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# Default Risk and Transition Dynamics with Carbon Shocks

Sujan Lamichhane

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**Default Risk and Transition Dynamics with Carbon Shocks****Prepared by Sujan Lamichhane**

Authorized for distribution by Hiroko Oura

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WORKING PAPERS

# **Default Risk and Transition Dynamics with Carbon Shocks**

Prepared by Sujan Lamichhane

# Default Risk and Transition Dynamics with Carbon Shocks\*

Sujan Lamichhane<sup>†</sup>

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## Abstract

Climate mitigation policies are being introduced around the world to limit global warming, generating new risks to the economy. This paper develops a continuous time heterogeneous agents model to study the impact of carbon pricing policy shocks on corporate default risk and the consequent transition dynamics. We derive a closed-form solution to corporate default probability based on firms' intertemporal optimization decisions and explicitly characterize the transition speed. This allows for studying policy implications in an analytically tractable way. The model is calibrated to different US corporate sectors to quantify the heterogeneous effects of carbon price shocks. While carbon-intensive sectors face increased default risks, there are notable asymmetric effects within sectors. Higher carbon prices increase default risk but also induce faster transition towards the new post-shock steady state with a highly non-linear impact. Our results suggest that once a range of possible price shocks are accounted for, the increase in the cost of capital/risk premiums might be sharply different across sectors.

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# 1 Introduction

Climate-related policies are being considered around the world to limit the impact of climate change on human society. Nearly 200 countries signed the Paris Agreement in 2015, pledging to limit the global warming increase to well below 2°C compared to pre-industrial levels. Achieving this goal requires the substantial reduction in greenhouse gas (GHG) emissions and the global transition to a low-carbon economy. However, there is uncertainty regarding how these climate-related policies will be implemented, thereby generating new risks to the economy.<sup>1</sup>

One of the main strategies proposed for limiting emissions is to introduce carbon pricing policies (HLCCP 2017). Carbon price shocks could lead to a potentially sharp increase in corporate default risks. In this regard, understanding the heterogeneous and asymmetric effects of carbon shocks across and within various sectors of the economy becomes crucial from a financial and economic stability perspective. Another critical aspect for policymakers is to understand how fast different sectors might transition to the new steady state of heightened default risks after the carbon price shocks and the relation between default risk and transition dynamics.

Despite the importance of these issues, there is a lack of analytically tractable framework that characterizes the heterogeneous and non-linear effects of climate policy shocks on corporate default risks and related transition dynamics. This paper makes some progress in this direction. Our contributions are twofold. First, we provide *analytical characterizations* of corporate default risks and the transition dynamics of the economy given the carbon price shock and generate some new theoretical predictions. In particular, we derive a closed-form solution to corporate default probability based on firms' intertemporal optimization decisions. We also provide explicit expression for the transition speed of the entire net worth distribution across firms. Given the parsimonious model that allows for closed-form solutions, it can be calibrated easily. These allow us to study policy implications in an analytically tractable way.<sup>2</sup> Second, calibrating the model to US data, we quantify the heterogeneous effects of carbon shocks across and within two broad economic sectors: transportation and manufacturing. In particular, we find highly non-linear and asymmetric effects of carbon

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<sup>1</sup>The literature differentiates between physical risk—arising from the impact of climate change and climate-related hazards—and transition related risks—arising from a shift to a low carbon economy. This paper abstracts away from the impact of physical risk and focuses on transition-related risks only.

<sup>2</sup>Understanding the theoretical aspects of default risks and transition dynamics is important. In particular, closed-form expressions (when available) are useful to policymakers, as they allow for easy experiments with various policy responses in a parsimonious and efficient way.

price shocks on default risks, transition speed, and the cost of capital/risk premiums, even after accounting for potential mitigation measures to reduce carbon emissions intensity.<sup>3</sup>

We build a continuous time model of firms heterogeneous in their cash flows and a financial sector that lends to the firms. Firms have the necessary production technology to generate cash flows where carbon emissions are assumed proportional to output/revenue. The impact of carbon pricing policy is modeled as an additional balance sheet cost, i.e. carbon tax for each unit of emission produced. As such, firms are exposed to carbon policy risk based on their emissions and carbon price per unit of emission.<sup>4</sup>

Firms maximize the lifetime value of their consumption/dividend payouts as they face idiosyncratic shocks to their cash flows and consequent default risks defined as the probability of net worth/equity value going below zero. This occurs when earnings, net of carbon taxes, can no longer cover the debt expenses. Hence, in our setup, default risks can also be viewed as insolvency risks. Further, an additional feature of the model considers the case when firms may reduce risks by downsizing their business. This can be seen as equivalent to production and hence, indirect emissions cuts. Thus, exogeneous differences in cash flows lead to endogenous decisions, thereby generating a cross sectional distribution of firms over their net worth which represents the state of the economy.

Given the uncertainty regarding exact carbon prices that maybe eventually implemented, we consider three carbon price shocks: \$25, \$50, and \$75 per ton of CO<sub>2</sub>-equivalent (tCO<sub>2</sub>e) emission. These reflect various degrees of potential policy responses and are consistent with those recommended by policymakers around the world.<sup>5</sup> This helps us explore a range of possibilities and highlight the non-linear impact of carbon shocks. Thus, the carbon pricing

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<sup>3</sup>From a practical perspective, our model can also be directly applied in policy analysis settings. For example, in the context of scenario analysis/climate risk stress testing, the closed-form solutions are particularly useful for directly computing projections of default risks for firms/sectors given carbon shocks, analogous to the use of Merton (1974)-based models. This output could further be used in assessing financial institutions' vulnerabilities given their exposures to various firms/sectors. Thus, our model can be used in conjunction with more granular non-public/regulatory data to understand the effects of various policy responses.

<sup>4</sup>As noted in Ramadorai and Zeni (2021), "carbon tax" appears as one of the most frequently used terms in firms' descriptions of the potential climate regulatory risks they face. This motivates our modeling of carbon policy as additional costs to the firms' balance sheets. This is also consistent with approaches to financial stability assessment at central banks and policy institutes (see Adrian et al. 2022; Grippa and Mann 2020; Battiston et al. 2021; and references therein).

<sup>5</sup>For example, the HLCCP (2017) recommended a carbon price range of \$40-\$80 by 2020 and \$50-\$100 by 2030 to achieve the Paris Agreement temperature targets. The Biden administration in the US has calculated \$51 per tCO<sub>2</sub>e as the interim value of the social cost of carbon (IWG 2021). Similarly, the Network for Greening the Financial System (NGFS 2020), a consortium of central banks around the world, has developed various scenarios of carbon price paths consistent with given temperature and emissions targets. The above range of shocks also accounts for the uncertainty regarding the magnitude of emissions reduction required to limit global warming (Barnett et al. 2020).

policy is modeled in a simple and parsimonious way in order to study its implications for corporate default risks and transition dynamics in an analytically tractable way.

The pre-shock steady state, i.e. the baseline of the model without carbon prices, is calibrated to the U.S. corporate data in two broad sectors: transportation and manufacturing. Data show that the transportation sector is highly carbon intensive relative to the manufacturing sector. With our closed form solutions we can conveniently calibrate the model to quantify the impact of such policy shocks, compare the post-shock to the pre-shock steady state, and highlight the interplay between default risk and transition speed. We note that one of the key objects of interest is *changes* (i.e. increase or decrease) in risk because this helps quantify the *differential impact* of the carbon pricing policy across firms/sectors.<sup>6</sup>

Results show the asymmetric impact of the carbon shocks on firms, both across and within sectors where even the large net worth firms face non-trivial increases in default risks with increasingly larger shocks. For carbon price shocks of \$25, \$50, and \$75, the average default rates in the transportation sector *increases* (relative to the baseline) by 2.66%, 5.21%, and 9.17%, respectively. Further, the average default rates of the bottom 10% of the firms (based on their net worth) increase by 2.58%, 5.34%, and 8.31%, respectively, whereas those for the top 10% of the firms increases by 0.47%, 1.37%, and 3.04%, respectively.<sup>7</sup> Consequently, there is a sharp shift in the distribution of default probabilities across firms leading to a heavy right tail. The impact on the manufacturing sector, on the other hand, is relatively mild. For the same set of \$25, \$50, \$75 shocks, the average increase in default rates are 0.20%, 0.39%, and 0.59%, respectively. This reflects lower emissions intensity relative to that in the transportation sector. These results sharply highlight the heterogeneous impact of carbon price shocks: while higher carbon emissions are intuitively related to higher default risk, there are non-linear effects of the shocks across and within sectors.

Another important economic issue is understanding how various sectors of the economy transition towards the new steady state after the carbon price shocks.<sup>8</sup> We show that the

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<sup>6</sup>These two sectors are chosen largely for illustration purposes and also due to some data constraints. Nevertheless, our model can be readily applied to any sector or firm (and various countries) depending on data quality and availability (see section 2 for more discussions).

<sup>7</sup>By construction the bottom 10% of the firms face higher levels of default risks—hence the need to quantify *changes* in default rates in order to understand the effects from carbon price shocks.

<sup>8</sup>Note that transition dynamics in a heterogeneous agents model generally involves characterizing the evolution of the entire cross-sectional distribution of agents over their state variable(s) from some initial conditions/steady state to a new steady state given some exogenous shocks (see section 3.3 for more details). In our case, it is the net worth distribution across firms that evolves into a new steady state after exogenous carbon price shocks. Hence, interpreting our analysis warrants some caution. The statements on transition are with respect to this specific feature of the model and not necessarily about the general and idealized notion of transition characterized by a (fully) green/decarbonized economy.

higher the default rate, the faster the transition speed. This might appear counterintuitive, considering higher default rates could imply that the economic activity is stalling. So, a clear economic intuition follows from a simple observation that, in our model, default rates can also be interpreted as average discount rates of the agents. Higher discount rates imply that future payoffs are less valuable, so it may be better to consume sooner than later, and that the value function settles down to its stationary value faster because the future is discounted more. Since the same dynamical system governs the evolution of the net worth, it also settles down faster to its new stationary distribution.<sup>9</sup> Alternatively, a direct interpretation is that higher default induces more agitation in the distribution, making it settle down faster.

To quantify the effects, we transform the transition speed into the corresponding half-life, i.e. half the time it takes for the net worth distribution to converge to the new steady state. Thus, faster transition implies lower half-life and vice versa. For \$25, \$50, and \$75 shocks the half-life is 1.4500 years, 1.4411 years, and 1.3919 years, respectively, in the transportation sector. But for the same shocks, the half-life is just 0.8295 years, 0.8295 years, and 0.8273 years, respectively, in the manufacturing sector. These findings highlight two key points. First, the larger the carbon price shocks, the faster the transition speed, where non-linear impact becomes sharper with larger shocks. Second, transition speed across sectors can be starkly different for the same set of carbon price shocks. Our calibrations show that the transition speed in the manufacturing sector is higher after the shock relative to that in the transportation sector, implying that the manufacturing sector settles down to its new steady state faster.

These results provide new insights into the interplay between transition and default risks due to carbon shocks with cross-sectoral implications that are largely unexplored in the literature. Higher carbon price shocks increase corporate default risks in any given sector. But such policies also induce faster transition towards the new post-shock steady. Further, the rate of transition is generally non-linear and the degree of non-linear impact is largely driven by the emission intensities across sectors. In particular, the transportation sector, with higher emission intensity, transitions slowly relative to the manufacturing sector. As such, the results underscore a sharp connection and potential tradeoffs between default risk and transition dynamics.<sup>10</sup>

We also explore the possibility that firms might respond to carbon pricing policies by

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<sup>9</sup>More formally, the evolution of the value function and net worth distribution are intimately related because the operators governing their dynamics are adjoint, i.e. the infinite-dimensional analogue of a matrix transpose.

<sup>10</sup>Note that we do not make any welfare statements, but just state the implications of our results. Analyzing the welfare implications are beyond the scope of this paper.



introducing various measures to reduce their emission intensity. However, it is unclear to what extent such mitigation measures might reduce emission intensities. To account for such possibilities and also to keep the model tractable given our goals, we also consider a 10% reduction in emission intensity that could arise from a variety of reasons.<sup>11</sup> Considering percentage reduction in emission intensity, as opposed to total emissions, allows us to compare the results across firms/sectors. We find that the overall default rates across both sectors are reduced, as expected. It is notable that there is a distinct asymmetric impact across sectors. The transportation sector sees a material decline in default rates whereas manufacturing sector sees only a small decline, for the same percentage reduction in emission intensity.

While direct carbon pricing is one of the main policy tools proposed, risks might also materialize from other channels. For example, forward looking financial sector/investors might also price in potential increases in corporate risks, even if carbon pricing policies are not yet implemented. Changes in investor sentiment could lead to a relatively higher cost of capital for emission intensive sectors, reduced investment flows and higher risk premium to compensate for the increased possibility of future carbon pricing policies and risks. Given the analytically tractable model, we can explore such implied effects. In particular, we ask the following questions: First, what is the increase in the cost of capital/carbon risk premium, that is consistent with and reflects the increased corporate risks from potential direct carbon price policy? Second, how does such risk premium change after accounting for potential mitigation measures to reduce emission intensity?

We find that mitigation measures have marginal impact in reducing the default risk and, hence, the cost of capital in the manufacturing sector relative to the significant impact in the transportation sector. The implied increase in cost of capital in the transportation sector from potential carbon price shocks of \$25, \$50, and \$75 are 224 basis points (bps), 449 bps, and 674 bps, respectively. For the manufacturing sector, these effects are relatively mild, at 16 bps, 31 bps, and 47 bps, respectively. These rates are also easily interpreted as required increase to corporate spreads or risk premium to account for the heightened risks from carbon pricing policies, even if such carbon price policies are not yet implemented. However, after accounting for the implied drop in default risk following potential mitigation measures to reduce emission intensity by 10%, the transportation sector continues to face the high cost of capital. But the same mitigation measure only marginally reduces default risk

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<sup>11</sup>For example, firms might adopt renewable technologies and switch to greener energy sources in their production process. Mitigation efforts could also increase from alternative decarbonization policies such as subsidies that promote green technologies or technological progress. Further, participation in emission trading schemes may also help firms to manage their risks which could eventually have equivalent impact as direct reduction in emission intensity.

and therefore, the cost of capital, in the manufacturing sector. An interesting implication is that the manufacturing sector might have weak incentives to substantially reduce emissions given the marginal impact of such actions on reducing default risk and hence funding costs. These results show that our model can be used to study a variety of climate-related policies in addition to the direct impact of carbon shocks.

**Related literature:** This paper contributes to the literature on climate change and financial markets, a rapidly growing area of research, as highlighted in the recent survey articles by Giglio et al. (2020) and Hong et al. (2020). On a broad level, our results suggest that once a range of possible carbon price shocks in the future and corresponding impact on default risks are accounted for, the increase in the cost of capital/risk premium might need to be sharply higher and asymmetric across sectors, relative to that suggested by current estimates in the literature, even after accounting for potential mitigation efforts. In other words, risk premium could also be starkly different depending on sectoral exposure to future carbon policy shocks.

