii) **An implicit scheme** for the PDE can be written as:

$$\frac{V_n^{m+1} - V_n^m}{\Delta\tau} = a(y)(\frac{V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1}}{(\Delta y)^2} - \frac{V_{n+1}^{m+1} - V_{n-1}^{m+1}}{2\Delta y}) - (r-\gamma)\frac{V_{n+1}^{m+1} - V_{n-1}^{m+1}}{2\Delta y}$$

or, equivalently: $D_t V^{m+1} = a(y)(D_{yy} - D_y)V^{m+1} - (r-\gamma)D_y V^{m+1}$. After rearranging:

$$V^m = -\Delta\tau a(y)(D_{yy} - D_y)V^{m+1} + (I + \Delta\tau(r-\gamma)D_y)V^{m+1} + \overline{V}^m \quad (24)$$

Let $B_1 = \Delta\tau(D_{yy} - D_y)$, and $B_3 = I + \Delta\tau(r-\gamma)D_y$, then the implicit scheme becomes:

$$V^m = (-A(y)B_1 + B_3)V^{m+1} + \overline{V}^m \quad (25)$$

where $\overline{V}^{m-1}$ is an $(N+2,1)$ vector given by the boundary values as transpose of $(\overline{V}_{N+1}^m, 0.....,0, \overline{V}_0^m)$.

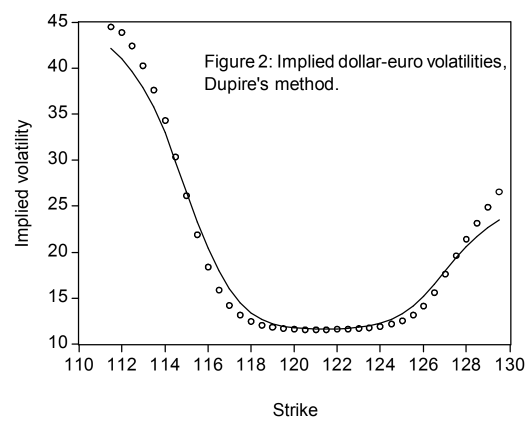
(iii) **The Crank-Nicolson scheme** is defined as: $\theta(\exp licit) + (1-\theta)(implicit)$.

$D_t V^{m+1} = \theta[a(y)(D_{yy} - D_y)V^m - (r-\gamma)D_y V^m] + (1-\theta)[a(y)(D_{yy} - D_y)V^{m+1} - (r-\gamma)D_y V^{m+1}]$.

After rearranging and using the matrix notation, the PDE becomes:

$$\{\theta - (1-\theta)[(-A(y)B_1 + B_3)]\}V^{m+1} - \theta\overline{V}^{m+1} = \{\theta[(A(y)B_1 + B_2)] - (1-\theta)\}V^m + (1-\theta)\overline{V}^m \quad (26)$$

In all three approximations, a recursive scheme is obtained: $V^{m+1} = \Upsilon V^m + \overline{V}^{m+1}$.
The matrix $\Upsilon$ depends on local volatilities $A(y)$, the only unknown parameters, as well as on the finite difference scheme. Forward induction was used, updating volatilities at each time step based on the interpolated grid whose initial and terminal values are the observed market values. The estimation of the smile can be seen as a two boundary PDE with unknown parameters. For $\tau = 0$, the boundary condition is given by the payoff vector; and for $\tau_0 = T - t_0$, the boundary condition is given by the current market values $V_0$. The results, based on a Crank-Nicolson scheme, are shown in Figure 2.




