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A Framework for Assessing the Costs of Pension Reform Reversals

By Daniel Baksa, Zsuzsa Munkacsi, and Carolin Nerlich

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A Framework for Assessing the Costs of Pension Reform Reversals

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Abstract

Several European countries are currently considering reversing parts of their pension reforms that were adopted previously to improve sustainability. In this paper we present a framework that allows us to quantify the macroeconomic and fiscal costs of such reversals. We thereby integrate the country-specific information from the latest Ageing Report into a dynamic general equilibrium model with overlapping generations. Focusing on Germany and Slovakia as country cases, our model replicates the Ageing Report's pension expenditure projections very well. We calculate the macroeconomic impact of first the additional pension reforms needed to contain the public debt pressures arising from population ageing and second the costs of reform reversals. Our model results show that undoing past pension reforms would generate substantial adverse macroeconomic costs and could pose challenges for fiscal sustainability.

JEL Classification Numbers: H55, J11, J26

Keywords: public pension, reform reversals, population ageing, overlapping generations model, Ageing Report

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1 Introduction

Population is ageing in most advanced economies, including in Europe, and this trend is expected to continue in the coming decades. Population ageing is widely seen to pose adverse fiscal and macroeconomic challenges that render policy action.[3] Against this background, many countries in Europe have implemented significant pension reforms in the past two decades to contain the adverse consequences of ageing. More recently, however, reform progress has stalled in Europe and despite an unchanged demographic outlook several countries have reversed, or plan to do so, parts of previously adopted pension reforms. For example, Germany recently decided to cap the decline in the benefit ratio and the increase in the contribution rate until 2025 at certain levels, and is considering whether to extend this cap even until 2040. Slovakia decided to break the automatic link between changes in life expectancy and retirement age, by capping the retirement age at 64 years. In Spain, the government inter alia postponed the implementation of the sustainability factor to 2023, which links the replacement rate to changes in life expectancy. In Italy it was decided to partly backtrack previous reform achievements by inter alia temporarily facilitating early retirement. Greece is facing significant risks related to court decisions reversing past pension cuts. In the Netherlands discussions are ongoing whether to postpone the foreseen increase in the retirement age, and thereafter to loosen the agreed link between changes in life expectancy and retirement age. All these reform reversals cause potentially substantial fiscal and macroeconomic costs in the long run, thereby putting additional burden on future generations. Moreover, this may lead to debt level pressures which are an area of concern particularly for countries with already elevated public debt-to-GDP ratios.

In this paper, we offer a framework that allows us to evaluate the adverse macroeconomic and fiscal impact of a sudden and not foreseen reversal of pension reforms. We thereby use a dynamic general equilibrium model with overlapping generations (OLG) and combine it with the 2018 Ageing Report to exploit the country-specific information contained in it.[4] We focus on Germany and Slovakia as country cases. Concretely, on the basis of our model we try to replicate the Ageing Report's pension expenditure projections for these two countries. Looking at

[3]For an overview of the macroeconomic and fiscal implications of ageing see for example ECB (2018).

[4]See European Commission (2018). The Ageing Report's long-term projections are published every three years. They are jointly prepared by the Ageing Working Group and the European Commission. While the Ageing Report's pension expenditures are a central element of the projection exercise, total ageing cost projections also cover other public expenditure items, such as health and long-term care costs.

a variety of fiscal and macroeconomic variables, we can disentangle the future impact that is purely related to population ageing from the projected impact of pension reforms that have been adopted in the past, but are not yet fully implemented. Within our framework we are also able to quantify the size of additional pension reform measures needed to compensate for the expected ageing-induced increase in the public debt-to-GDP ratio. Even more importantly with respect to our main policy question, the framework allows us to quantify the possible macroeconomic and fiscal effects of pension reform reversals. We also offer an indication of what would be needed in terms of policy measures to compensate for these reform reversals.