Bolton and Kacperczyk (2021) find that a one-standard deviation increase in scope one emissions is associated with a rise in expected returns of 1.8% per annum, reflecting the view that investors are already screening to limit carbon risk in their portfolios. However, our results suggest that such a risk premium might need to be significantly higher and heterogeneous across sectors once risks from a range of possible future carbon price shocks are accounted for. For example, carbon risk premium in the transportation sector might be significantly higher relative to that in the manufacturing sector.

Chava (2014) finds that lenders charge significantly higher interest rates to firms with environmental concerns such as emissions. Kolbel et al. (2021) find that corporate disclosure of transition risks increases CDS spreads after the Paris Agreement of 2015, i.e. climate risk disclosure is associated with higher credit spreads. In particular, a one-standard-deviation increase in transition risk (climate policy risk) leads to an increase of 6.99 bps in the average five-year CDS spread. Our results indicate that such an impact on cost of debt capital and corporate spreads is highly heterogeneous across and within sectors, such that the risk premium increase might need to be much larger than suggested by these estimates for emission-intensive sectors, even after accounting for potential mitigation measures.<sup>12</sup> Thus,

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<sup>12</sup>Relatedly, Ilhan et al. (2020) find that climate policy uncertainty is priced in the equity option markets and that the cost of option protection against downside tail risk is higher for the more carbon-intensive sectors of the S&P 500. Our results also show the heterogeneous impact of various carbon price shocks on the default risks across sectors, which ultimately affects the valuation of equity derivatives. Once again, our results suggest that the effects of climate risk may not be fully priced, especially considering heterogeneous

this paper complements the climate finance research by showing the asymmetric and heterogeneous impact of carbon price policies and that different sectors will require different risk compensation to account for the increased default risks under various plausible carbon price shocks.

This paper is also related to the corporate credit risk literature that studies firm default risk (e.g. Merton 1974; Leland 1994; Jarrow and Turnbull 1995; Duffie and Singleton 1999). See Jarrow 2009 and Sundaresan 2013 for surveys). In particular, our definition of “default” is similar in spirit to that in forward-looking structural credit risk models/literature following Merton (1974) where default occurs given non-positive equity value in the future. In our model, default occurs when net worth, i.e. the equity value, is non-positive. The main difference is that our default risk measure is based on firm’s intertemporal optimization decision in an infinite horizon model, whereas default risk in Merton (1974)-type models is generally based on no-arbitrage models over a finite horizon. Hence, our notion of default risk is quite relevant for climate risks, as it considers risks over a long horizon and is based on firms’ optimal intertemporal decisions.<sup>13</sup> Nevertheless, both approaches complement each other as both provide forward-looking but structurally different perspectives in corporate credit risks.

This paper also contributes to the continuous time heterogeneous agent literature in terms of analytically characterizing transition dynamics. See Kaplan and Violante (2018) for a survey. Given technical challenges, numerical computations are used almost exclusively for studying transition dynamics. A notable exception is Gabaix et al. (2016), who analytically study the transition dynamics of income distribution. More recently, Lamichhane (2020) studies the transition dynamics of household debt distribution. We focus on transition via net worth distribution of firms while also studying default risks from carbon shocks.<sup>14</sup>

The rest of the paper is organized as follows. Section 2 presents the model and calibration. Section 3 discusses the results. Section 4 concludes.

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impacts across sectors.

<sup>13</sup>On a practical side, our model mainly needs corporate balance sheet data and does not rely on market-based data to quantify default risks. This feature makes it even more applicable to various emerging and developing economies where a comprehensive financial market structure/data may be lacking.

<sup>14</sup>The economic mechanisms/implications we explore are quite different from those in the standard Aiyagari-Bewley-Huggett heterogeneous agent models (e.g. Achdou et al. 2020 and Kaplan et al. 2018), or in the heterogeneous firm models (e.g. Hopenhayn 199) and Luttmer 2007). These models consider borrowing/saving, labor/capital, production/investment, and exit/entry decisions. Because of the heterogeneous agents structure, the common object of interest is the cross-sectional distribution of agents. However, we consider a different economic problem where firms control their consumption/payout and risk exposures in order to optimize the expected discounted value of consumption until default. In this regard, our paper is also related to the literature that followed Miller and Modigliani (1961) and Radner and Shepp (1996). See Cadenillas et al. (2007) for references on models of dividend optimization.

## 2 Model

First we briefly summarize the model structure. Let  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  denote the probability space characterizing randomness in the economy where  $\mathbb{P}$  is the probability measure,  $\mathbb{F} = \{\mathcal{F}\}_{t \geq 0}$  is the filtration of the  $\sigma$ -field  $\mathcal{F}$ . The economy consists of a continuum of firms of mass one and a financial sector, represented by a bank. Firms face uncertainty in their cash flows which generates potential default risks. The carbon pricing policy is modeled as a shock to the balance sheet, where we explore various degrees of such shocks and their consequent effects on corporate default risks and transition dynamics. Thus, the model is purposefully kept simple in order to gain some new theoretical insights and explore their quantitative implications.

**Firms:** There is a continuum of firms of mass one indexed by  $f \in [0, 1]$ , heterogeneous in their cash flows  $Z^f$ . We assume that firms have the necessary production technology to generate these cash flows, which evolve as:

$$dZ_t^f = (\mu - \theta P \mu - \gamma)dt + \sigma dW_t^f \quad (1)$$

The diffusion term  $\sigma dW_t^f$ , with standard Brownian motion  $W_t^f$ , represents idiosyncratic shocks to cash flows with volatility  $\sigma$ .<sup>15</sup>  $\mu$  is the average income per-unit time, which can also be interpreted as the production output or equivalently, sales revenues (net of other operating costs).  $\theta$  is the emission intensity, i.e. the ratio of total carbon emissions ( $E$ ) to the output. The impact of carbon pricing policy is modelled as an additional cost (a carbon tax) where firms pay carbon price  $P$  for each unit of emission produced.

Thus, carbon emission is proportional to the output of the firm which is conceptually captured via  $E = \theta \mu$ . This formulation is also flexible as it also allows us to explore the impact of potential mitigation measures, defined as reduction in the emission intensity  $\theta$  since emissions are proportional to the production where  $\theta$  represents the proportionality constant. Reduction in the intensity could occur for a variety of reasons, such as adoption of greener technologies, technological progress, increased use of renewable energy etc. As such, any reduction in  $\theta$  implies less emission produced per-unit of output and reflects mitigation efforts.<sup>16</sup>

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<sup>15</sup>These Brownian shocks can also be interpreted as reflecting idiosyncratic productivity shocks drawn by the firms. As such, volatility  $\sigma$  captures the cross-sectional dispersion of these productivity shocks across firms, i.e.  $\sigma$  is the productivity dispersion.

<sup>16</sup>Since the main focus of the paper is exploring the impact on default risk and transition dynamics, the model implicitly assumes a constant technology for simplicity and tractability. As such, results of

The impact of carbon shock is captured by the carbon price parameter  $P$ . Since  $P = 0$  initially in the baseline, firms face no additional negative shocks due to their emissions as seen from the equation above. However, a positive price  $P$  set by policymakers acts as shock to the cash flows and consequently affects the endogenously determined net worth. And, by varying carbon prices  $P$ , we can trace out effects of various policy responses. Thus, the impact of carbon pricing policy shock is modeled in a simple and parsimonious way to study its implications in an analytically tractable way.

For notational simplicity, let us define  $\delta = 1 - \theta P$ . This means, for a given intensity  $\theta$ , the larger the carbon price shocks  $P$ , the lower the  $\delta$ . Then we have  $\mu - \theta P\mu = \mu(1 - \theta P) = \mu\delta$  and we can rewrite equation (1) more compactly as:

$$dZ_t^f = (\mu\delta - \gamma)dt + \sigma dW_t^f \quad (2)$$

Firms also have liabilities/debt expenses in their balance sheet, captured by  $\gamma$ , that must be paid each period. For example, this could be interest expenses on corporate debt, loans etc. Thus,  $\gamma$  can be interpreted as capturing average cost of debt capital. Higher  $\gamma$  would imply increasing funding costs and financing constraints faced by the firms. To see this concretely, note that we can represent  $\gamma = R \times B$  where  $B$  is the total debt and  $R$  is the interest rate, i.e. the cost of capital. As such, for a given borrowing level  $B$ , higher  $\gamma$  signifies increased funding costs and, thus, potentially higher risk premium/corporate spreads demanded by investors. The interpretation of  $\gamma$  as reflecting the cost of debt capital will be important later when we explore market responses that are consistent with the effects of direct carbon pricing policies, even if such policies are not yet implemented.

We can view the firms as being run by decision-making specialists/managers. They are insiders who represent firms' investors or shareholders.<sup>17</sup> Hence, consumption can also be interpreted as payout. Hereafter, we use the terms consumption and payout interchangeably.

Let  $C_t^f = \int_0^t c_u^f du$  denote cumulative payout up to time  $t$  where  $c_t^f$  is the payout rate.  $C_t^f$  is a non-negative, non-decreasing process. We make two assumptions motivated by reasonable practical considerations. First, firms pay dividends only when the net worth  $X_t$  reaches beyond a certain threshold level  $\bar{x}$ . Second, the payout rate is bounded above i.e.  $c_t^f \leq m < \infty$ .<sup>18</sup> These assumptions are also practical, as they reflect the fact that

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our experiment with reduction in  $\theta$  can also be seen as capturing the potential impact of changes in firm technology that help lower emission intensity.

<sup>17</sup>This is analogous to the setting in He and Krishnamurthy (2013) where specialists manage the intermediaries. Also, we abstract away from any potential principal-agent issues.

<sup>18</sup>While it is possible to derive the optimal  $\bar{x}$  we abstract away from such considerations given our goals

firms accumulate internal capital from retained earnings as buffers for exogeneous shocks and impose restrictions on dividend distributions, bonuses, share purchases etc., so as to reduce default risks. Thus, the firms in the model can also be differentiated as those paying dividends or not, allowing us to capture more layers of relevant heterogeneity.

Since heterogeneity of firms is now understood, for notational simplicity we drop the superscript “f” hereafter. Thus, firms maximize the expected discounted value of consumption by optimally choosing  $c_t$  and the amount of risk to control  $\bar{\alpha}_t$ . The optimal value function is defined as:

$$V(x) = \max_{\{c_t, \bar{\alpha}_t\}} \mathbb{E} \int_0^\infty e^{-\rho t} c_t dt \quad (3)$$

where  $\rho$  is the discount rate. Let  $X_t$  be firms’ net worth or, equivalently, equity. Then the net worth/dynamic budget constraint evolves as:

$$dX_t = (\bar{\alpha}_t(\mu\delta - \gamma) - c_t)dt + \bar{\alpha}_t\sigma dW_t \quad (4)$$

Before discussing  $\bar{\alpha}$  in equation (4) above, we first define the default or insolvency risk of the firms.<sup>19</sup> Let  $\tau$  denote the first passage time of net worth  $X_t$  going below the default threshold level zero:

$$\tau = \inf\{t > 0 : X_t \leq 0\}$$

As seen,  $\tau$  is the random time of default. If net worth reaches zero firms are declared insolvent and they exit the market (further discussion appears below).<sup>20</sup> This condition can be characterized by the failure probability  $\mathcal{P}(X_t \leq 0)$  and the distribution of first passage time  $\mathcal{P}(\tau < \infty)$ , both of which are the same here:

$$\mathcal{P}(X_t \leq 0) \equiv \mathcal{P}(\tau < \infty)$$

Hereafter, we use the terms default risk, insolvency risk, and failure probability interchangeably. Note that in general, if debt expense  $\gamma$  is greater than income  $\mu$  ( $\gamma > \mu$ ), then probability of default is one. Thus, we assume  $\mu > \gamma$  for practical purposes.<sup>21</sup>

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in this paper. Instead, we calibrate this as a parameter.

<sup>19</sup>We do not consider jumps in the net worth evolution, because jumps would eventually give us integro-differential equations where clean closed-form solutions to default risk and transition dynamics become difficult. Thus, we work with a diffusion process without jumps to simplify the analysis.

<sup>20</sup>We abstract away from strategic default decisions, i.e.  $\tau$  is not endogenous; instead it depends on other optimal decisions and shocks.

<sup>21</sup>Intuitively, if the expected income per-unit time is less than the debt expenses then the initial value is

As a simple extension, we also consider the case where firms can reduce their insolvency risks by directly reducing  $\sigma dW_t$  which captures the total risk to their balance sheets. To achieve this, firms can shut down a certain portion of their business, for example, by cutting off some production or laying off workers etc. However, shutting down production implies that firms also need to give up an equivalent fraction of net income  $\alpha_t(\mu - \gamma)$ . This is indeed equivalent to keeping a fraction  $\bar{\alpha}_t(\mu - \gamma)$  of the business income along with the remaining risk  $\bar{\alpha}_t\sigma dW_t$ .<sup>22</sup> Alternatively stated, firms keep  $\bar{\alpha}_t Z_t$  of the stochastic cash flow of equation (2). This is a *proportional risk-reduction mechanism* because the net cash flows and risks are reduced in the same proportion.<sup>23</sup> Thus, firms reduce their risk exposures to  $(1 - \alpha_t)\sigma dW_t = \bar{\alpha}_t\sigma dW_t$  which also entails giving up corresponding income flows and retaining  $\bar{\alpha}_t(\mu - \gamma)$ .

The optimal problem is solved via a standard continuous time dynamic programming approach using the Hamilton-Jacobi-Bellman (HJB) equation which characterizes optimal decision (3) for the dynamic budget constraint (4) as:

$$\rho V(x) = \max_{\{c, \bar{\alpha}\}} c + \eta(x)V'(x) + \frac{1}{2}\hat{\sigma}^2 V''(x) \quad (5)$$

where  $\eta(x) = (\bar{\alpha}(\mu\delta - \gamma) - c)$  is the optimally chosen drift,  $\hat{\sigma} = \bar{\alpha}\sigma$ .

Let  $g_t(x)$  denote the time  $t$  cross-sectional distribution of firms over their net worth and let  $\bar{g}(x)$  denote its ergodic distribution. Let  $\phi$  denote the *default rate*, i.e. the average rate at which firms become insolvent. Since our notion of default/insolvency is similar to that of bankruptcy, we assume that after insolvency, firms leave the market (or equivalently die out). Thus,  $\phi$  captures the average outflow rate of firms or, equivalently, the mass of firms that become insolvent at each instant. Because the mass of firms is normalized to one,  $\phi$  is also the rate at which new firms enter the market, i.e. the birth rate.<sup>24</sup> Hence, every insolvent firm is replaced by a new firm drawn from  $\bar{g}(x)$ .<sup>25</sup>

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essentially the maximum attainable value. Hence, it is optimal to consume everything immediately, implying  $\tau = 0$  and immediate default.