Many important findings can be easily noted. Most strikingly, the smile was strongly evident, and the RND can deviate considerably from the lognormal distribution. Traders had wide-ranging expectations, and attempted consequently to protect their exposure to exchange rate risk. While a segment of the market seemed to expect volatility to remain close to the ATM level computed at $\sigma = 11.58$, there were however traders who held the belief that large moves in the exchange rate in either direction could not be ruled out. The volatility was excessively high on both tails, compared with the ATM implied volatility, but more so at lower strikes. The forex market was, thus, populated with groups of traders holding opposite beliefs regarding large moves in the exchange rate. Expectations were far from settling in one direction. There was a group of traders who would not be surprised by a sudden appreciation of the dollar with respect to the euro. Volatility at low strikes was very high, exceeding 40 percent, indicating high value for OTM euro put options. In contrast, there was an opposite group of traders who would not rule out a significant depreciation of the dollar with respect to the euro. Volatility at high strikes was rising rapidly, reaching 31 percent, indicating thus high value for OTM euro call options. The risk reversal, estimated at 0.25, provides an indication that the market was still valuing OTM euro call more than OTM euro put options, anticipating thus a further depreciation of the dollar.

### B. Estimation of the Implied RND

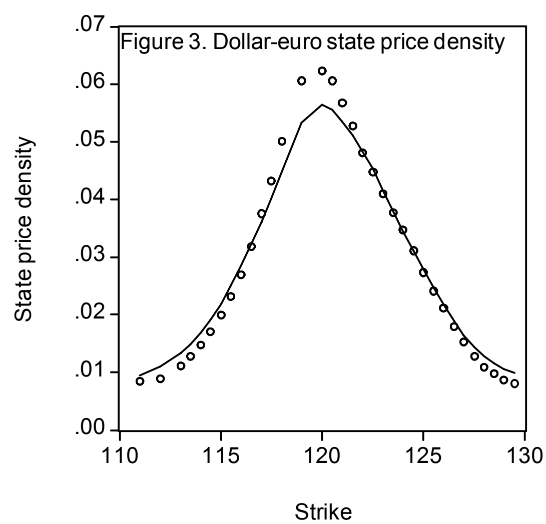
As is well known, the estimation of the smile amounts to estimating the probability distribution. Two methods are adopted here to recover the implied RND. The first method consists of solving a linear system in the state price vector,[8] and the second method solves for the Fokker-Planck equation. The merit of the first method is to rely only on actual market data without any interpolation. The second method, however, had to rely on interpolating market data in order to be able to use a finite difference grid.

**A Linear System in the State Price Vector**

The asset pricing equations can be expressed by a linear system:

$$u = Dp \qquad (27)$$

Where $u$ = an asset price vector at time $t = 0$ for assets maturing at time $T$; $D$ = payoff matrix at the maturity time $T$; $p$ = state price, or Arrow-Debreu price vector; giving the price at time $t = 0$ for primitive securities maturing at time $T$. It is required that $p$ prices all calls, puts, straddles, risk-reversals, and forward contracts. The least squares method subject to $p$ >0 yields: $\hat{p} = (D'D)^{-1} D'u$. The estimates are shown in Figure 3.




The implied distribution of the exchange rate conforms to stylized features of financial distributions and corroborates the estimates of the smile, namely this distribution was

[8] See Duffie (2001) and Neftci (2000).

skewed and leptokurtotic. Estimates of the state price vector pointed to a right skewness, with a longer right tail, implying higher probabilities for options with strikes higher than the forward rate to finish in money. The median was slightly lower than the mean. This finding corroborates the risk reversal quotes in Figure 1 which pointed out that one-month euro calls continued to be priced higher than corresponding euro puts, nonetheless, the quote premium was narrowing. The density was also leptokurtotic, implying that the probability of extreme events was higher than under the lognormal distribution. Tentatively, the market sentiment on May 5, 2004 could be described as favoring greater stability around the forward, however with a bias for a dollar depreciation. This is only a snap shot of expectations. As is well known, depending on the news, expectations could be extremely volatile and may change dramatically during intra-day trading or from one day to another.