The model used in our framework is called OGRE (Baksa and Munkacsi 2016a, and 2016b), which is an acronym for Overlapping Generations and Retirement. It is a Gertler-type (Blanchard-Weil-Yaari-type) dynamic general equilibrium model with OLG households, demography, unemployment and wage bargaining.[5] The model assumes two generations: the workers and the retirees. Workers either work, in which case they receive labour income and pay income taxes, or are unemployed, in which case they receive unemployment benefits. The retirees do not work, but they receive pension benefits. The size and structure of the population is changing over time, and changes are induced by the probabilities to be born, to retire and to pass away. The model has a rich fiscal sector which covers several public revenue and expenditure items, including pensions. For simplicity reasons we use the closed-economy version of the model.

The general fit of our model with the Ageing Report projections is very good for the two countries analysed. While both countries are facing challenges due to population ageing, they differ in terms of demographic driving forces and projected pension expenditures. We do not expand the set of countries further as this would go beyond the scope of this paper. Yet, the framework can be easily applied to other countries, including those in which fully-funded schemes play a more prominent role, as OGRE can handle both pay-as-you-go and fully-funded regimes.[6]

Our results can be summarised as follows: in line with the literature we find that population

[5]See Gertler (1999), Blanchard (1985), Weil (1989), and Yaari (1965). The labour market rigidities and wage bargaining are based on Blanchard and Gali (2010).

[6]In this paper we only focus on pension reforms that adjust the parameters of existing pay-as-you-go systems, while we disregard any reforms that would imply a switch from a pay-as-you-go to a fully-funded regime. Baksa and Munkacsi (2016b) and Baksa et al. (2016) examined the latter, for instance. While not reported in the paper, we also calibrated the model for Spain and Portugal, the results of which could be shared upon request.

ageing has major macroeconomic and fiscal adverse implications if left unaddressed through reforms.[7] However, different from most other studies, we do not only look at the demographic transition, but also account for already adopted pension reforms, as captured in the 2018 Ageing Report. This is an important aspect when calculating the additional pension reforms that would be needed to fully contain the adverse public debt impact of ageing. In line with the literature, we find evidence that pension reforms help to contain the adverse implications, although by a varying degree, depending on the concrete measures adopted and the country-specific circumstances. In particular, increases in the retirement age appear to help to alleviate ageing pressures most. The analysis also shows, though, that reform packages that consist of various pension reform measures help to spread the adjustment burden more equally across generations. Finally, we find strong evidence for the presumption that reversals of pension reforms are potentially very costly. In fact, reform reversals would not only result in higher aggregate pension expenditure and public debt-to-GDP ratios, but would in most cases also exacerbate the adverse macroeconomic impact of ageing. If the reversed reform elements were to be compensated by other reform measures, this would improve fiscal sustainability, but might disproportionally impact one generation.

Our main contribution to the literature is twofold: First, our paper offers a framework that allows us to replicate the long-term dynamics of pension expenditures, as projected in the Ageing Report, very well, which we combine with the behavioural and feedback effects of our OLG model. We use this framework to evaluate the macroeconomic and fiscal impact of population ageing and pension reforms. The Ageing Report projections have the advantage that they are detailed, country-specific and account for implementation delays, i.e., the future impact of already adopted pension reforms. In particular the latter is important in view of the numerous pension reforms adopted in past years, as not accounting for them would overstate and bias the adverse ageing-related consequences.[8] Yet, the Ageing Report projections are based on a simple accounting framework and thereby do not capture feedback effects between changes in

[7] On the macroeconomic impact of population ageing and pension reforms, although far from exhaustive, see for example Fehr (2000), Börsch-Supan et al. (2006), Kilponen et al. (2006), Diaz-Gimenez and Diaz-Saavedra (2009), Kara, E. and L. von Thadden (2010), Karam, et al. (2010), Keuschnigg et al. (2013), de la Croix et al. (2013), Goraus et al. (2014) and Baksa and Munkacsi (2016b). See also Conesa and Krueger (1999), Galasso (2008), Heijdra and Romp (2009), and Beetsma et al., (2013) who discussed the political feasibility of pension reforms and optimal pension policies.

[8] On importance of implementation delays of pension reforms for their macroeconomic impact, see Bi and Zubairy (2019).

macroeconomic variables, age-related expenditures and pension parameters.