<sup>22</sup>Here we are assuming that as the firms downsize, they concurrently reduce their income  $\mu$  and also associated expenses  $\gamma$ . This is a reasonable assumption, as we consider firms shutting down a portion of their businesses, which generally entails reduction in both income as well as expenses.

<sup>23</sup>An alternative interpretation is that the firm liquidates/sells a portion of business in a proportional manner. However, the interpretation of production cuts is more natural in our setting considering the carbon price shocks.

<sup>24</sup>It is also possible to assume that insolvent firms are revived as shareholders inject outside capital. However, market exit appears a more relevant assumption.

<sup>25</sup>More formally, default rate  $\phi$  is also the *killing rate* of the diffusion process in equation (4). For some random (default) time  $\tau$ ,  $\hat{X}$  is a diffusion process with killing rate  $\phi$  such that  $\hat{X} = X_t$  for  $t < \tau$ . For  $t \geq \tau$ , we can either leave  $\hat{X}$  undefined (since it is killed) or we can define  $\hat{X} = \partial$  where  $\partial$  is the *cemetery state*, i.e.

Finally, the evolution of the net worth distribution  $g_t(x)$  is characterized by the following Kolmogorov Forward (KF) equation:

$$\partial_t g_t(x) = \mathcal{L}^* g(x) + \phi \bar{g}(x) \quad (6)$$

where  $\mathcal{L}^* g(x) = -\partial_x(\eta(x)g_t(x)) + \frac{1}{2}\partial_{xx}(g_t(x)\hat{\sigma}^2) - \phi g_t(x)$  is the Kolmogorov forward operator. At each time  $t$ , the state of the economy can be described by the distribution of net worth.

**Financial sector:** We now outline the financial sector's problem which is represented by a bank. For completeness and to highlight that our framework can be easily used to study financial sector risks as well, we present a general structure of the financial sector's problem. However, we mainly consider it to motivate the source of corporate lending and cost of capital related to  $\gamma$  in the firm's problem and to close the model.

The bank also maximizes its expected discounted value of payout by optimally choosing  $c_t^b$ . Given this choice, the amount allocated to investing by lending to the firms is also known. The optimal value function is:

$$V(x^b) = \max_{c_t^b} \mathbb{E} \int_0^\infty e^{-\rho^b t} c_t^b dt \quad (7)$$

where  $\rho^b$  is the discount rate. Bank's net worth  $X_t^b$  evolves as follows:

$$dX_t^b = (\tilde{\mu} + RX_t^b - c_t^b)dt + \tilde{\sigma}dW_t^b \quad (8)$$

where  $RX_t^b$  denotes the income from lending to the firms at the interest rate  $R$  and  $\tilde{\mu}$  denotes income from all other sources (e.g. from loans, credit lines, mortgages to households). The bank also has its own balance sheet risk  $\tilde{\sigma}dW_t^b$  where  $\tilde{\sigma}$  is the volatility (and  $dW_t^b$  is independent of  $dW_t$ ). This could arise from various sources (e.g. household defaults on loans and mortgages) and allows us to consider potential insolvency risk in the financial sector.

The general structure above also shows that interactions between the financial sector and the corporate sector, including potential feedback effects from firm default, can also be studied by further extension of our model. However, given our focus on corporate default risks we largely abstract away from this consideration.<sup>26</sup> Thus, the bank's optimal problem

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the state of default in our case. This means the net worth process  $X_t$  is sent to the cemetery state  $\partial$  at the first hitting time  $\tau$  when  $X_t \leq 0$ . Then,  $\mathcal{L}^* g(x)$  is the forward operator of the diffusion process with killing rate  $\phi$ . See Oksendal (2002) for more details.

<sup>26</sup>More concretely, we can assume that the financial sector is implicitly guaranteed by the government such that it does not go bankrupt.



is also solved via HJB equation as:

$$\rho V(x^b) = \max_{c^b} c^b + (\psi + \tilde{\mu} - c^b)V'(x^b) + \frac{1}{2}\tilde{\sigma}^2V''(x^b) \quad (9)$$

**Equilibrium:** The economy can be closed in a simple way in our model. The equilibrium simply equalizes the total corporate-based income of the banks and the total debt expenses of the firms  $\gamma$ . Let  $\mathbb{T}^f = -\mathbb{E}[\bar{\alpha}\gamma]$  denote aggregate corporate debt expenses. Recall that  $\bar{\alpha}$  is a function of net worth  $x$ . Let  $\mathbb{T}^b$  denote total bank income from corporate lending. Then, in equilibrium, the borrowing and lending market clears, i.e.  $\mathbb{T}^b + \mathbb{T}^f = 0$ , and the decisions of the bank and firms are optimal.

We take this simplified approach since our focus is not on exploring the equilibrium dynamics of borrowing/lending via  $R$ , but on the corporate default risks given carbon price shocks. A benefit of this simplification is that we can consider various comparative statics exercises with the cost of capital  $R$ . For example, once we know how carbon price shocks of varying magnitudes affect the default risks, we can explore changes in the cost of capital  $R$  that is consistent with and reflects the heightened corporate risks from potential carbon policies/shocks, even if such policies are not yet implemented. This would reflect potential impact on default risks even in the absence of explicit carbon price shocks, due to forward-looking financial sector/investors anticipating increased future risks. Section 3, on results, presents detailed discussions.

To summarize, given the focus of this paper, we make simplifying assumptions, as above, to consider the partial equilibrium setting where firms optimize their payouts (and also control total risk), given carbon price shocks. However, the general structure presented shows that our model can also be extended in future research to study other financial stability-related issues stemming from equilibrium borrowing and lending interactions between the firms and banks and potential feedback effects.

## 2.1 Calibration

We calibrate the model on two broad sectors of the US economy: (i) the transportation sector; and (ii) the manufacturing sector. Transportation belongs to the Standard Industrial Classification (SIC) industry group 5 with SIC codes between 4000 and 4800. Manufacturing belongs to SIC industry group 3 with SIC codes between 2000 and 4000. As discussed above, these sectors are chosen largely to illustrate the application of our model, as both sectors emit a large share of aggregate GHG emissions in the US but also have heterogeneous

emission intensities (emissions per-unit output). Additionally, the choice is also motivated by some limitations on the availability of emissions data. For example, self-reported firm level emissions disclosure data is not yet comprehensive and/or reliable. But the estimates of aggregated emissions at the sector level are generally more reliable and comprehensive. And to emphasize the broad applicability of our model, we only use publicly available GHG emissions data (see details below). A limitation is that publicly available sources may not necessarily contain disaggregated GHG emissions for all the various (sub) sectors of the economy. Nevertheless, reliable disaggregated emissions data for these two broad sectors are easily available, together with corresponding sectoral output data needed to construct emission intensities. Regardless, we emphasize that the model can be directly applied following the calibration steps discussed below to any sector (or even at the firm level) given data reliability and availability.<sup>27</sup>

**Corporate Data:** Corporate data is obtained from Datastream and we focus on the asset-weighted averages of the relevant variables for the 2012-19 sample period. Since climate regulation and carbon policies gained strong momentum since the past decade, focusing on the data averages over this period is appropriate for calibrating the baseline, which represents the pre-shock steady state. Once the baseline is established, we then simply apply carbon price shocks as per the equations in the model. The main corporate variables used are: earnings before interest and taxes (EBIT), total shareholder’s equity, interest expenses, dividend payout ratio, and total debt. To make comparisons easier, we normalize the net worth  $X$ , which is a proxy for the total shareholder’s equity, to lie on a unit interval  $[0, 1]$ . This also scales down all other variables accordingly while preserving the heterogeneity across net worth. As such default risk  $\mathcal{P}$  calculation is not affected by this normalization. We also winsorize these variables at 5% and 95% to remove the effect of outliers.

Given this normalization, we then obtain the model parameters  $\{\mu, \gamma, \sigma, m, \bar{x}, \delta\}$  where asset-weighted averages are computed each year and then averaged.  $\mu$  is the average EBIT for each sector. EBIT is a proxy for free cash flow and excludes the impact of capital structure on profits. Further, since EBIT represents operating profit, it embeds revenues and cost

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<sup>27</sup>Note that our calibrations and ensuing results are based on the economic structure reflected in the US data. However, the model can also be calibrated to other economies, from advanced to emerging/developing. Consequently, the impact across different countries would likely exhibit different patterns depending various factors such as economic structure, corporate balance sheet composition, differing emission intensities across sectors. Such exhaustive cross country study could be a new avenue for future research where a rich set of market based data are generally scarce but our model could still be applied to uniformly quantify default risks.

of goods sold.<sup>28</sup> This means that once appropriate carbon price shocks are calibrated via  $\delta$  the impact of shocks can be directly analyzed/interpreted as carbon tax or, equivalently, additional cost of carbon emissions.

To calibrate  $\gamma$ , we first calculate the average/effective interest rates  $R$  of the the firms. In the data, this is simply the ratio of total interest expenses to total debt  $B$ :  $R = \frac{\text{Interest Expenses}}{\text{Total Debt}}$  such that  $\gamma = R \times B$ . This approach to calculating  $\gamma$  allows us to later consider the implicit increase in cost of debt capital that would generate the same impact as the equivalent carbon price shocks. The payout rate  $m$  is calibrated to match the average fraction of EBIT paid as dividends, i.e.  $m = \text{average dividend payout ratio} \times \mu$ . And  $\bar{x}$  is the average shareholder equity value of the companies that paid positive dividends.

The volatility parameter  $\sigma$  is calibrated to match the average fraction of firms in the sample period whose interest coverage ratio ( $ICR = \text{EBIT}/\text{Interest Expenses}$ ) falls below the critical threshold of one ( $ICR < 1$ ). Note that  $ICR$  is a widely used solvency/debt ratio for corporate risk assessment. It is considered a good proxy for the ability of a firm to service its debt. The lower the  $ICR$  ratio, the greater the default risk. And  $ICR < 1$  generally implies financial distress and the potential for imminent default, because the firm may be unable to cover debt expenses from its earnings.<sup>29</sup> As such, the average fraction of firms with  $ICR < 1$  in a given sector over our sample period should be a good proxy for the sectoral default risk. These default rates are 16% and 36% for the transportation and manufacturing sectors, respectively.<sup>30</sup> Thus, using the parameters  $\{\mu, \gamma, \bar{x}, \delta\}$  and our closed-form solution of the default risk  $\mathbb{P}$  (equations (10) and (11) in section 3 results), we calculate optimal  $\sigma$  to match the default rate proxies of the corresponding sectors. Table 1 shows all the calibrated parameters discussed above.

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<sup>28</sup>Note that the GDP of a given sector (or economy) is also conceptually analogous to the sum of the EBIT across all the firms since both are value added concepts.

<sup>29</sup>This is also consistent with our model where  $\mu < \gamma$  implies immediate default because earnings are no longer able to meet the debt expenses. Thus, our calibration of  $\mu$  and  $\gamma$  against EBIT and interest expenses, respectively, also captures the risk via  $ICR$ .

<sup>30</sup>Note that the model could also be calibrated using historical bankruptcy rates. However, such a data, especially across various granularity of sectoral classifications, is generally not readily available. Chava and Jarrow (2004) have computed historical bankruptcy rates over the period 1962-99 period for one-digit SIC industry groups, which combines transportation, communications, and utilities sub-sectors into a single SIC industry group 5. Such an aggregation makes it hard to apply it to our case. Nevertheless, Chava and Jarrow (2004) document a historical bankruptcy rate of 37% in the manufacturing sector. Interestingly, this is close to the proxy of the manufacturing sector's default rate of 36% from our calculations. Further, as we show later, our results are robust to this calibration since the *changes* in default risks, the key object of interest, are largely driven by carbon emissions and price shocks.

Table 1: Calibrated Parameters for Two Sectors (Baseline)

| Parameters | Transportation | Manufacturing |
|------------|----------------|---------------|
| $\mu$      | 0.1615         | 0.1350        |
| $\gamma$   | 0.0250         | 0.0183        |
| $\sigma$   | 0.1977         | 0.2886        |
| $m$        | 0.0344         | 0.0140        |
| $\bar{x}$  | 0.2738         | 0.2578        |

Note: Calibrated parameter values using the US corporate data for the two economic sectors over the 2012-19 sample period.

**Carbon emissions and price shocks:** As seen from equation (1), the carbon price  $P$  acts as a shock to firm earnings in the form of additional costs via  $\delta$ . As discussed earlier, there is some uncertainty as to what the price of carbon emissions should be in order to achieve the climate change goals. Additionally, there is uncertainty regarding the magnitude of emissions reduction required to limit global warming. As such, central banks and policy organizations around the world consider a certain range of carbon price per tCO<sub>2</sub>e. For example, the HLCCP (2017) recommended a carbon price range of \$40-\$80 by 2020 and \$50-\$100 by 2030 to achieve the Paris Agreement temperature targets. The Biden administration in the US has calculated \$51 per tCO<sub>2</sub>e as the interim value of the social cost of carbon (IWG 2021). Similarly, the Network for Greening the Financial System (NGFS 2020), a consortium of central banks around the world, has developed various scenarios of carbon price paths consistent with given temperature and emissions targets. Given the above motivation on the uncertainty regarding carbon regulation and the consequent prices, we consider three carbon price shocks: \$25, \$50, and \$75. These shocks allow us to explore a range of future possibilities and quantify the impact of climate-related policy.<sup>31</sup>

We obtain emissions data for the transportation and manufacturing sectors from the publicly available data source by Ritchie and Roser (2020). The GHGs are generally reported in standardized measurement units as tons of carbon dioxide equivalents (tCO<sub>2</sub>e). Carbon prices are also generally set in dollar price per-unit tCO<sub>2</sub>e.<sup>32</sup> For each year during the period

<sup>31</sup>While one could also explore stochastic paths of carbon prices, it would render the model analytically tedious and also would not lead to particularly stronger insights on default risks than what is already obtained. Since the uncertainty is regarding the expected range of future carbon prices, our consideration of a plausible range of carbon prices representing various degrees of shocks in a parsimonious stochastic model already captures the potential impact associated with various possible carbon prices.

<sup>32</sup>The standardized measurement of tCO<sub>2</sub>e accounts for the differential warming effects from various GHGs, i.e. the amount of greenhouse gas that leads to same equivalent warming as a ton of CO<sub>2</sub>.

Table 2: Calibration of Carbon Shocks.

|                | Baseline (no shock)                  | \$25 shock                    | \$50 shock                    | \$75 shock                    |
|----------------|--------------------------------------|-------------------------------|-------------------------------|-------------------------------|
| Transportation | $\delta = 1$<br>( $\theta P = 0\%$ ) | $\delta = 0.92$<br>(-8%)      | $\delta = 0.84$<br>(-16%)     | $\delta = 0.76$<br>(-24%)     |
| Manufacturing  | $\delta = 1$<br>( $\theta P = 0\%$ ) | $\delta = 0.9948$<br>(-0.52%) | $\delta = 0.9896$<br>(-1.04%) | $\delta = 0.9844$<br>(-1.56%) |

Note: Calibrated values of  $\delta = 1 - \theta P$  reflecting shocks under different carbon prices.