### The Fokker-Planck Equation

Another snap shot of the market was studied; the method now solves for the Fokker-Planck equation (21) using the volatilities shown in Figure 2. The implied transition density in Figure 4 has much fatter tail than a lognormal distribution; traders, therefore, were wary of the likelihood of sudden jumps in the exchange rate. Skewness was positive, but not pronounced, implying closeness to normality and some symmetry around the forward rate. The median was slightly lower than the mean. This finding was supported by the relatively low and diminishing values of both strangles and risk reversals. In sum, the Fokker-Planck transition density conveys similar impression on the market sentiment as the state price density, namely the market expected some stability around the forward rate, without dismissing sudden large moves in the rate. However, in contrast to the state price density, the Fokker-Planck density seemed to imply that traders were more ambivalent about the direction of the rate and the bias for a dollar depreciation was much smaller.

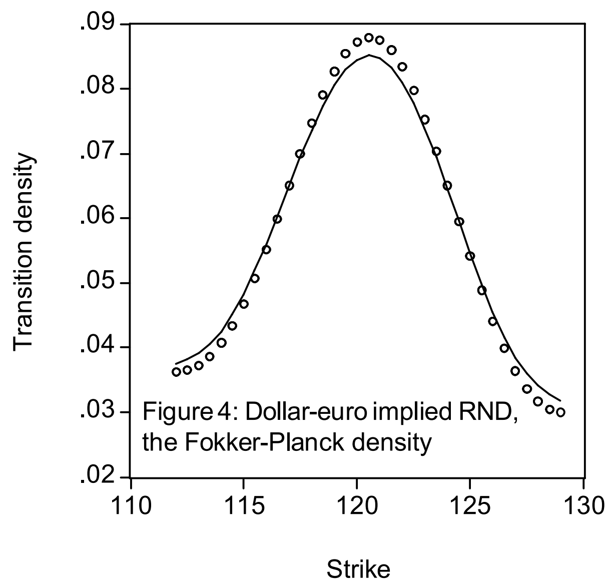


Figure 4: Dollar-euro implied RND, the Fokker-Planck density

## IV. Conclusion

Gauging market sentiment and extracting expectations provides a valuable information for both traders and policy-makers. In this respect, the present paper has addressed the inverse problem in option theory, or equivalently, the calibration of option models; it attempted to extract the smile and the implied RND for the dollar-euro exchange rate from option prices. It has shown that Markov diffusion processes can be reconstructed from observed option values using tree methods or dual PDEs. As only a limited number of exchange-traded options were available, calibration relied on interpolation and extrapolation of market data across strikes and maturities. The implied smile showed that traders had wide-ranging expectations and large moves in the exchange rate in either direction would not occur as a surprise. The tail events had significantly higher probability than under the normal distribution. Solving for both the state price density and the Fokker-Planck density, the paper has estimated an implied RND for the dollar-euro rate that had kurtosis and skewness. Robustness of estimates was addressed by attempting to infer RND using two different methods, namely a linear system in the state price vector which did not use any interpolation, and the Fokker-Planck method which had to rely on interpolation. Both methods proved similar facts, namely fatter tails and higher probability around the forward rate; however, skewness, even though positive, was less pronounced in the Fokker-Planck density.

The calibration methods described in this paper are highly pertinent for the IMF in its surveillance role. It is very important for the IMF to assess market sentiment regarding exchange rates and commodity prices such as oil prices from derivatives' prices. This will enable the IMF to determine macroeconomic and financial policies that will enhance financial stability. For central banks, knowledge of the RND of assets' returns is important. If markets become too volatile and pressures are building in one direction, then an intervention by the monetary authorities would be required. Shifts in the RND would determine the impact of the intervention and suggest whether maintaining the new policy thrust would be necessary to influence significantly the market's expectations.

## ELEMENTS OF OPTION PRICING THEORY

The standard theory of derivatives pricing assumes a Markov diffusion process for the price of the underlying asset given by a stochastic differential equation (SDE):