Second, our framework enables us to quantify the cost of reform reversals. There is a wide range of studies that analyse the economic and fiscal impact of population ageing and different kinds of pension reforms, including systemic reforms. To our knowledge, however, the macroeconomic costs of reform reversals have not been systematically analysed in the literature yet. The few exceptions include the papers by Börsch-Supan and Rausch (2018) and Dolls and Krolage (2019) which focus on the fiscal cost of pension reform reversals in Germany.

We do not look at the drivers of pension reform reversals in this paper. In fact, reform reversals can be, among other things, explained by moral hazard behaviours. For example, the median voter is ageing, which calls for swift policy action rather sooner than later. In addition, the political pressure to adopt reforms is distinctly vanishing in good economic times (Beetsma et al. (2013)). Yet, these aspects go well beyond the scope of this paper and would deserve a paper on its own.

The paper is organised as follows: In section 2 we briefly outline the ageing challenge. In section 3 we present our framework and explain how we link the OLG model to the 2018 Ageing Report projections for our two country cases, Germany and Slovakia, to generate the baseline scenario. Moreover, we present the motivation for our policy scenarios, including the one on reform reversals. In section 4 we first present the results for our baseline scenario. We then show the impact in case additional pension reforms were adopted to neutralize the public debt pressures arising from population ageing. In section 5, we assess the adverse impact of reform reversals by using Germany and Slovakia as illustrative examples. In section 6 we conclude.

2 The demographic challenge

Europe's population is rapidly ageing and the demographic challenges are expected to become even more pressing over time. In particular in the next one and a half decades, the baby boom generation, i.e., the sizeable cohort of those born between 1955 and 1970, will enter retirement. The change in the relative size of the age cohorts is well captured by the old-age dependency ratio, which sets the elderly population (of age 65 and older) in relation to the working-age population (of age 15-64). According to Eurostat, the old-age dependency ratio of the euro area will almost double in the next half century, from 30% today to 52% in 2070, with two thirds of this increase concentrated in the next two decades (Figure 1, left-hand side).[9] The demographic transition differs, however, across countries. By 2070, the old-age dependency ratio is projected to be highest in Portugal and Greece, while it will be lowest in Ireland and France (Figure 1, right-hand side).

Population ageing is being driven by a number of demographic trends. Life expectancy will continue to rise, as people tend to live longer. According to Eurostat, remaining life expectancy at the age of 65 will average at 23.6 years for men and 26.9 years for women by 2070, which corresponds to around 5 years more than today. Moreover, fertility rates are already today well below the natural recovery level in most European countries. Looking ahead, they are expected to either remain low or even in some countries decline further. Finally, these negative demographic trends are expected to be only partly mitigated by net inward migration flows of workers, in particular as these flows are likely to fluctuate over time.

Population ageing has considerable adverse macroeconomic implications. These materialise inter alia in form of a shrinking labour force as well as a drop in the growth rates of consumption, investment and GDP growth.[10] The precise impact is strongly determined by the starting position, and whether ageing mainly stems from higher life expectancy, lower fertility rates, or a combination of both. With respect to the labour force, the pool of workers will gradually age and, in particular, if fertility rates are very low, eventually shrink. A gradual contraction in the labour force and aggregate employment will, in turn, drag down GDP growth.[11] Moreover, ageing will affect aggregate consumption and saving rates, as workers and pensioners have different marginal propensities to consume and save. Following the life-cycle-hypothesis, workers tend to accumulate savings and consume less than the elderly, but once in retirement people are likely