2012-19, we also obtain the industry-level GDP data from the Bureau of Economic Analysis (BEA), which represent an industry’s contribution to the total GDP and is a value-added metric. The emissions and the output of the industry allow us to construct carbon emission intensity for each industry, i.e. emissions per-unit of output for each year. We compute the average of such carbon intensities over our sample period to proxy for the emission intensity  $\theta$  of the two sectors.<sup>33</sup>

Table 2 shows the calibrated values of  $\delta$  and  $\theta P$  for each sector and each carbon price shock. As seen, the transportation sector faces large shocks relative to the manufacturing sector. This is directly driven by heterogeneous carbon emissions/intensity across these sectors given the same carbon prices.<sup>34</sup>

### 3 Results

This section presents our results. In section 3.1 we discuss implications of carbon shocks on corporate default risks. Section 3.2 discusses implicit impact via cost of capital. Section 3.3 discusses the transition dynamics and section 4 concludes. All proofs are contained in the appendix.

<sup>33</sup>The implicit assumption here is that the emission intensity for firms in a given sector is the same. This means, given the same emission intensity, the cost of emissions for firms in the same sector is proportional to the firm-level output/earnings. This is a reasonable assumption and approach to calculating cost of emissions, especially given the current lack of reliable/complete disclosures or estimates of historical emissions at the firm level. As such, we calibrate the parameters at the sector level, where it is relatively easy to obtain the total GHG emissions and sectoral emission intensity as discussed above. Nevertheless, our model can also be directly calibrated to the individual firm level whenever data allows.

<sup>34</sup>The ratio of emission intensity in the transportation sector to manufacturing sector is about 15, i.e. transportation sector’s emission intensity is 15 times higher than that of the manufacturing sector. Also note that any mitigation measures to reduce emission intensity  $\theta$  would lessen the impact of shock  $\delta$  by the same proportion.

### 3.1 Carbon Price Shocks and Default Risks

We first provide a closed-form solution to the default risk  $\mathcal{P}(X_t \leq 0)$  and illustrate the heterogeneous effects of the shocks across and within sectors. After that, we also discuss how firms might reduce their business risk (or equivalently, downsize the business), given the carbon shocks.

**Proposition 1. (Default Risk):** *Let  $c^*$  be optimal consumption payout rate. For some net worth level  $\bar{x}$ ,  $c^* = 0$  whenever  $x \leq \bar{x}$  and  $c^* = m$  whenever  $x > \bar{x}$ . Then, the default probability  $\mathcal{P}(X_t \leq 0) = \mathcal{P}$  for firms with net worth  $x \leq \bar{x}$  is:*

$$\mathcal{P} = \frac{m + (\mu\delta - \gamma - m)e^{\frac{2(\mu\delta - \gamma)}{\sigma^2}(\bar{x} - x)}}{m + (\mu\delta - \gamma - m)e^{\frac{2(\mu\delta - \gamma)}{\sigma^2}\bar{x}}} \quad (10)$$

*The default probability  $\mathcal{P}$  for firms with net worth  $x > \bar{x}$  is:*

$$\mathcal{P} = \frac{(\mu\delta - \gamma)e^{-\frac{2(\mu\delta - \gamma - m)(x - \bar{x})}{\sigma^2}}}{m + (\mu\delta - \gamma - m)e^{\frac{2(\mu\delta - \gamma)}{\sigma^2}\bar{x}}} \quad (11)$$

This result sharply captures various economic factors and carbon price shocks that drive corporate default risks. Simple observation shows that firms with higher (lower) net worth  $x$  face a lower (higher) level of failure probability  $\mathcal{P}$ , as expected. Further, if  $m \rightarrow (\mu\delta - \gamma)$ , then  $\mathcal{P} \rightarrow 1$ . This means that the maximal consumption rate cannot exceed the available funds  $(\mu\delta - \gamma)$  since that would lead to the failure probability of one. We now discuss various implications of carbon price shocks arising directly from the above default risk equations.

Figure 1 shows the effects of carbon price shocks of \$25, \$50, and \$75 on the default risk  $\mathcal{P}$  for the transportation sector. The blue solid line shows the risks for baseline calibration corresponding to the pre-shock steady state. As seen, the larger the carbon price shock, the higher the default risk. However, note the highly non-linear effect of carbon price shocks on default risks across the firms with varying levels of net worth. Under \$25 shock the effect is mild. However, under \$75 shock, even the high net worth firms in the right tail of the distribution show a non-trivial increase in default risks.

Figure 2 shows the density of the default rates, i.e. the distribution of the failure probabilities  $\mathcal{P}$  corresponding to four different cases in Figure 1. As seen, with increasingly larger shocks, the distribution begins shifting to the right. In particular, under \$75 shock, the shift in distribution is significant with a large increase of mass around the right tail of the distribution. This implies a significant increase in the fraction of firms facing high default

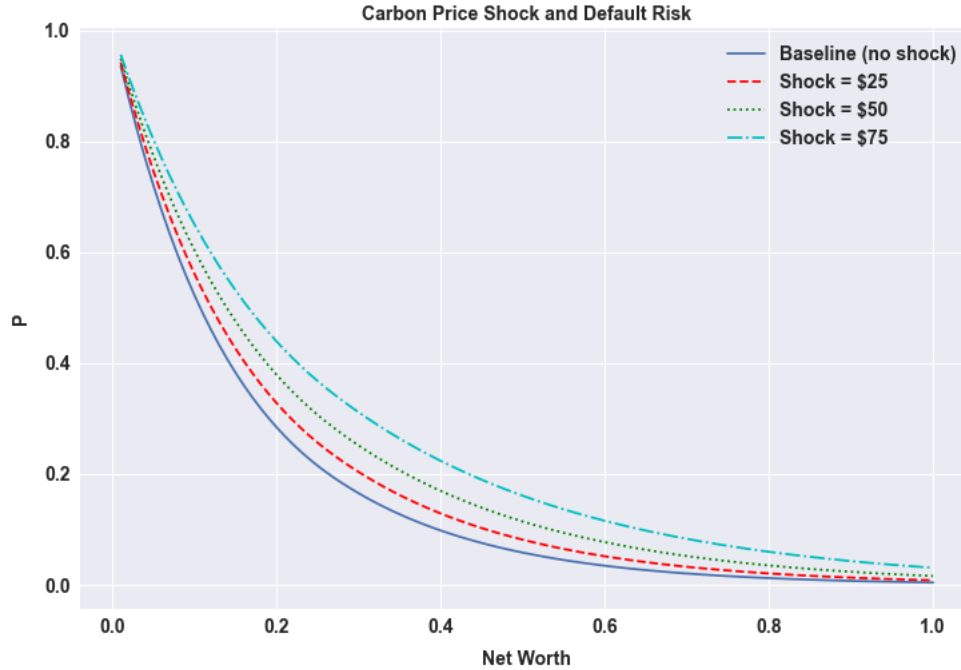


Figure 1: Default Risk in the Transportation Sector from Carbon Price Shocks

risks. Further, this sharply highlights the asymmetric and non-linear impact of carbon price shocks.

Table 3 shows the magnitude of these shocks to the average industry default rates as well to for the bottom and top 10% of the firms in the transportation sector. Panel A shows the average baseline default risk where 16% is the proxy calibrated as the pre-shock steady state described before. Recall that the key object of interest is *changes* in the default rates relative to the baseline where carbon price shocks are the main drivers of variation in default rates for a given sector.

The three columns show the new default rates. Given carbon shocks of \$25, \$50, and \$75, the average default rates increase by 2.66%, 5.21%, and 9.17%, respectively. This shows a substantial increase in failure risk in the transportation sector. While a large impact is expected given the already high emissions profile of the firms in this sector, our model is able to capture the highly non-linear increase in default risk that would otherwise be ignored in linear approximations.

We know that default risks across firms in a given sector will be different based on their net worth. As such, it is important to consider within sectoral heterogeneity also to better understand the differential effects of carbon price shocks. Table 3, Panel B shows

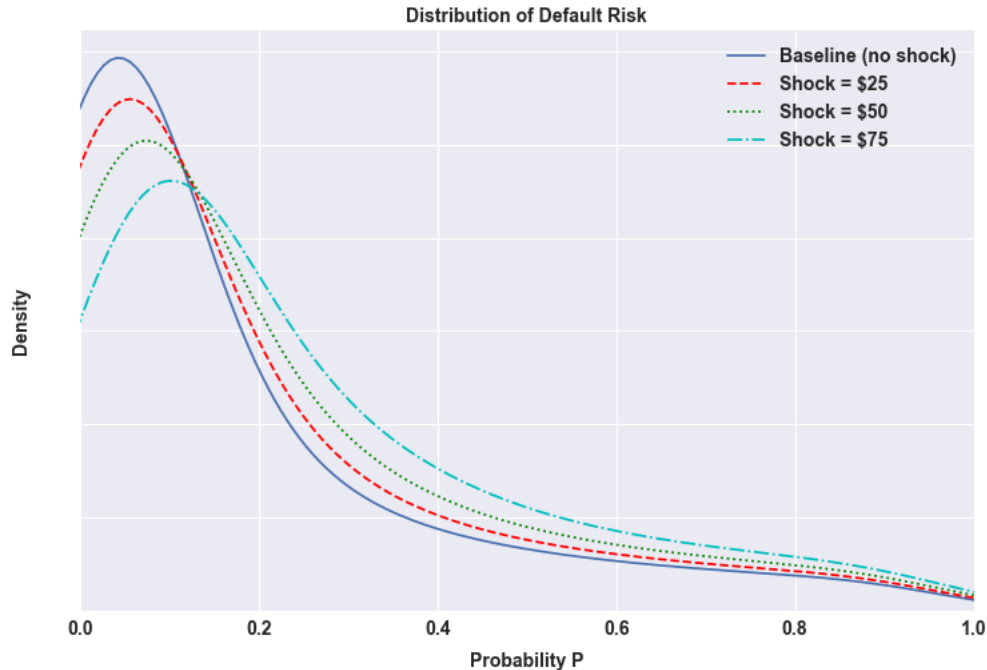


Figure 2: Distribution of Default Rates in the Transportation Sector

such differential effects. As seen, in the baseline, the bottom 10% of the firms (based on net worth) have higher default rates (70.98%) relative to the top 10% of the firms (0.55%). Under shocks of \$25, \$50, and \$75, the average default rates of the bottom 10% of the firms increase by 2.58%, 5.34%, and 8.31%, respectively. In comparison, the average default rates of the top 10% of the firms increases by 0.47%, 1.37%, and 3.04%, respectively. This shows the *asymmetric* impact of the carbon shocks within a sector. As the carbon shocks become larger, even the top 10% of the firms face a significant increase in default risk, as seen by the 3% rise in failure risk under the \$75 shock. Thus, even the large firms with a large equity capital cushion could find themselves in increased financial distress. This is also sharply seen in Figure 1 with a significant increase in the mass around the right tail of the default curve under the \$75 shock. To reiterate, this result highlights the importance of understanding the heterogeneous impact within sectors from the carbon pricing policies.<sup>35</sup>

We now discuss the implications of a possible mitigation measure, defined as the potential

<sup>35</sup>We emphasize that the default rate levels for the bottom and top 10% of the firms are model implied, unlike the sector level default rates used for calibrating volatility parameters. Considering low-net-worth firms will have higher default rates relative to the high-net-worth firm by construction, this again shows the importance of focusing on the changes induced by carbon shocks for isolating the impacts across various parts of the distribution.



Table 3: Carbon Price Shocks and Default Risks: The Transportation Sector

|   | Baseline (no shock) | \$25 shock | \$50 shock | \$75 shock |
|---|---------------------|------------|------------|------------|
| Panel A: Average Effect                         |                     |            |            |            |
| $\mathcal{P}$                                   | 0.16                | 0.1826     | 0.2121     | 0.2517     |
| $\Delta\mathcal{P}$                             |                     | 0.0266     | 0.0521     | 0.0917     |
| $\Delta\mathcal{P}$ with $\Delta\theta = -10\%$ |                     | 0.0201     | 0.0456     | 0.0785     |
| Panel B: Heterogeneous Effects                  |                     |            |            |            |
| Bottom 10% of Firms                             | 0.7098              | 0.7356     | 0.7632     | 0.7929     |
| $\Delta\mathcal{P}$                             |                     | 0.0258     | 0.0534     | 0.0831     |
| $\Delta\mathcal{P}$ with $\Delta\theta = -10\%$ |                     | 0.0231     | 0.0477     | 0.0740     |
| Top 10% of Firms                                | 0.0055              | 0.0102     | 0.0191     | 0.0359     |
| $\Delta\mathcal{P}$                             |                     | 0.0047     | 0.0137     | 0.0304     |
| $\Delta\mathcal{P}$ with $\Delta\theta = -10\%$ |                     | 0.0041     | 0.0114     | 0.0242     |

Note: This table presents the effects of carbon price shocks on default risks for the transportation sector using equations (10) and (11). Panel A shows the average effects and Panel B shows the heterogeneous effects across firms.  $\Delta\mathcal{P}$  shows the changes in default risk.

reduction in emission intensity, i.e. less carbon emissions per-unit output. This could occur as firms adopt new/greener technologies, change their production structure etc. Since we do not know the degree to which firms might potentially adapt to reduce their emission intensity we consider a 10% reduction in emission intensity  $\theta$  to simplify the analysis, i.e.  $\Delta\theta = -10\%$ . Considering the percentage reduction in emission intensity, as opposed to the total emissions, also allows us to compare the outcome across sectors.

Table 3 also shows corresponding updated default risks in the transportation sector. The default risk declines from {2.66%, 5.21%, and 9.17%} to {2.01%, 4.56%, and 7.85%} after reducing emission intensity  $\theta$  by 10%. As seen, even after a large reduction in intensity, the corresponding reduction in default risk is not as material, except under the \$75 carbon price. Equivalent effects for the top and bottom 10% of the firms are also shown in the table, revealing similar patterns. All these findings again highlight notable asymmetric effects within the sector despite the same set of carbon policy shocks and same percentage reduction in emission intensity.