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dZ_t \quad \text{(A.1)}$$

$S_t$ is the underlying asset price at time $t$, $Z_t$ is a Wiener process, with $dZ_t = Z_{t+dt} - Z_t \sim N(0, dt) = \sqrt{dt}N(0,1)$. The coefficients $\mu(S_t, t)$ and $\sigma(S_t, t)$ are, respectively, the drift and the diffusion coefficients of the stochastic process.[9] A derivative written on the underlying asset has its price given by $f(S_t, t)$. Invoking Ito's lemma, which is a fundamental theorem in asset pricing, $f(S_t, t)$ satisfies Ito's formula:

$$df(S_t, t) = \left\{ \frac{\partial f(S_t, t)}{\partial t} + \mu(S_t, t)\frac{\partial f(S_t, t)}{\partial S_t} + \frac{1}{2}\sigma^2(S_t, t)\frac{\partial^2 f(S_t, t)}{\partial S_t^2} \right\} dt + \left\{ \sigma(S_t, t)\frac{\partial f(S_t, t)}{\partial S_t} \right\} dZ_t$$

(A.2)

### Pricing by No-Arbitrage

Derivatives can be priced by a no-arbitrage or portfolio replication method. A portfolio is formed, consisting of a long position in $\Delta$ (delta) units of the underlying asset, with

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[9] Let $p(t, S_t, T, S_T)$, $T \geq t$, denote the transition probability density function corresponding to $dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dZ_t$. Assuming $\frac{\partial p}{\partial S}$ and $\frac{\partial^2 p}{\partial S^2}$ exist and are continuous with respect to time, then $p(t, S_t, T, S_T)$ is a so-called fundamental solution (Green's function) of the Kolmogorov backward equation: $\frac{\partial p}{\partial t} + \mu(S_t, t)\frac{\partial p}{\partial S_t} + \frac{1}{2}\sigma(S_t, t)\frac{\partial^2 p}{\partial S_t^2} = 0$, satisfying the end condition: $\lim_{t \uparrow T} p(t, S_t, T, S_T) = \delta(S_t - S_T)$ for fixed $(T, S_T)$, $\delta(S_t - S_T)$ is a Dirac function concentrated at $S_T$. For fixed $(t, S_t)$, the density $p(t, S_t, T, S_T)$ is also a fundamental solution of the Kolmogorov forward, or the Fokker-Planck, equation:
$\frac{\partial p}{\partial T} + \frac{\partial(\mu(S_T, T)p)}{\partial S_T} - \frac{1}{2}\frac{\partial^2(\sigma(S_T, T)p)}{\partial S_T^2} = 0$, satisfying the initial condition:
$\lim_{T \downarrow t} p(t, S_t, T, S_T) = \delta(S_T - S_t)$, $\delta(S_T - S_t)$ is the Dirac function concentrated at $S_t$. Both equations are derived from the Chapman-Kolmogorov theorem and can be stated in terms of the infinitesimal generator of the continuous-time Markov process. The knowledge of $p(t, S_t, T, S_T)$ allows to compute, by integration, expected values of an arbitrary function $g(S_T)$, as $f(S_t, t) = E_{t, S_t} g(S_T) = \int_0^\infty g(S_T)p(t, S_t, T, S_T)dS_T$.

$\Delta \equiv \frac{\partial f(S_t,t)}{\partial S_t}$, and a short position in one unit of the derivative. The value of this portfolio at time $t$ (i.e., now) is: $\Pi(S_t,t) \equiv \Delta S_t - f(S_t,t)$. Using Ito's lemma, the change in the value of the portfolio is:

$$d\Pi(S_t,t) \equiv \Delta dS_t - df(S_t,t) = \Delta\mu S_t dt + \Delta\sigma S_t dZ_t -$$
$$\left\{\frac{\partial f(S_t,t)}{\partial t} + \mu(S_t,t)\frac{\partial f(S_t,t)}{\partial S_t} + \frac{1}{2}\sigma^2(S_t,t)\frac{\partial^2 f(S_t,t)}{\partial S_t^2}\right\}dt - \left\{\sigma(S_t,t)\frac{\partial f(S_t,t)}{\partial S_t}\right\}dZ_t =$$
$$-\{\frac{\partial f(S_t,t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f(S_t,t)}{\partial S_t^2})\}dt\ .$$