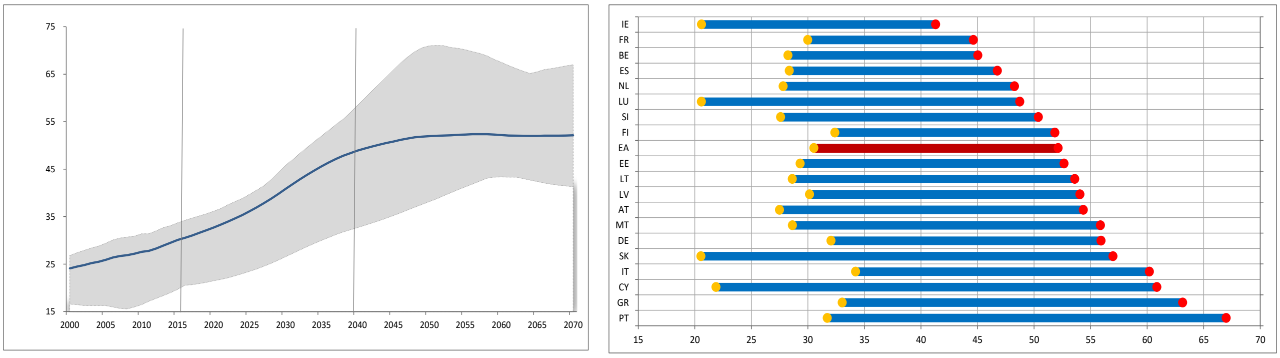
[9] The old-age dependency ratio is, however, only a rough proxy for the effective old-age dependency ratio, as the later takes on top of the statutory retirement age also incentives for earlier or later retirement into account. Moreover, the indicator does not acknowledge that the countries' statutory retirement age might be different from the 65-year-threshold.

[10] For an overview of the main channels through which ageing can affect the economy, see e.g. Börsch-Supan and Ludwig (2006), Keuschnigg et al. (2013), de la Croix et al. (2013), ECB (2018), and Brand et al. (2018).

[11] The adverse growth impact is expected to be even larger if one believes that productivity and innovation is age-dependent and hump-shaped, as shown e.g. in Aksoy et al. (2019). In our model, however, we do not assume productivity to be endogenous.

Figure 1: Old-age dependency ratio

(%, 2000-70) (in %, 2016 and 2070)



Sources: Eurostat, and authors' calculations

Note: The old-age dependency ratio (OADR) is defined as the number of people aged 65 or older as a percentage of the working-age population (i.e., people aged 15 to 64). In the chart on the left-hand side, the blue solid line shows the ratio for the euro area aggregate and the grey shaded area is determined by the largest and lowest values of the euro area countries over time. The chart on the right-hand side shows the OADR for 2016 (yellow dots) and 2070 (red dots) in all euro area countries

to dissave and increase consumption. At the same time, the prospect of rising life expectancy is likely to cause households to change their saving behaviour by fostering precautionary savings. At aggregate level, the impact can be expected to vary over time in line with changes in the underlying demographic structure and behavioural responses. The capital-labour ratio can be expected to increase with labour supply shrinking. As this will exert downward pressure on the equilibrium real interest rate, also investment growth is likely to be dampened.

Ageing poses considerable fiscal challenges. Public pension spending in Europe is elevated already today, accounting for more than half of total public expenditures. In several countries pension spending can be expected to further increase following a rising number of pensioners and given that most European countries have a pay-as-you-go (PAYG) system in place. This would aggravate the intergenerational burden sharing. Indeed, the 2018 Ageing Report projects public pension spending in the euro area to increase on average from 12.3 percentage points of GDP in 2016 to 13.5 percentage points of GDP in 2040, before falling back to 11.9 percentage points of GDP in 2070, notwithstanding large cross-country differences. Moreover, rising pension expenditures can pose challenges for the financing of pension systems, in particular for countries with a mandatory PAYG regime. The outlook of the old-age dependency ratio indicates that the number of workers potentially available to finance one pensioner will shrink from three to two by 2070. Assuming no additional changes to the main three pension parameters - i.e., the retirement age, the benefit ratio or the contribution provisions - population ageing can be expected to result in a widening of the financing gap. Yet, challenges vary across countries, reflecting differences in the set-up of their social security systems, the underlying change in the age profile and national preferences. In addition, population ageing may also affect tax revenues. Most prominently, a shrinking labour force will ceteris paribus limit indirect tax revenues and social security contributions. Taking all these aspects together, ageing can be expected to push public debt-to-GDP ratios too. This, in turn, is likely to pose long-term fiscal sustainability risks, particularly in those countries with already elevated levels of public debt today.