We now discuss the impact of carbon price shocks on the manufacturing sector shown in Table 4. Panel A shows that under shocks of \$25, \$50, and \$75, the average default rates

Table 4: Carbon Price Shocks and Default Risks: The Manufacturing Sector

|   | Baseline (no shock) | \$25 shock | \$50 shock | \$75 shock |
|---|---------------------|------------|------------|------------|
| Panel A: Average Effect                         |                     |            |            |            |
| $\mathcal{P}$                                   | 0.36                | 0.3620     | 0.3639     | 0.3659     |
| $\Delta\mathcal{P}$                             |                     | 0.0020     | 0.0039     | 0.0059     |
| $\Delta\mathcal{P}$ with $\Delta\theta = -10\%$ |                     | 0.0018     | 0.0035     | 0.0053     |
| Panel B: Heterogeneous Effects                  |                     |            |            |            |
| Bottom 10% of Firms                             | 0.8687              | 0.8695     | 0.8703     | 0.8711     |
| $\Delta\mathcal{P}$                             |                     | 0.0008     | 0.0016     | 0.0024     |
| $\Delta\mathcal{P}$ with $\Delta\theta = -10\%$ |                     | 0.0007     | 0.0014     | 0.0022     |
| Top 10% of Firms                                | 0.0931              | 0.0947     | 0.0962     | 0.0978     |
| $\Delta\mathcal{P}$                             |                     | 0.0015     | 0.0031     | 0.0046     |
| $\Delta\mathcal{P}$ with $\Delta\theta = -10\%$ |                     | 0.0014     | 0.0027     | 0.0041     |

This table presents the effects of carbon price shocks on default risks for the manufacturing sector using equations (10) and (11). Panel A shows the average effects and Panel B shows the heterogeneous effects across firms.  $\Delta\mathcal{P}$  shows the changes in default risk.

increase by 0.20%, 0.39%, and 0.59%, respectively. These are quite mild compared with the average increases in the transportation sector as seen in Table 3. Further, under shocks of \$25, \$50, and \$75, the average default rates of the bottom 10% of the firms increase by 0.08%, 0.16%, and 0.24%, respectively. The average default rates of the top 10% of the firms increase by 0.15%, 0.31%, and 0.46%, respectively. As seen, while the default risk levels are lower for the top 10% of the firms, the *changes* in the default rates induced are slightly larger. This is due to the non-linear nature of the probability, as seen in equations (10) and (11). And, despite the low magnitude of changes in default rates relative to that in the transportation sector, the heterogeneity within the sector remains.

The impact of mitigation measures to reduce emission intensity by the manufacturing sector is also seen in Table 4. Such measures seem to have marginal impact on reducing the default risk relative to the significant impact in the transportation sector. This indicates that the manufacturing sector might have less incentive to significantly reduce emissions given the relatively small impact on the default risks due to additional mitigation measures.<sup>36</sup>

<sup>36</sup>We abstract away from the issue of costs of mitigation measures. However, such a small impact on the default risk from mitigation measures raises the question whether the cost of mitigation measure would be

The above results for the transportation and manufacturing sectors sharply highlight how carbon price shocks affect firms differently across and within sectors.<sup>37</sup> We see that implementation of direct carbon tax policies uniformly across the different segments of the economy can lead to starkly heterogeneous responses across and within economic sectors. Further, given the low impact on the overall default risks, the manufacturing sector might not be as sensitive to carbon price policies (relative to the transportation sector). As such, the sector might have weak incentive to reduce emission intensity in the long run. This calls for policies that might need to be more targeted considering inter-sectoral and intra-sectoral heterogeneity. The total volume of GHG emissions, irrespective of sectoral origin, affects the global warming outcomes. Hence, if certain segments of the economy, despite emitting a significant portion of emissions (in absolute terms), are not as sensitive to carbon pricing policies, as measured by changes induced in the default risks, policymakers might need to explore other measures and incentives to reduce emissions from such sectors.

In the discussions above, we abstracted away from the optimal risk reduction/downsizing decision of the firms by setting  $\bar{\alpha} = 1$ . We now briefly discuss how carbon shocks might affect the decision to downsize the business to reduce default risks. As shown below, in our model, the downsizing and consequently the implied production/emissions cuts are mostly driven by firms with lower net worth because they face the highest risk of default.

**Proposition 2. (*Optimal Risk Reduction*):** *For a firm with net worth  $x$  the optimal risk reduction and hence downsizing is given by  $\alpha_t = 1 - \bar{\alpha}_t$  where*

$$\bar{\alpha} = \frac{(\mu\delta - \gamma)x}{\sigma^2(1 - \beta)} \quad (12)$$

for some constant  $\beta \in (0, 1)$ .

First, notice that the optimal risk reduction is linear in net worth  $x$ . As net worth increases  $\bar{\alpha}$  also increases linearly, implying that firms optimally choose to keep more risks on their balance sheets and hence not downsize or cut back on their production as much. This is intuitive because with higher net worth, firms are able to withstand more risks. As net worth rises to some critical level  $\hat{x} = \frac{\sigma^2(1-\beta)}{(\mu\delta-\gamma)}$  firms retain all the risks on their balance

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greater than accepting the implications of a small increase in default risks. We leave this another issue to the future research.

<sup>37</sup>We can also view the results in the manufacturing sector as a robustness check on the results in the transportation sector (or vice versa). Despite the high initial (baseline) default rate of 36.0% relative to 16.0% in the transportation sector, the overall *increase* in default risk due to the carbon shock is relatively mild compared with those in the transportation sector in Table 3. This shows that the results are indeed driven by the carbon shocks and not by initial high/low levels of sectoral default risks.

sheets, i.e.  $\bar{\alpha} \rightarrow 1$  as  $x \rightarrow \hat{x}$ . Consequently, low net worth firms optimally choose to downsize relatively more to avoid default and maximize their lifetime value.

As seen from equation (12), if carbon shock  $\delta$  is larger,  $\bar{\alpha}$  is lower and  $\alpha$  is higher. This implies that firms facing a higher impact from carbon shocks must downsize their business or, equivalently, cut back on production and associate risks to avoid default. Since reduced production is directly proportional to reduction in emissions, the carbon shocks do result in lower emissions. However, this is not without costs because it entails reduced production.<sup>38</sup>

This result also highlights an important limitation of direct carbon price policies. While carbon price shocks  $\delta$  does induce emission reductions, our calibrations show that the policy impact could be limited because it might largely affect the low net worth firms, at least initially. This is because firms with net worth beyond the threshold  $\hat{x} = \frac{\sigma^2(1-\beta)}{(\mu\delta-\gamma)}$  do not cut back on production and hence do not contribute much towards reducing total emissions. Further, even if low-net-worth firms drastically reduce their production/emissions, their total carbon footprint might still be relatively small compared with those of the high-net-worth firms.<sup>39</sup> As such, the uniform carbon pricing policy might not deliver the desired reduction in emissions. The implication is that the direct carbon price policies need to be more targeted and need to also account for the size of the firms so that low-net-worth firms do not face undue burden. Fully accounting for implications of carbon policies to production and emissions is beyond the scope of this paper. Nevertheless, our results point to potential limitations of the carbon pricing policy so as to inform better climate-related policy design.

### 3.2 Impact on Cost of Capital

While direct carbon pricing is one of the main policy tools proposed for limiting GHG emissions and moving towards a low-carbon economy, risks might also materialize from other channels. For example, the financial sector/investors might also price in a potential increase in corporate risks from carbon pricing policies, even if such policies are not yet implemented. Investors might re-assess borrowing costs of various industries due to increased insolvency risks from potential climate policies, including carbon pricing policy. In other words, changes in investor sentiment could lead to a relatively higher cost of capital or risk premium in

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<sup>38</sup>These reductions in productions could lead to lower economic growth as well. Even though the formal analysis of such an impact on the overall economy/productions is outside our model, the inference is straightforward.

<sup>39</sup>This can also be seen from the definition of carbon intensity  $\theta$  as emissions per-unit output. The total emissions of a given firm can be approximated by multiplying  $\theta$  by a suitable firm-level measure of output/income. This approximation of total emissions should generally be higher, on average, for large-net-worth firms.

emission-intensive sectors, resulting in reduced investment flows as investors demand more premium to compensate for the high exposure to carbon policy risks. Further, the higher cost of capital could also result from reduced subsidies and/or funding from the government to the firms/sectors with high emission intensity.<sup>40</sup> Thus, corporate default risks could also increase in the event investors respond to the possibility of future carbon pricing policies, thereby leading to an increased cost of capital to account for potentially higher default risk. Given the analytically tractable and parsimonious model, we can explore such responses that are consistent with the implied effects of a potential direct carbon price policy.

Motivated by these considerations, we ask the following questions: First, what is the increase in the cost of capital that is consistent with and reflects the increased corporate risks from potential direct carbon price policy in the future? Second, how does such cost of capital change after accounting for the potential mitigation measures by firms/sectors to reduce emission intensity? These are *implicit* changes in the cost of capital that generate the same effects as the corresponding carbon price shocks. In other words, even in the absence of carbon price shocks (i.e.  $\delta = 1$  in our model), shocks to the cost of capital  $R$  could also lead to the same increase in default risks as those due to direct carbon price shocks, as seen in Tables 3 and 4.

For each sector, these implicit impacts are calculated as follows. Since  $R$  is the effective corporate borrowing rate, we now calibrate  $R$  to match the increased average default rates under various carbon prices, as shown in Panels A of Tables 3 and 4, while turning off the direct carbon shock with  $\delta = 1$ . The resulting new/increased  $R$  would then reflect the increased risk and hence higher cost of capital due to potential carbon price policies, even if such policies are not explicitly in place yet. Similarly, we calibrate new  $R$  considering 10% reduction in emission intensity  $\theta$  where the resulting  $R$  would account for the impact of these efforts.

Table 5, Panel A shows the calibrated values of  $R$  for the transportation sector. As seen, the cost of capital rises from the pre-shock baseline of 4.51% to {6.76%, 9.01%, and 11.26%} under the potential increase in carbon prices by \$25, \$50, and \$75 carbon shocks, respectively. These are 224 bps, 449 bps, and 674 bps *increase* in borrowing costs and represent a substantial increase in the cost of capital for the transportation sector. Panel B

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<sup>40</sup>Equivalently, a lower cost of capital could result from increased investment flow and/or increased subsidies and government funding in the firms/sectors that attempt to adapt by lowering their carbon emission intensities. For example, if firms in the transportation sector deliver on their commitment to reduce emissions by, for instance, switching to energy efficient vehicles, then the differences in the cost of capital against both possibilities can be assessed. Firms may also participate in emission trading schemes to manage their risks which could lower their effective funding costs.

Table 5: Effects on Cost of Capital

|  | Baseline (no shock) | \$25 shock | \$50 shock | \$75 shock |
|--|---------------------|------------|------------|------------|
| Panel A: Transportation sector         |                     |            |            |            |
| Funding Cost R                         | 4.51%               | 6.76%      | 9.01%      | 11.26%     |
| $\Delta R$                             |                     | 224 bps    | 449 bps    | 674 bps    |
| $\Delta R$ with $\Delta\theta = -10\%$ |                     | 202 bps    | 405 bps    | 607 bps    |
| Panel B: Manufacturing sector          |                     |            |            |            |
| Funding Cost R                         | 4.05%               | 4.21%      | 4.36%      | 4.52%      |
| $\Delta R$                             |                     | 16 bps     | 31 bps     | 47 bps     |
| $\Delta R$ with $\Delta\theta = -10\%$ |                     | 15 bps     | 28 bps     | 42 bps     |

Note: Implicit increase in corporate cost of debt capital  $R$  under potential carbon price shocks.

shows the analogous impact on the manufacturing sector. As seen, the increases in funding costs are 16 bps, 31 bps, and 47 bps, which are relatively mild compared with those in the transportation sector.

Table 5 also shows the corresponding impact of the mitigation measures that reduce the carbon emission intensity by 10%. As Panel A shows, these efforts have a relatively limited impact on lowering the funding cost for firms in the transportation sector. The cost of capital rises by 202 bps, 405 bps, and 607 bps. Despite the mitigation measures, the funding cost is still relatively elevated. This follows since the average default rates, as seen in Table 3 are relatively high even after mitigation efforts. Thus, the transportation sector is highly exposed to carbon policy shocks and the sector would need to undergo a significant change to reasonably lower its default risk and, thereby, cost of capital.

In contrast, Table 5, Panel B shows that the reduction in cost of capital in the manufacturing sector from mitigation measures is not as material. Even under a 10% reduction in emission intensity, the new cost of capital is still close to the one before mitigation measures. This suggests a relatively subdued effect of carbon pricing policies in the manufacturing sector. An interesting implication of this finding is that the manufacturing sector faces weaker incentives to substantially reduce emissions/intensity given the limited impact on default risk and, hence, funding costs from such mitigation actions.

These results highlight new dimensions of the impact of carbon price shocks and again underscore the importance of sectoral heterogeneity. These results also have sharp implications for recent findings in the asset pricing literature, as outlined in the introduction

section. The large implicit shocks to the cost of capital in the transportation sector relative to the manufacturing sector, even after accounting for potential mitigation measures to reduce emissions, suggest that an increase in corporate spreads/risk premium might need to be sharply different across sectors. Our results show the highly non-linear and asymmetric impact of carbon price policies across sectors. As such, different sectors will require different risk compensation to account for the heterogeneous effects on default risks. Importantly, the implication is that the current estimates in the literature might be significantly underestimating the impact of carbon emissions/disclosures on risk premium, CDS/corporate spreads (e.g. Bolton and Kacperczyk 2021, Kolbel et al. 2021) as they do not account for a range of possible carbon shocks in the future.<sup>41</sup>

### 3.3 Transition Dynamics

In previous sections, we analyze the effects of carbon price shocks on corporate default risks. The goal of this section is to understand the transition dynamics after the shock. Transition dynamics generally studies how the economy transitions and adjusts to a new long-run steady state after some shock. As discussed previously, most existing models generally conduct comparative statics on pre-shock and post-shock steady states. However, it takes time to reach the new post-shock steady state. As such, it is important to also understand and quantify the time taken to reach the post-shock regime.

The key difficulty in studying transition dynamics in a full heterogeneous agents model like ours is that it involves characterizing the time evolution of the entire cross-sectional distribution of agents over their state variable(s) (net worth distribution in our case). This presents considerable analytical challenge, and thus, the analysis of transition dynamics has been largely numerical, as discussed above. A notable exception is Gabaix et al. (2016), who use operator theory approach to study transition properties of income distribution. More recently Lamichhane (2020), studied the transition dynamics of debt distribution using the Feynman path integral approach. In this paper, we use the operator theory approach but with a somewhat different treatment from Gabaix et al. (2016). More details are contained in the appendix. We now make some simplifying assumptions while still preserving the

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<sup>41</sup>It is also possible that markets are not pricing in large carbon price shocks because such risks are not deemed material. However, this argument assumes that all these risks are already embedded in the current markets prices. If these risks are quite realistic, this would imply gross mispricing of risks in the markets, especially for the carbon intensive sectors. Given the worldwide momentum building behind various climate policies that include considerations of non-trivial carbon pricing policies, it is reasonable to assume that carbon pricing policies and hence carbon price shocks, at least in the lower end of the range recommended by HLCCP (2017) that we have explored here, are a distinct future possibility.

micro-foundation of the problem developed earlier in this paper.