Note that $dZ_t$ has cancelled, the portfolio is therefore riskless and by no-arbitrage must earn the risk-free rate of interest $r : d\Pi = r\Pi dt$. The no-arbitrage condition is written as:

$$-\{\frac{\partial f(S_t,t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f(S_t,t)}{\partial S_t^2}\}dt = r\{\frac{\partial f}{\partial S}S - f\}dt\ .$$

Hence, the no-arbitrage condition requires that the derivative's price satisfies a partial differential equation (PDE):

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0 \qquad (A.3)$$

This equation shows the price of a derivative as a combination of its hedging parameters; it must be satisfied by every derivative security whose underlying asset price is $S$ (i.e., not just a call or put option). What distinguishes derivative securities is the type of boundary conditions. For a call option with strike $K$, $f(S,T) = \max(0, S-K)$, and for a put with strike $K$, $f(S,T) = \max(0, K-S)$. If the asset pays dividends at a continuous rate $\gamma$, then the no-arbitrage condition becomes:

$$d\Pi + \Delta\gamma S dt = r\Pi dt,$$

leading to the PDE:

$$\frac{\partial f}{\partial t} + (r-\gamma)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0 \qquad (A.4)$$

Using the Feynman-Kac formula, which stipulates that if $f(S,t)$ satisfies a PDE of the form:

$$\frac{\partial f}{\partial t} + \mu(S,t)\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2(S,t)\frac{\partial^2 f}{\partial S^2} - rf = 0,$$

with final condition: $f(S,T) = g(S)$ for all S, then the solution is given by:

$$f(S_t,t) = E_t[g(S_T)\exp\{-\int_t^T r(u)du\} \mid S_t = S]$$

The conditional expectation is computed with respect to the transition probability density $p(S_T \mid S_t)$ inferred from the dynamics: $dS = \mu(S,t)dt + \sigma(S,t)dZ$. The price of the derivative is:

$$f(S_t,t)=\int_0^{\infty}\exp\{-\int_t^T r(u)du\}g(S_T)p(S_T\mid S_t)dS_T\,.^{10}\quad \text{(A.5)}$$

**The Black-Scholes (BS) Model**

The BS model (1973) illustrates the no-arbitrage method. The asset price is assumed to be log-normally distributed and its dynamics are given by a geometric Brownian motion with constant drift and diffusion coefficients: $dS_t=\mu S_t dt+\sigma S_t dZ_t$.[11] Let $C(S_t,t;K,T)$ denote the price at time $t$ of a European call option on an underlying asset with price $S_t$, an exercise price $K$, and maturity date $T$. The price of the call satisfies a PDE:

$$\frac{\partial C}{\partial t}+rS\frac{\partial C}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}-rC=0\quad \text{(A.6)}$$

with boundary condition: $C(S_T,T;K,T)=\max[0,S_T-K]$.[12] The value of $C(S_t,t;K,T)$ that solves this PDE is given by the BS formula: [13]

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[10] Note when $g(S_T)=e^{i\phi S_T}$ this expectation yields the characteristic function of the state price density. Moreover, binomial, trinomial trees, Monte Carlo, finite-difference, or fast Fourier transforms methods are used to compute the price of a derivative when a closed-form for this expectation is not available. In the case of a binomial tree approximation, the up move $u$, the down move $d$, and the transition probability $p$, are, respectively: $u.d=1$, or $d=1/u$ ; $u=e^{\sigma\sqrt{\Delta t}}$ ; $d=e^{-\sigma\sqrt{\Delta t}}$ ; and $p=\dfrac{(1+r\Delta t)-d}{u-d}$. Thus, knowledge of $\sigma$ implies knowledge of $p$ and vice-versa.