To address the adverse implications of population ageing, many countries have adopted pension reforms.[12] Especially countries that underwent an economic adjustment programme, such as Greece, Spain, Cyprus and Portugal, have adopted fundamental changes to their pension systems. These included a wide range of measures, affecting pension system rules as well as pension parameters. In contrast, systemic pension reforms that foresee a full or partial shift from PAYG schemes to fully-funded schemes have been limited to a few countries, mainly in Eastern Europe.

More recently, however, the reform progress has stalled and several countries are planning or have already reversed parts of their previously adopted pension reforms. For example, Germany decided in 2018 to cap both the expected decline in the benefit ratio and the expected increase in the contribution rate until 2025 at pre-fixed levels. It is contemplating whether to extend this cap until 2040, in which case the strong cohort of baby boomer would benefit from this cap. More recently, the government supported the idea to introduce a basic pension as of 2021, conditional on 35 contributory years. Slovakia decided to implicitly break the automatic link between changes in life expectancy and the statutory retirement age, by capping the retirement age at 64 years. Moreover, the government legislated generous changes to minimum pensions and the Christmas bonus. In Spain, the government inter alia postponed the implementation of the sustainability factor to 2023, which should link the replacement rate to changes in life expectancy. In Italy previous reform achievements were partly abandoned by inter alia temporarily facilitating early retirement. Greece is facing significant risks related to court decisions

[12] For a detailed overview of past pension reforms in various EU countries see Carone et al. (2016) and European Commission (2017).

that aim at undoing past pension cuts. In the Netherlands discussions are ongoing whether to postpone the foreseen increase in the retirement age, which thereby would loosen the agreed link between changes in life expectancy and retirement age. None of these reform reversals mentioned above were reflected in the 2018 Ageing Report projections, as they were not adopted by the time.

3 The framework

3.1 Key elements of the OGRE model

We estimate the macroeconomic and fiscal impact of population ageing and changes of the pension parameters on the basis of a dynamic general equilibrium OLG model. We use the model by Baksa and Munkacsi (2016a and 2016b), called OGRE (Overlapping Generations and Retirement), which is summarised in Annex 1.[13]

OGRE is a Gertler-type (Blanchard-Yaari-Weil-type) model.[14] Its central element is the demographic block which considers two types of households, both with perfect foresight: workers (i.e., people between the age of 20 and the retirement age) and retirees (i.e., people who have reached the retirement age). The absolute and relative size of these two cohorts changes over time, and so does their sum, i.e., total population. The size of the cohorts is determined by three probabilities: the probability to be born, to retire and to pass away. All three probabilities may change over time.

According to the model, workers either work, in which case they receive labour income and pay income taxes, or are unemployed, in which case they receive unemployment benefits. Workers use their disposable income or unemployment benefits for consumption and savings in risk-free bonds, in line with the Euler equation. They retire, once they have reached retirement age. This is determined by the probability to retire.[15] Retirees do not work, but receive pension benefits. Per capita pension payments are set in real terms at the start of retirement, and are

[13] For a more detailed description of the closed-economy version of the model see Baksa and Munkacsi (2016a). Compared to the original version of the model, we disregard the informal sector in this paper and the ageing shock is differently modelled. In fact, in the original version of OGRE, the ageing shock was defined as a 10 percentage point increase in the old-age dependency ratio.

[14] See Gertler (1999), Blanchard (1985), Yaari (1965) and Weil (1989).

[15] In the model we use the effective retirement instead of the statutory retirement age.

kept constant thereafter. Retirees die with a certain probability. By the end of their lifetime, retirees have consumed all their savings.

On the production side, firms hire workers and use physical capital. Firms take account of price adjustment costs when setting prices. Moreover, the model assumes labour market rigidities in form of hiring costs and wage bargaining, which have an influence on the level of unemployment.[16] Moreover, the model has a rich fiscal sector with different kinds of public revenues (personal income tax, social security contributions, VAT) and public expenditure items (pension benefits, unemployment benefits, government consumption).[17] The public pension system is a PAYG system. Governments may issue government bonds to finance any funding gap, for example in the pension system. We assume the economy to be closed. Monetary policy is characterised by a Taylor rule.