Recall that  $\phi$  in equation (6) represents the average default rate (in a given sector or economy), capturing the outflow rate of firms.<sup>42</sup> And each defaulting firm is replaced with new firms drawn from  $\bar{g}(x)$  at the same rate  $\phi$ . Hence, restating the evolution of  $g_t(x)$  as in KF equation (6), we have:

$$\partial_t g_t(x) = \mathcal{L}^* g_t(x) + \phi \bar{g}(x) \quad (13)$$

where  $\mathcal{L}^* g(x) = -\eta \partial_x g_t(x) + \frac{1}{2} \sigma^2 \partial_{xx} g_t(x) - \phi g_t(x)$  where  $\eta = (\mu\delta - \gamma - m)$  with some initial condition  $g_0(x)$ .

We also assume that the net worth  $X_t$  evolves in  $[0, u]$ . Following the normalization of net worth in the unit interval  $[0, 1]$ , as discussed above, we set  $u = 1$ . From practical purposes also, considering an upper bound is a reasonable approximation, especially since numerical computations eventually require net worth to lie in a bounded domain.<sup>43</sup> These reasonable simplifications allow us to analytically characterize the transition dynamics while also retaining the economic structure to be able to compare with the default risk results above. As such, this section takes as given the various default rates corresponding to various carbon shocks, as in the previous section and derives additional implications for transition dynamics in a simplified way. The analysis is internally consistent with the results on the default risk in the previous section.

We now ask the following questions. How fast is this transition process? How does carbon policy shock affect the transition dynamics across various sectors? What is the relation between default risk and transition speed? The following result formalizes the above intuitive discussions by characterizing the average speed of convergence of the cross-sectional distribution of net worth.

**Proposition 3. (*Transition Speed*):** *Let  $g_0(x)$  be some initial condition and let  $\|g_t(x) - \bar{g}(x)\|$  denote the  $L^1$ - norm distance between the distribution of net worth at time  $t$  and its*

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<sup>42</sup>We also abstract away from the downsizing decision of the firms, given the objective to understand the transition dynamics given the carbon price shocks.

<sup>43</sup>Note that if  $x \in [0, \infty)$ , we may need to impose  $\eta < 0$  for the existence of a stationary distribution. However, that would imply  $(\mu\delta - \gamma) < m$  as discussed earlier, making the failure probability one. Hence, the assumption of a bounded interval also has economic motivation, allowing the sign of  $\eta$  to be unrestricted. Also, the proof of Proposition 3 discusses how the assumption of a reflecting boundary at  $[\epsilon, u]$ , with  $\epsilon \leq 0$ , for a diffusion process can be compatible with a diffusion killed at random time  $\tau$  upon hitting the boundary zero of interval  $[0, u]$ . The key idea is to consider diffusion without killing on  $[\epsilon, u]$  and diffusion with killing on  $[0, u]$ . Since we already capture default risks via  $\phi$ , we simplify the problem by assuming reflecting boundaries on  $[0, u]$ . Thus, default means that at random times  $\tau$  the diffusion process is killed and sent to the *cemetery* state.



stationary distribution  $\bar{g}(x)$ . Let  $\mathbb{S}$  denote the average speed of convergence. Then,

$$\mathbb{S} = \phi + \frac{(\mu\delta - \gamma - m)^2}{2\sigma^2} + \frac{\pi^2\sigma^2}{2} \quad (14)$$

Equation (14) allows us to sharply quantify the impact of carbon price shocks on the transition speed.<sup>44</sup> We first discuss the general theoretical predictions of the above result to better understand the sector-specific quantitative implications later.

First, the higher the default rate  $\phi$ , the faster the transition speed  $\mathbb{S}$  towards the new steady state. This might appear counterintuitive since higher default rates could imply that the economy is stalling. A clear economic intuition follows from the observation that in our model default rates can also be interpreted as agents' discount rates.<sup>45</sup> Higher discount rates, and equivalently higher default rates, imply that future payoffs are less valuable. Hence, it may be better to consume sooner than later, and, thus, the value function settles down to its stationary value faster. Since the same dynamical system governs the evolution of the net worth distribution, it also settles down faster to its new steady state.<sup>46</sup> Alternatively, a direct interpretation is that higher default rates induce more agitation in the distribution, thereby making it settle down faster. Since many agents/firms eventually become insolvent during the transition, higher default rates simply mean these insolvencies occur faster, making the transition faster.

This result has an interesting policy implication. While larger carbon policy shocks increase the overall corporate default risk, they also induce faster transition towards the new steady state. This represents a potential tradeoff that policy makers might face. Higher carbon prices increase defaults but also induce faster transition. Relatively smaller carbon price shocks do not lead to a significant rise in default risks but such policies also generate slower transition.<sup>47</sup>

Notice that carbon shocks  $\delta$  and implicit shocks to the cost of capital enter the equation

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<sup>44</sup>Characterizing speed as the rate of convergence in  $L^1$ - norm is appropriate for our setting since  $L^1$ - norm is more robust than  $L^2$ - norm to the outliers. This is also the case in Gabaix et al. (2016).

<sup>45</sup>In particular, it can be shown that the killing rate of the diffusion process  $X_t$  (i.e. the default rate) is analogous to the discount rate (see proof for more details).

<sup>46</sup>More formally, the infinitesimal generator  $\mathcal{L}V(x)$  is associated with the value function, whereas its adjoint, the KF operator  $\mathcal{L}^*g(x)$  is associated with the distribution. These operators jointly govern the dynamics of the system towards the new steady state.

<sup>47</sup>Note that these results do not make any welfare statements as that analysis is out of the scope of this paper. Our results simply state the theoretical and quantitative implications between default risks and transition speed. Further, it is the net worth distribution across firms that evolves to a new steady state after exogenous carbon price shocks. Hence, we again emphasize caution in interpreting our analysis. The statements on transition are with respect to this specific feature of the model and not necessarily about the highly general and idealized notion of transition characterized by a fully green/decarbonized economy.

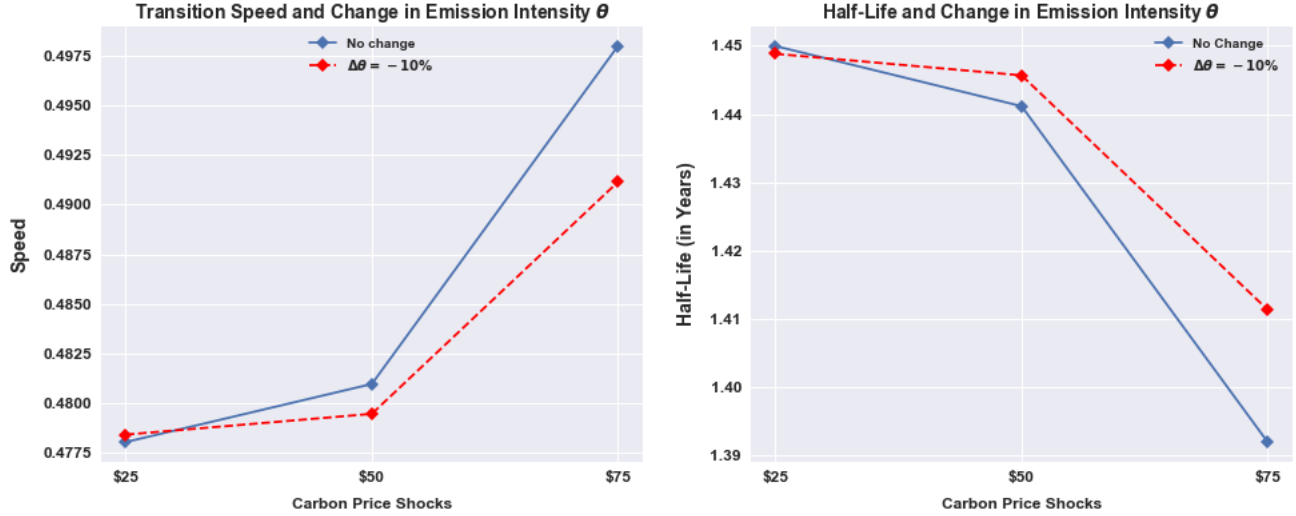


Figure 3: Transition Speed and Half-Life ( $t_{1/2}$ ) in the Transportation Sector

in a highly non-linear way where the magnitude of these quantities and corresponding default rate  $\phi$  govern the transition speed. For example, if the default rate of a given sector is much larger relative to the two other terms in equation (14), then the average transition speed will appear approximately linear with higher default risk  $\phi$  associated corresponding shock  $\delta$ . However, if  $\phi$  is relatively low, then the speed will more distinctly show the non-linear impact. We will see these distinct patterns in the quantitative analysis of two sectors that we now discuss.

To understand the quantitative implications, it is useful to discuss the average transition speed in terms of its corresponding half-life  $t_{1/2}$ . This is equal to half the time it takes for the distribution to converge to the new steady state distribution from some initial conditions. The half-life of the corresponding speed  $S$  is simply  $\log(2)/S$ .

Figure 3, left panel shows the transition speed, and the right panel shows the corresponding half-life  $t_{1/2}$  for the transportation sector. We also show the impact from mitigation measures that reduces emission intensity by 10% as discussed in previous section. The y-axis on the left panel shows the computed speed from equation (6) and y-axis on the right panel shows the corresponding half-life. As seen, the larger the carbon price shocks, the faster the transition speed and the smaller the half-life. Interestingly, after accounting for potential mitigation measures, the transition speed decreases and, thus, the half-life  $t_{1/2}$  increases. Further, the highly non-linear impact of carbon shocks is evident from the sharp increase in the slope between \$50 and \$75 shock. Figure 4 shows almost linear impact for the manufacturing sector. This highlights the heterogeneous impact on transition speed across

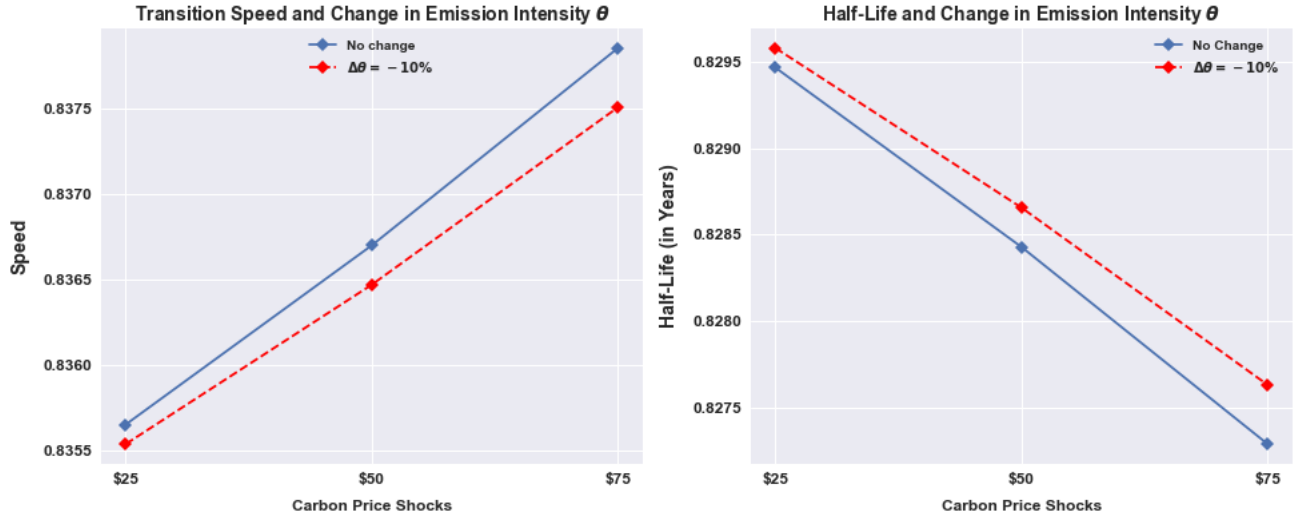


Figure 4: Transition Speed and Half-Life ( $t_{1/2}$ ) in the Manufacturing Sector

these two sectors.<sup>48</sup>

Table 6 shows these quantitative effects concretely. Panel A shows the effects for the transportation sector, directly corresponding to those in Figure 3. As seen, under \$25, \$50, and \$75 carbon shocks, the half-life is 1.45 years, 1.4411 years, and 1.3919 years, respectively. However, if we consider 10% mitigation measures the corresponding transition speed is reduced and half-life increased. This is seen in Panel A as smaller half-life values of 1.4488 years, 1.4457 years, and 1.4113. Panel B shows the effects for the manufacturing sector, directly corresponding to those in Figure 4. Under the direct carbon shocks of \$25, \$50, and \$75, the half-life is 0.8295 years, 0.8284 years, and 0.8273 years, respectively. However, if we consider mitigation measures, the speed also decreases.

Note that the relative increase in speed (and consequent decline in half-life) is not as pronounced in the manufacturing sector as it is in the transportation sector. This is because, despite the high levels of default rates  $\phi$  in the manufacturing sector, the non-linear effects from the carbon price shocks in equation (14) are not as large, thereby generating approximately linear impact. In other words, the non-linear changes induced by carbon shocks are relatively small. In contrast, in the transportation sector, shocks from carbon prices are substantial, thereby generating non-linear effects in transition speed, especially as

<sup>48</sup>Here we illustrate the average transition speed of the entire net worth distribution. While one could also analyze the speed of the various parts of the distribution, we abstract away from such considerations given our goals and also the fact that such analysis would introduce significant technical complexities and additional assumptions.

Table 6: Half-Life ( $t_{1/2}$ ) of Transition

|   | \$25 shock | \$50 shock | \$75 shock |
|---|------------|------------|------------|
| Panel A: Transportation                         |            |            |            |
| Half-life $t_{1/2}$ (in years)                  | 1.4500     | 1.4411     | 1.3919     |
| Half-life $t_{1/2}$ with $\Delta\theta = -10\%$ | 1.4488     | 1.4457     | 1.4113     |
| Panel B: Manufacturing                          |            |            |            |
| Half-life $t_{1/2}$                             | 0.8295     | 0.8284     | 0.8273     |
| Half-life $t_{1/2}$ with $\Delta\theta = -10\%$ | 0.8296     | 0.8287     | 0.8276     |

Note: Half-life  $t_{1/2}$  (in years) from direct carbon price shocks, with and without mitigation.

the carbon prices get larger. Further, from the half-life values in Table 6, it is clear that the manufacturing sector settles down to its new steady state much faster relative to the transportation sector, given the same set of carbon shocks.

The results above, in conjunction with the impact on the default risk discussed in the previous section, highlight new dimensions and implications to the climate-related policies. First, larger carbon price shocks increase the default risks in any given sector. But such policies also increase the transition speed towards the new steady state where the rate of transition is generally non-linear. Second, the degree of non-linear impact is largely driven by the emission intensities across sectors for the same carbon price shocks. We find that the transportation sector with high emission intensity transitions slowly relative to the manufacturing sector. These results underscore a sharp connection and interplay between default risk and transition dynamics with cross-sectoral implications that is largely unexplored.

Another important issue is regarding the *default spiral* firms might face during transition, i.e. once firms get to the lower tail of the net worth distribution (say, due to a series of negative cash flow shocks), escaping insolvency might become increasingly difficult. The following result formally characterizes this notion.