[11] Assuming $f(S_t,t)=Ln(S_t)$ and applying Ito's Lemma:

$df(S_t,t)=\{\mu-\frac{1}{2}\sigma^2\}dt+\sigma dZ_t$. It follows:

$dLn(S_t)=\{\mu-\frac{1}{2}\sigma^2\}dt+\sigma dZ_t\sim N(\{\mu-\frac{1}{2}\sigma^2\}dt,\sigma^2dt)$

[12] In order to solve this PDE analytically or numerically, it is helpful to make the equation dimensionless. Let $x=\ln(S/K)$, $\tau=\frac{1}{2}\sigma^2(T-t)$, $C/K=\upsilon(x,\tau)$, a dimensionless equation is obtained: $\dfrac{\partial\upsilon}{\partial\tau}=\dfrac{\partial^2\upsilon}{\partial x^2}+(k-1)\dfrac{\partial\upsilon}{\partial x}-k\upsilon$, where $k=r/\frac{1}{2}\sigma^2$. The initial condition becomes $\upsilon(x,0)=\max(e^x-1,0)$. This equation contains only one dimensionless parameter, $k=r/\frac{1}{2}\sigma^2$. A further change in variables: $\upsilon=e^{\alpha x+\beta\tau}u(x,\tau)$, $\alpha=-\frac{1}{2}(k-1)$, and $\beta=-\frac{1}{4}(k+1)^2$, yields the well-known heat equation: $\dfrac{\partial u}{\partial\tau}=\dfrac{\partial^2 u}{\partial x^2}$ for $-\infty<x<\infty$, and $\tau>0$.

(continued)

$$C(S_t,t;K,T)=S_tN(d_1)-Ke^{-r(T-t)}N(d_2) \quad \text{(A.7)}$$

N(.) is the cumulative normal distribution,

$$d_1=\frac{Ln\frac{S_t}{K}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}};\ \ d_2=d_1-\sigma\sqrt{T-t}$$

The inputs for the BS formula are: $S_t,K,T-t,r,\sigma$. All these variables are observed except for $\sigma$ which measures the risk and has to be estimated. The delta of the call option is $\Delta=\frac{\partial C}{\partial S_t}=N(d_1)$, it measures the number of shares of the underlying asset in the hedging portfolio; whereas $Ke^{-r(T-t)}N(d_2)$ is the number of riskless bonds in the hedging portfolio.[14]

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The Fourier transform of this equation is $\tilde{u}(\tau,\phi)=\int_{-\infty}^{\infty}e^{i\phi x}u(\tau,x)dx$; it satisfies an ordinary differential equation (ODE): $\frac{\partial\tilde{u}}{\partial t}=\phi^2\tilde{u}$. Knowledge of $\tilde{u}$ enables to compute $u$ as an inverse Fourier transform: $u(\tau,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\phi x}\tilde{u}(t,\phi)d\phi$. The Laplace transform $\hat{u}(t,\phi)=\int_{0}^{\infty}e^{-\phi t}u(t,x)dt$ yields also an ODE in the transform. Its solution $\tilde{u}$ enables to compute $u$ as an inverse Laplace transform.

[13] The price of a European put option $Put(S_t,t;K,T)$ is obtained from the put-call parity for European options: $Call(S_t,t;K,T)-Put(S_t,t;K,T)=S_t-K.B(t,T)$, where $B(t,T)$ is the price at time $t$ of a bond maturing at time $T$.

[14] Observing that the characteristic function determines uniquely the probability distribution, may have closed-form expressions, and is infinitely differentiable, whereas the corresponding distribution function may not be available in closed-form and its boundary condition may not be differentiable, Heston (1993), Scott (1997), Carr and Madan (1999), and Bakshi and Madan (2000) used the characteristic function of the state price density to price options. Let $\Phi(\phi,t)$ be the conditional characteristic function:

$$\Phi(\phi,t)=\int_0^{\infty}e^{i\phi S_T}\exp\{-\int_t^T r(u)du\}g(S_T)p(S_T\mid S_t)dS_T$$

then $\Phi(\phi,t)$ satisfies the fundamental PDE (the Fokker-Planck equation) with terminal condition $\Phi(\phi,t)=e^{i\phi S_T}$. Expressing the option price as: $C(S,t)=\Pi_1S_t-B(t,T)K\Pi_2$, where $B(t,T)$ is the time $t$ price of a discount bond maturing at $T$, Heston (1993) showed that $\Pi_1$ and $\Pi_2$ are probability functions that satisfy the Fokker-Planck equation. Their respective characteristics $\Phi_1$ and $\Phi_2$ satisfy also the Fokker-Planck equation with boundary conditions:

(continued)

## Risk-Neutral Pricing

Considering a contingent claim with boundary condition: $f(S_T, T) = g(S_T)$, Cox and Ross (1976) introduced risk-neutral pricing by expressing the value of the claim at time $t$ as:

$$f(S_t, t) = e^{-r(T-t)} E\{g(S_T) \mid S_t\} = e^{-r(T-t)} \int_0^\infty g(S_T) dP(S_T, T \mid S_t, t) \quad \text{(A.8)}$$

where $P(S_T, T \mid S_t, t)$ is the risk-neutral probability distribution of the underlying asset price $S_T$ at time $T$, given $S_t$ at time $t < T$. Knowing that $f(S_t, t)$ satisfies the PDE (A.3), it follows immediately that the probability transition function $P(S_T, T \mid S_t, t)$ satisfies two central equations, the forward (Fokker-Planck) equation and the Kolmogorov backward equation:

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial S} rS + \frac{1}{2} \frac{\partial^2 P}{\partial S^2} \sigma^2 S^2 = 0 \qquad \text{(A.9)}$$

The SDE with which the forward and backward equations are associated is: $dS_t = rS_t dt + \sigma S_t d\hat{Z}_t$. This SDE is obtained by rewriting the actual SDE as: $dS_t = rS_t dt + (\mu - r) S_t dt + \sigma S_t dZ_t$. The new Wiener process is therefore: $d\hat{Z} = \frac{(\mu - r)}{\sigma} dt + dZ_t = \lambda dt + dZ_t$. The coefficient $\lambda = \frac{(\mu - r)}{\sigma}$ is called the market price of risk. The transformation of the actual SDE into a new SDE is obtained via Girsanov's theorem. The latter deals with the construction of a martingale measure $\xi_t(\lambda) = \exp(-\frac{1}{2} \int_0^t \lambda^2 ds + \int_0^t \lambda dZ_s)$, which transforms the actual transition probability density into an equivalent martingale density. Let a riskless bond be chosen with an initial value $B_0 = 1$ and a deterministic evolution: $dB = rBdt$. The price of the risky security S, discounted by $B_t$, gives a discounted price process: $y = \frac{S}{B}$. Then in a risk-neutral world, the discounted price has a zero drift and is therefore a martingale.[15]

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$\Phi_j(\phi, t) = e^{i\phi S_T}$, $j = 1, 2$. A closed-form for $\Phi(\phi, t)$ can be derived from a set of ordinary differential equations. $\Pi_1$ and $\Pi_2$ are computed numerically as inverse Fourier transforms of the characteristic functions: namely: $\Pi_j = \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \frac{\Phi_j(-\phi) e^{i\phi K} - \Phi_j(\phi) e^{-i\phi K}}{i\phi} d\phi = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}[\frac{e^{-i\phi K} \Phi_j}{i\phi}] d\phi$. Along the same line, Laplace transform of the Green's function can be used to price options.

15 $dy = \frac{BdS - S.dB}{B^2} = \frac{rBSdt + \sigma BS\hat{Z} - rBSdt}{B^2} = \sigma y \hat{Z}$. Therefore, $E(dy/y) = 0$.

The risk-neutral probabilities, discounted at the risk-free rate of interest, are interpreted economically as the prices of Arrow-Debreu (AD) securities, or the state prices. An AD security is a primitive security associated with a particular future state of the world; it pays \$1 if that state occurs, and nothing otherwise.[16] All contingent claims and derivatives can be expressed in terms of a portfolio of AD securities and priced accordingly. Given a vector of state prices, the price of any contingent claim may be determined by multiplying the claim's payoff in each state by the corresponding state price, and then summing over all states.

[16] An AD security can be replicated by investing in a suitable combination of European call options, known as a butterfly spread. The state price at any given state is the cost of the butterfly spread centered on that particular state.

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