The population ageing shock enters the model via a combination of a declining or low probability to be born (equivalent to a declining or very low fertility rate) and a falling probability to pass away (equivalent to an increase in life expectancy). Thereby, an ageing shock results in a lower share of workers relative to pensioners. In particular the fertility rate determines how persistently the labour force will shrink over time. In our model, a shrinking labour force results in lower aggregate employment while the unemployment rate will decline. As labour input becomes scarcer, this would adversely affect the production process. We assume per capita labour productivity and the labour force participation rate not to change with age. Compensation per employee tends to go up as the labour force gets scarcer.

Moreover, the model accounts for other general equilibrium effects of population ageing. On the demand side, growth in aggregate consumption can be expected to be depressed by a smaller number of workers and lower income growth. This effect is partly compensated by higher aggregate consumption resulting from a rising cohort of retirees. Moreover, a more subdued growth outlook triggers a decline in aggregate investment, which would partly reduce the projected rise in the capital-labour ratio.

[16] The modelling of the labour market and the wage bargaining process is based on Blanchard and Gali (2010).

[17] In principle, the model allows to differentiate between PAYG, fully funded, and mixed pension schemes. Yet, this differentiation is not further considered in this paper.

In our model, population ageing will affect public finances through several channels: first, public pension spending will grow with a rising cohort of retirees. Subsequently, this would result in higher public expenditure, particularly if there are no built-in adjustments on the side of social security contributions or pension benefit payments. Second, lower economic activity and a shrinking labour force reduces government revenues via lower indirect taxes by households and firms. To finance a budget deficit, the government can issue risk-free bonds. This together with lower GDP growth rates will push up the public debt-to-GDP ratio.

3.2 Ageing Report projections: generating the baseline scenario

In the baseline scenario we quantify the impact of population ageing and already enacted pension reforms by taking the Ageing Report projections. These projections offer many advantages. First, the projections are based on a very detailed, country-specific set of data that allows us to capture the main characteristics of a country's pension system. Second, the projections account explicitly for the future impact of already enacted, but not yet fully implemented social security reforms on age-related spending. This is important in view of the numerous pension reforms adopted in past years, as their impact is not yet fully reflected in actual data and will only become fully visible after some time. Instead, not accounting for their impact would overstate the adverse ageing-related consequences. Third, the projections are done under a no-policy change assumption. Concretely, in the baseline scenario it is assumed that no additional pension policy measures will be adopted over the projection horizon. This provides a more accurate picture of a country's actual adjustment needs. Finally, the projections are based on a set of common macroeconomic and demographic assumptions across countries. This allows for a cross-country comparison of the demographic transition and the adjustment needs.

However, the Ageing Report projections as such are not suitable for evaluating the macroeconomic implications of population ageing, pension reforms or their reversals. In fact, the projections are based on a simple accounting framework that ignores general equilibrium behavioural reactions. Thus, feedback effects between changes in macroeconomic variables, age-related expenditures, and pension parameters cannot be captured. For example, it cannot be assessed how changes in the retirement age will affect employment, or how consumption growth will react to a lower pension benefit ratio. By integrating the Ageing Report projections in our model allows us to benefit from the information inherent in the Ageing Report while overcoming the limitations on the feedback effects.

The baseline scenario relies on two sets of information: the ageing shock and the pension expenditure dynamics projected in the Ageing Report. First, we define the ageing shock by targeting the old-age dependency ratio from 2016 to 2070, as stated in the Ageing Report. Thereby, we translate the ageing shock into the model structure by allocating the level and change in the old-age dependency ratio to the probabilities to be born, to be retired and to have passed away.[18] Thus, the projected size of the two age cohorts (i.e., the workers and the retirees) in a country over time is derived from the probabilities in the initial period and the changes over time. Thereby, we can explain to what extent changes in the old-age dependency ratio are due to changes in fertility, mortality and retirement. This is important for our analysis, as the impact of population ageing depends on the prevailing driving forces that trigger the demographic transition, as explained above. Moreover, with this approach we are able to capture changes in the demographic structure such as the sizeable cohort of the baby boom generation entering retirement, rising longevity and low fertility, as well as changes to the effective retirement age due to already enacted pension reforms.