**Proposition 4. (*Exit from Lower Tail*):** Let  $[x_1, x_2]$  be a set near the lower tail of net worth distribution. Let  $P(\tau_{x_2}) = P(\tau_{x_2} < \tau_{x_1})$  denote the probability that  $X_t$  with initial  $x \in (x_1, x_2)$  reaches the upper boundary  $x_2$  before it reaches the lower boundary  $x_1$ . Define  $\kappa = \frac{2(\mu\delta - \gamma - m)}{\sigma^2}$ . Then,

$$P(\tau_{x_2}) = \frac{e^{-\kappa x_1} - e^{-\kappa x}}{e^{-\kappa x_1} - e^{-\kappa x_2}} \quad (15)$$

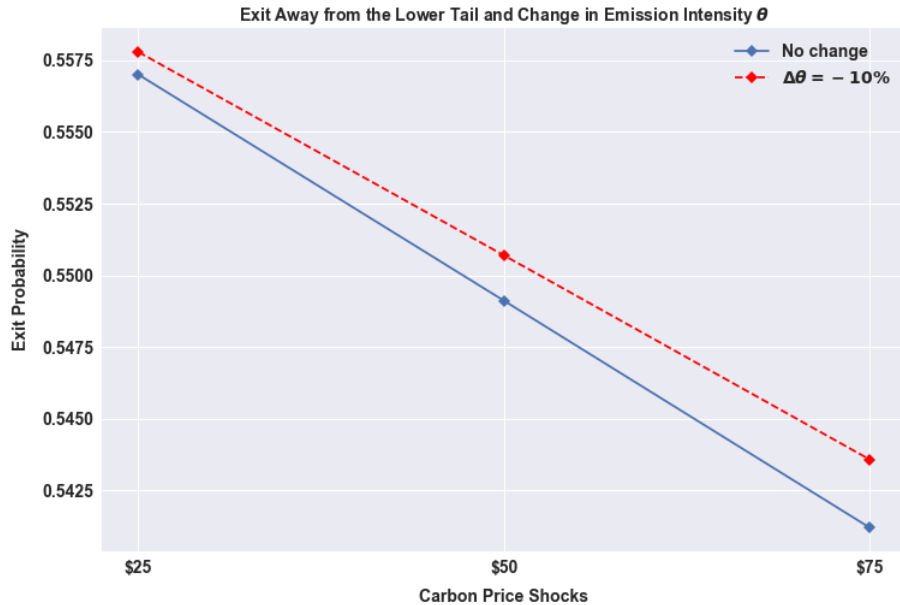


Figure 5: Carbon Shocks and Exit from Lower Tail in Transportation Sector.

Figure 5 illustrates the exit probability  $P(\tau_{x_2})$  in equation (15) for the transportation sector.<sup>49</sup> As seen, the larger the carbon price shock, the lower the probability of exiting away from the low-net-worth states. This is consistent with the fact that default risk increases under large carbon shocks because remaining in the lower tail of the net worth longer is related to a greater risk of insolvency. This also implies that after the shock, the mass of firms in the lower tail of the net worth distribution increases. Figure 5 also shows the effect of mitigation measures to reduce emission intensity where the exit probability is increases with mitigation measures. This is consistent with the fact that mitigation efforts allow firms to reduce their default risks and exit away from the low-net-worth states with high probability. Thus, this result can also be seen as largely an extension of the previous results on default risk from a different perspective.

## 4 Conclusion

This paper develops a continuous time heterogeneous agents model and studies the impact of climate-related policy risk on corporate default risks and the transition dynamics. Given

<sup>49</sup>This exercise considers the lower tail of  $[x_1, x_2] = [0.05, 0.15]$ , with  $x = 0.1$ . Also, we only present the result for the transportation sector for brevity since it has the largest impact on default risk. The effect on the manufacturing sector follows similar patterns, but magnitudes are quite small.

the closed-form solutions for default risks and transition dynamics, our model allows one to study policy implications in an analytically tractable way. The model is calibrated to different US sectors to quantify the heterogeneous effects of carbon price shocks. The analysis shows that while carbon-intensive sectors face increased default risks, there are notable asymmetric effects within sectors. Higher carbon prices increase corporate default risk but also induces faster transition towards the new post-shock steady state, with highly non-linear impact largely driven by emission intensities across sectors. Our results suggest that once a range of possible price shocks are accounted for, the increase in the cost of capital might need to be much higher than the current estimates in the literature suggest, even after accounting for potential mitigation efforts. As such, our framework and results should be useful to policymakers, as they explore the impact of various new climate-related policies. In particular, closed-form solutions from our model can be directly used in a practical policy analysis setting, such as in climate risk stress testing and financial stability analysis.

## Appendix: Proofs

*Proof. Proposition (2):*

Here, we are primarily interested in optimal response  $\alpha = 1 - \bar{\alpha}$  and the interior solution  $\alpha \in (0, 1)$ . Substitute  $\eta(x) = (\bar{\alpha}(\mu\delta - \gamma) - c_t)$  in firms' HJB equation:

$$\rho V(x) = \max_{\{c, \bar{\alpha}\}} c + (\bar{\alpha}(\mu\delta - \gamma) - c_t)V'(x) + \frac{1}{2}\hat{\sigma}^2 V''(x)$$

Given the linearity of HJB in  $c$  and due to concavity of  $V(x)$  (as the HJB equation is quite standard, see Fleming and Sonar 2006), we know there is some  $\bar{x}$ , such that  $\bar{x} = \inf\{x : V'(x) = 1\}$ . Let  $c^*$  be the optimal consumption/payout. Then,  $c^* = m$  for  $x > \bar{x}$  and  $c^* = 0$  for  $x < \bar{x}$ . At  $V'(x) = 1$ ,  $c^* \in [0, m]$  where we set  $c^* = 0$  for convenience.

Then, for all  $x < \bar{x}$  over the set  $[0, \bar{x})$  we have:

$$\rho V(x) = \max_{\bar{\alpha}} (\bar{\alpha}(\mu\delta - \gamma))V'(x) + \frac{1}{2}\bar{\alpha}^2\sigma^2 V''(x)$$

Taking FOC w.r.t.  $\bar{\alpha}$ , we get:  $\bar{\alpha} = -\frac{(\mu\delta - \gamma)V'(x)}{\sigma^2 V''(x)}$ . Define  $p = \frac{(\mu\delta - \gamma)^2}{2\sigma^2}$  and  $\beta = \frac{\rho}{p + \rho}$ .

Then, plugging  $\bar{\alpha}$  into the above HJB equation gives:

$$-p \frac{V'(x)^2}{V''(x)} - \rho V(x) = 0$$

This is a standard differential equation with known general solution form as  $V(x) = K_1 x^\beta$  for some constant  $K_1$ . Note that  $0 < \beta < 1$  given that  $V$  is concave.

Plugging using  $V(x) = K_1 x^\beta$  into the expression above, using  $V(0) = 0$ , we get:

$$\bar{\alpha} = \frac{(\mu\delta - \gamma)x_t^f}{\sigma^2(1 - \beta)}$$

For the interior solution we have  $\bar{\alpha} < 1$ . Then the above solution is valid on  $[0, \hat{x})$  where  $\hat{x} < \bar{x}$  (see Fleming and Sonar 2006). Then, using applying  $\bar{\alpha} < 1$  to the expression above, we get:

$$\hat{x} = \frac{\sigma^2(1 - \beta)}{(\mu\delta - \gamma)}$$

where  $\hat{\alpha} \rightarrow 1$  for  $x > \hat{x}$ . □

*Proof. Proposition (1):*

The relation between the threshold net worth level  $\bar{x}$  and optimal consumption/payout

$c^*$  is already established above in proof of Theorem 2. For default risk, we first solve the case with maximal payout  $m$ , which occurs when  $x > \bar{x}$ . The alternate case with  $x < \bar{x}$  follows immediately.

For initial wealth  $x$ , let  $\hat{P}(x, \bar{x}) = E[e^{-\rho\tau}]$  denote the Laplace transform of the density of failure time  $\tau$  for net worth process  $X_t$ . As a function of  $\rho$ ,  $\hat{P}$  is a Laplace transform of  $\tau$ . Further,  $\hat{P}(x, \bar{x})$  can also be interpreted as expected discounted value of 1 due at failure time.

Given the default indicator function  $\mathbb{I}_{\tau < \infty}$  we have,

$$\hat{P}(x, \bar{x}) = E[e^{-\rho\tau}] = E[e^{-\rho\tau}\mathbb{I}_{\tau < \infty}].$$

Then, taking limit as  $\rho \rightarrow 0$ , we get the failure probability as:

$$\hat{P}(x, \bar{x}) = E[\mathbb{I}_{\tau < \infty}] = P(\tau < \infty) = P(X_t \leq 0).$$

Define  $\hat{\sigma} = \bar{\alpha}\delta_2\sigma$  and  $\xi = \bar{\alpha}(\mu\delta - \gamma)$ .

As seen above,  $\hat{P}(x, \bar{x})$ , taken as a function of  $x$ , satisfies the following homogeneous second-order differential equation (analogous to the HJB equation).

$$\frac{\hat{\sigma}^2}{2}\hat{P}'' + \xi\hat{P}' - \rho\hat{P} = 0, \quad \text{for } 0 < x < \bar{x} \quad (16)$$

$$\frac{\hat{\sigma}^2}{2}\hat{P}'' + (\xi - m)\hat{P}' - \rho\hat{P} = 0, \quad \text{for } x \geq \bar{x} \quad (17)$$

We know that solutions to these types of homogeneous second-order differential equations can be obtained following well-known standard differential equation techniques that we outline below (we skip writing down routine algebra steps). Once the solutions are obtained, we take  $\rho \rightarrow 0$  to get the failure probability.

First, note the boundary conditions that apply to equations (16) and (17). As  $x \rightarrow \infty$ ,  $\tau \rightarrow \infty$ , i.e. there is no failure, and  $\hat{P}(x, \bar{x}) = 0$ . Alternatively, if  $x \rightarrow 0$ , then  $\tau \rightarrow 0$  due to fluctuations of Brownian motion and  $\hat{P}(x, \bar{x}) = 1$ , i.e. failure is immediate.

Thus, for  $x \geq \bar{x}$ , the solution is of the form  $\hat{P}(x, \bar{x}) = e^{n(x-\bar{x})}\hat{P}(\bar{x}, \bar{x})$  where  $n$  is the negative root of the characteristic equation of (17), analogous to that for (16), shown below.

For  $x < \bar{x}$ , the solution is:  $\hat{P}(x, \bar{x}) = e^{k_2x} + k(e^{k_1x} - e^{k_2x})$  where  $k_1$  and  $k_2$  are the positive and negative roots, respectively, of the characteristic equation of (16):

$$\frac{\hat{\sigma}^2}{2}\zeta^2 + \xi\zeta - \rho = 0$$

Coefficient  $k$  is determined by continuity property of  $\hat{P}(x, \bar{x})$  and  $\hat{P}'(x, \bar{x})$  at  $x = \bar{x}$ . Then, substitute  $x = \bar{x}$  in  $\hat{P}(x, \bar{x})$  and differentiate w.r.t.  $\bar{x}$ . After some algebra, we obtain



coefficient  $k$  as:

$$k = \frac{(n - k_2)e^{k_2\bar{x}}}{(n - k_2)e^{k_2\bar{x}} + (k_1 - n)e^{k_1\bar{x}}}$$

Substituting this to  $\hat{P}(x, \bar{x}) = e^{k_2x} + k(e^{k_1x} - e^{k_2x})$ , we get:

$$\hat{P}(x, \bar{x}) = \frac{(n - k_2)e^{-k_1(\bar{x}-x)} + (k_1 - n)e^{-k_2(\bar{x}-x)}}{(n - k_2)e^{-k_1\bar{x}} + (k_1 - n)e^{-k_2\bar{x}}}$$

This also gives  $\hat{P}(\bar{x}, \bar{x})$ .

Let  $q = \frac{2\xi}{\sigma^2}$ . Then, taking limit as  $\rho \rightarrow 0$ , the roots becomes  $k_1 = 0$ ,  $k_2 = -q$  and  $n = -q + \frac{2m}{\sigma^2}$ .

Assuming  $\bar{\alpha} = 1$  gives  $\xi = (\mu\delta - \gamma)$  and  $\hat{\sigma} = \delta_2\sigma$

Substituting these to above  $\hat{P}(x, \bar{x})$  and recalling from above that it is the failure probability when  $\rho \rightarrow 0$ , for  $x > \bar{x}$  we get:

$$P(X_t \leq 0) = \frac{\xi e^{-\frac{2(\xi-m)(x-\bar{x})}{\sigma^2}}}{m + (\xi - m)e^{\frac{2\xi\bar{x}}{\sigma^2}}}$$

$$P(X_t \leq 0) = \frac{(\mu\delta - \gamma)e^{-\frac{2(\mu\delta-\gamma-m)(x-\bar{x})}{\sigma^2}}}{m + (\mu\delta - \gamma - m)e^{\frac{2(\mu\delta-\gamma)\bar{x}}{\sigma^2}}}$$

And for  $0 \leq x \leq \bar{x}$ , we get:

$$P(X_t \leq 0) = \frac{m + (\xi - m)e^{\frac{2\xi}{\sigma^2}(\bar{x}-x)}}{m + (\xi - m)e^{\frac{2\xi}{\sigma^2}\bar{x}}}$$

$$P(X_t \leq 0) = \frac{m + (\mu\delta - \gamma - m)e^{\frac{2(\mu\delta-\gamma)}{\sigma^2}(\bar{x}-x)}}{m + (\mu\delta - \gamma - m)e^{\frac{2(\mu\delta-\gamma)\bar{x}}{\sigma^2}}}$$

If we set  $m = 0$  in the above expression, we get the special case:

$$P(X_t \leq 0) = e^{-\frac{2\xi x}{\sigma^2}} = e^{-\frac{2(\mu\delta-\gamma)x}{\sigma^2}}$$

□

*Proof.* **Proposition (3):**

**Overview:** Before going into the proof, we briefly discuss a few background concepts. The proof involves the application of the spectral theory to diffusion processes. This application

is originally due to McKean (1956) who constructed a spectral representation for a general one-dimensional diffusion.

The transition speed or the rate of convergence of the distribution  $g_t(x)$  in  $L^1$ - norm  $\|g_t(x) - \bar{g}(x)\|$ , which captures the average speed of convergence, is given by the second eigenvalue of the differential operator of the diffusion process. The first or the principal eigenvalue (which is zero) is associated with the stationary distribution. A similar definition of speed is also used in Gabaix et al. (2016). We will see this explicitly in the spectral representation via eigenfunction expansion below.

The analytical characterization is easily obtained via the spectral representation of our one-dimensional diffusion process of net worth in equation (4). In order to ensure that the eigenvalues are real numbers, the differential operators need to be self-adjoint. This can be achieved in different ways. In this paper, we use the results from the Sturm–Liouville (SL) literature.<sup>50</sup> This involves transforming the original problem to the corresponding Sturm–Liouville spectral expansions.

This transformation then allows us to apply the well-known spectral theorem for self-adjoint operators in Hilbert space that yield a unique spectral representation of the operator. This gives considerable simplification because the eigenvalues of such SL operator are real and correspond to those of the original operator. Then, we simply apply well-known results from the SL theory.

**Background Details:** Consider the net worth  $X_t$  which evolves in the state space  $[\epsilon, u]$  where  $\epsilon \leq 0$ . Even though we ultimately restrict ourselves to non-negative values only given default definition, this process can technically take negative values. We assume that  $\epsilon$  and  $u$  are the reflecting boundaries for  $X_t$ .

For some generic coefficients  $\mu(y)$  and  $\sigma(y)$  that are continuous and positive, let us define two functions  $\mathcal{E}(x)$  and  $\mathcal{G}(x)$  as:

$$\mathcal{E}(x) = e^{-\int_0^x \frac{2\mu(y)}{\sigma^2(y)} dy}, \quad \mathcal{G}(x) = \frac{2}{\sigma^2(x)\mathcal{E}(x)}, \quad \int_{\epsilon}^u \mathcal{G}(x)dx < \infty$$

The functions  $\mathcal{E}(x)$  and  $\mathcal{G}(x)$  are sometimes referred to as the scale density and speed density (different from the transition speed), respectively. See Borodin and Salminen (1996) and Karatzas and Shreve (1991) for more details. The assumption  $\int_{\epsilon}^u \mathcal{G}(x)dx < \infty$  ensures that a stationary density  $\bar{g}(x)$  exists, which is given by:  $\bar{g}(x) = \frac{\mathcal{G}(x)}{k}$  where  $k := \int_0^u \mathcal{G}(x)dx$

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<sup>50</sup>See Fulton and Pruess (1994), Linetsky (2004, 2005) for more details and applications.

is the constant such that  $\int_0^u \bar{g}(x)dx = 1$ .

Let  $L^2$  be the Hilbert space of square integrable real valued functions that are bounded and continuous with the speed density  $\mathcal{G}(x)$  as the weighting function, i.e. the inner-product is  $\langle f, h \rangle = \int_\epsilon^u f(x)h(x)\mathcal{G}(x)dx$ .

Now, consider the infinitesimal generator of the diffusion  $X_t$  (i.e. without killing) with generic drift and diffusion coefficients  $\mu(x)$  and  $\sigma(x)$ :

$$\hat{\mathcal{L}}f(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$$

Then, it is well known that the infinitesimal generator  $\hat{\mathcal{L}}g(x)$  is self-adjoint and non-positive operator in  $L^2$  (see McKean 1956 and Linetsky 2004). Then, the operator corresponding to a diffusion with killing rate  $\phi(x) \geq 0$  is

$$\mathcal{L}f(x) = \hat{\mathcal{L}}f(x) - \phi(x)f(x)$$

Thus,  $\mathcal{L}g(x)$  is the infinitesimal generator of a diffusion  $\hat{X}$  that is killed at a rate  $\phi(x)$ .

We now discuss two closely related techniques and the interpretation behind the appropriate infinitesimal generator of the net worth process. This will simplify our analysis later, when we get to the spectral expansion part.

Since we kill the process  $X_t$  upon first hitting time  $\tau$  of the boundary  $\epsilon$ , one immediate interpretation is that the diffusion with killing is simply the process where all negative values are removed (i.e. the insolvent firms are removed). Then, we only retain the positive part. And when the process is sent to the *cemetery state*  $\partial$  at the first hitting time of the boundary, we let  $f$  take value zero there as  $f(\partial) = 0$  by convention as it is the state of default in our model. Alternatively, we can also leave  $\hat{X}_t$  undefined for  $t \geq \tau$ , see Oksendal 2002).

So,  $X_t$  can technically take on negative values. And by killing it upon reaching the lower boundary, the effective new diffusion  $\hat{X}$  is obtained as it is killed at random time  $\tau$  with rate  $\phi(x)$ . And note that  $\epsilon$  still remains reflecting since the killing rate  $\phi(x) = 0$  below zero where  $\mathcal{L}f(x) = \hat{\mathcal{L}}f(x)$ . Hence, the operator  $\hat{\mathcal{L}}f(x)$  is a self-adjoint non-positive operator and its adjoint Kolmogorov forward operator is given by:<sup>51</sup>

$$\mathcal{L}^*g(x) = -\partial_x(\mu(x)g(x)) + \frac{1}{2}\partial_{xx}(g(x)\sigma^2(x)) - \phi(x)g(x)$$

Since we are interested in net worth distribution in  $[0, u]$  where mass of firms is declining

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<sup>51</sup>Gorovoi and Linetsky (2004) also use similar arguments as they model shadow spot rates that can take negative values whereas the nominal spot rates can only take non-negative values.

due to killing/default, we re-inject the state space with new firms drawn from the ergodic distribution  $\bar{g}(x)$ , such that the mass of firms remains constant at one. Thus, the resulting Kolmogorov forward equation is analogous to the one augmented with birth and death at rate  $\phi$ .<sup>52</sup>

$$\partial_t g_t(x) = \mathcal{L}^* g(x) + \phi(x) \bar{g}(x)$$

Alternatively, if we want to consider diffusion process  $X_t$  on  $[0, u]$  without killing but still want to capture the default, the steps are simple. First, we assume reflecting boundaries as before. Then, we view  $\phi$  as simply the rate at which the process is discontinued, i.e. the death rate capturing the outflow of firms. The interpretation is that, each firm in the net worth distribution faces some ongoing risk of default  $\phi$ . This captures the effect of various unanticipated and potentially large negative shocks, analogous to jump shocks, that renders  $\phi g(x)$  firms with net worth  $x$  default each instant, sending them to the cemetery state. Hence, it captures the equivalent effect of diffusion with killing.

Thus, we have discussed two closely related techniques and interpretation behind the appropriate infinitesimal generator of the net worth process. Given this, the next steps are greatly simplified, as we can now apply the relevant aspects of spectral theory to our setting. Hereafter, to simplify the analysis, we interpret the killing rate  $\phi$  as the rate of death/default and birth/re-entry of firms. Thus, the diffusion process evolves in the bounded interval  $[0, u]$  with reflecting boundary conditions.<sup>53</sup> Hereafter, we will interchangeably use the terms killing rate and default rate as they mean the same thing here.

**Main Steps:** Consider the following Sturm-Liouville (SL) equation:

$$\mathcal{A}h(x) = \lambda h(x), \quad x \in (\epsilon, u) \tag{18}$$

where  $\mathcal{A} = -\mathcal{L}$  is the SL operator defined as the negative of the infinitesimal generator  $\mathcal{L}$ , and  $\lambda$  is the standard notation for the eigenvalue. Since  $\mathcal{A}$  is non-negative, we have the spectral parameter  $\lambda \geq 0$ .

Since both boundaries are reflecting, we have the so-called *non-oscillatory* boundaries in terms of the SL theory of classification of boundaries.<sup>54</sup> It is well known since McKean

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<sup>52</sup>For example, as in Gabaix et al. (2016). But now the death rate is the default rate where the process is killed at some random time  $\tau$  and sent to the cemetery state.

<sup>53</sup>Alternatively, we could define diffusion without killing in  $[0, u + \epsilon]$  instead of current  $[\epsilon, u]$  for some arbitrarily small  $\epsilon \geq 0$  where the process is killed whenever  $X_t \leq 0$ . Then, points zero and  $u + \epsilon$  would still remain reflecting boundaries. Thus, it is just a matter of using modified state space and then preserving the mass of the distribution by considering birth rate as well.

<sup>54</sup>See Karlin and Taylor (1981), Borodin and Salminen (1996) for discussion of Feller's boundary classifica-

(1956) that non-oscillatory boundaries do not generate any continuous spectrum. Hence, the spectrum of the SL equation above is simple, non-negative, and discrete and that the set of all the eigenvalues forms the spectrum of the operator. This considerably simplifies our analysis.

In summary, the SL generator  $\mathcal{A}$  has a discrete spectrum in  $L^2$  where the eigenvalues  $\lambda$  are real and non-negative and the principal eigenvalue  $\lambda_0 = 0$  corresponds to the stationary density, and  $\lambda_1 < \lambda_2 < \dots$ . The corresponding eigenfunctions  $\{\varphi_n\}_{n=0}^\infty$ , normalized as  $\|\varphi_n\|^2 = 1$ , form a complete orthonormal basis on  $L^2$  that can be expanded with generalized Fourier series. Thus,  $\{\lambda_n, \varphi_n\}_{n=0}^\infty$  are the eigenvalue-eigenfunction pairs of the generator  $\mathcal{L}$ .

Let  $g(x, t|x_0) = g(x, t|x_0, 0)$  denote the transition density function with initial density  $g_0(x)$ . Note that the solution of KF equation with initial condition  $g_0(x)$  is  $g(x, t) = \int g(x, t|x_0)g_0(x)dx$ . Then, the spectral representation/eigenfunction expansion of  $g(x, t|x_0)$  is<sup>55</sup>

$$g(x, t|x_0) = \bar{g}(x) + \mathcal{G}(x) \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(x_0) \quad (19)$$

As seen, the first term corresponding to the principal eigenvalue  $\lambda_0 = 0$  is the stationary density  $\bar{g}(x)$ . Given that our problem with constant diffusion coefficients is standard, the eigenvalues  $\lambda_n$  are easy to compute. In our case, the speed is governed by the second eigenvalue  $\lambda_1 = \lambda$  (since  $\lambda_0 = 0$  is the first eigenvalue), i.e. the transition speed  $\mathbb{S} = \lambda$ .<sup>56</sup>

Directly working with the infinitesimal generator with default rate  $\phi$  to solve for  $\lambda$  can be cumbersome. To simplify, we transform the operator with  $\phi$  to the one without  $\phi$  and then transform it back again to obtain the final result.

Define  $\hat{g}_t = e^{\phi t}(g_t(x) - \bar{g}(x))$ . We know  $\partial_t g_t(x) = \mathcal{L}^*g(x) + \phi\bar{g}(x)$ . Then for  $\hat{g}_t$  we get  $\partial_t \hat{g}_t(x) = \hat{\mathcal{L}}^*\hat{g}(x)$  where  $\hat{\mathcal{L}}^*$  is the adjoint of  $\hat{\mathcal{L}}$  defined above for the diffusion process without killing. We get  $\partial_t \hat{g}_t(x) = \mathcal{L}^*\hat{g}(x)$ . Thus, we first obtain the eigenvalue  $\lambda$  for  $\hat{\mathcal{L}}$ . This step is quite standard since it involves generators of the standard diffusion process in a bounded interval. Finally, we will add back  $\phi$  to obtain the final result.<sup>57</sup>

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tions for diffusions. See Linetsky (2004) for discussion of oscillator/non-oscillatory boundary classifications and how to transform the SL equation to the Liouville normal form to determine the nature of the boundary.

<sup>55</sup>See Borodin and Salminen (1996), Oksendal (2002), and Pavliotis (2014) for more details.

<sup>56</sup>The intuition for why the second eigenvalue governs the speed is as follows. In equation (19), the terms in the series corresponding to higher eigenvalues  $\lambda_n$  are increasingly suppressed since  $\lambda_n$  is increasing in  $n$ . Additionally, as time  $t$  gets large, the terms beyond second eigenvalues quickly approach zero. Thus, the speed is sharply captured by the second eigenvalue which represents the rate of convergence in  $L^1$ - norm. Also see Gabaix et al. (2016) for discussion of this intuition.

<sup>57</sup>This step is analogous to transforming a stock price  $S_t$  with dividend yield  $\phi$  into an equivalent stock price  $S_t^*$  without dividends by reinvesting all cash flows into the stock itself. The original price process  $S_t$  is

Define  $\zeta = (\mu\delta - \gamma - m)/\delta_2\sigma$  and  $\hat{\sigma} = \delta_2\sigma$ . Since the drift and diffusion coefficients in our problem satisfy the smoothness assumptions (continuous and strictly positive), we can transform the SL equation to the Liouville normal form using Liouville transformation and apply the SL theory in our context to get the first non-zero eigenvalue as:<sup>58</sup>

$$\lambda = \frac{\zeta^2}{2} + \frac{\hat{\sigma}^2\pi^2}{2u^2}$$

This is the eigenvalue for diffusion process without killing/default. Since the diffusion without killing decays exponentially at rate  $\lambda$  as seen from equation (19) above, diffusion with killing rate  $\phi$ , corresponding to the generator  $\mathcal{L} = \hat{\mathcal{L}} - \phi$ , decays at the rate  $\lambda + \phi$ . Thus, we get the transition speed expression with default rate by simply adding  $\phi$  to above  $\lambda$  and substituting back  $\zeta$ ,  $\hat{\sigma}$  as:

$$\mathbb{S} = \phi + \frac{(\mu\delta - \gamma - m)^2}{2\sigma^2} + \frac{\sigma^2\pi^2}{2u^2}$$

□

*Proof. Proposition 4:* We know that the standard scale function  $S(x)$  of diffusion with generic drift and diffusion coefficients  $\eta(y)$  and  $\hat{\sigma}(y)$  is given by:

$$S(x) = \int_0^x \left( e^{-\int_0^z \frac{2\eta(y)}{\hat{\sigma}^2(y)} dy} \right) dz$$

Then, the result is obtained from a simple application of the well-known Dynkin's formula (see Karatzas and Shreve 1991), which gives, after some algebra steps, the following relation:

$$P(\tau_{x_2}) = \frac{S(x) - S(x_1)}{S(x_2) - S(x_1)}$$

Since in our model,  $\eta = (\mu\delta - \gamma - m)$  and  $\hat{\sigma} = \delta_2\sigma$ , we get:

$$S(x) = \int_0^x e^{-\frac{2\eta z}{\hat{\sigma}^2}} dz$$

$$S(x) = -\frac{\sigma^2}{2\eta} [e^{-\frac{2\eta z}{\hat{\sigma}^2}}]_0^x$$

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increased by  $\phi$  times the time period over which dividend is earned, i.e.  $S_t^* = S_t e^{\phi t}$ . Then, original price  $S_t$  is obtained as  $S_t = S_t^* e^{-\phi t}$ .

<sup>58</sup>Such a transformed SL equation resembles a one-dimensional Schrodinger equation. And given that the diffusion evolves in the reflecting bounded domain  $[0, u]$  the eigenvalue asymptotics for SL operators follow the results in Fulton and Pruess (1994). Also, see Linetsky (2004, 2005) for its extensions.

$$S(x) = \frac{\hat{\sigma}^2}{2\eta} (1 - e^{-\frac{2\eta x}{\hat{\sigma}^2}})$$

$$P(\tau_{x_2}) = \frac{e^{-\kappa x_1} - e^{-\kappa x}}{e^{-\kappa x_1} - e^{-\kappa x_2}}, \quad \kappa = \frac{2(\mu\delta - \gamma - m)}{\sigma^2}$$

□

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