Second, to sketch the macroeconomic and fiscal effect of population ageing in the baseline scenario we try to replicate the Ageing Report's projected path of public pension spending for the period 2016-70 in our model structure.[19] Our estimates under the baseline scenario show the long-term impact of population ageing while accounting for already enacted pension reforms. The impact is expressed as percentage point changes compared to the initial period, which is the year 2016, in line with the latest Ageing Report. Moreover, the modelling approach allows us to decompose the macroeconomic and fiscal impact into the part that is purely driven by the demographic process and the part that is affected by the implementation delay of previously adopted pension reforms.

[18] It is important to note that the way the ageing shock is modelled in this paper is different from the approach used in the earlier versions of OGRE. See footnote 13.

[19] We also reproduced the 4 underlying driving factors of the pension spending path used in the Ageing Report. The results can be interpreted as a sensitivity analysis for the fitness of the model and are shown in the Annex 2.

3.3 The policy scenarios: additional pension reforms and reform reversals

A central part of the paper relates to the policy scenario analysis. We focus on two kinds of policy scenarios, which are compared to the baseline: a scenario with additional pension reforms and one assuming a pension reform reversal.[20] The analysis is done for our two country cases: Germany and Slovakia.

In doing so, we first determine the required size of the additional pension policy instruments. Thus, under the additional policy reform scenario we calibrate the policy instruments such that they stabilise the public debt-to-GDP ratio at its initial level, i.e., in 2016. Assuming a full compensation of the public-debt impact of population ageing is obviously a radical choice. However, we use this approach for presentational purposes rather than to prescribe concrete policy advices. The main reason why we focus on changes in the public debt-to-GDP ratio rather than on changes in public pension spending is that this allows us to account for all inherent changes to the pension parameters, including the contribution rate, and for feedback effects though changes in tax revenues. We distinguish three different types of pension reforms: increase in the retirement age, decline in the benefit ratio and increase in the contribution rate. We assume additional reforms to be implemented at once. We also analyse the impact if all policy options were adjusted at the same time in comparison to the baseline scenario.

Thereafter, we define the reform reversal scenario and gauge its potential macroeconomic impact. We try to mimic, as much as possible within our model framework, the current policy discussions on reform reversals in the two country cases. Concretely, in the reversal scenario for Germany we freeze the benefit ratio at its current level of 48% and assume that the contribution rate would not exceed the threshold of 20% until 2040. With this reform reversal scenario we assume that the agreed freeze of the benefits ratio and contribution rate until 2025 will be extended until 2040.[21] In Slovakia, we assume the retirement age to stop increasing from the year 2045 onwards. This reflects the decision by the Slovakian government to freeze the retirement

[20]It should be recalled that neither the policy scenarios nor the reform reversal scenarios are based on the Ageing Report.

[21]In Germany, the so-called "double threshold" measure was adopted in mid-2018 to be effective until 2025. Members of the government suggested extending this measure until 2040, which is broadly in line with the recommendation by the Deutsche Rentenkommission issued in March 2020. As, this measure was adopted after the publication of the 2018 Ageing Report, it is not reflected in the Ageing Report projections.

age at 64 years.[22] Similar to the approach taken in the additional reform scenario, we determine the necessary compensatory measures under the reform reversal scenario by what is needed to stabilize the public debt-to-GDP ratio.

3.4 Data and calibration

The calibrated models for our two country cases match well the input data (see Table 1 and Annex 3). For the macroeconomic variables in the initial steady-state period, the calibrations are based on multi-year averages of national accounts data from Eurostat, in most cases covering the period 2009-2016, i.e., after the outbreak of the financial crisis. The targeted values include GDP growth and its main components, as well as variables related to the labour market and public finances. Moreover, we replicate all important country-specific pension system variables, as reported in the 2018 Ageing Report. These include gross pension expenditures, the (gross) replacement rate and the benefit ratio.[23] With the OGRE model we are able to reproduce the Ageing Report's change in public pension spending (including its driving factors) over the projection horizon until 2070 (see Annex 2).

The main advantage of our approach is that we are able to capture the country-specific dynamics of public pension expenditures, as projected in the Ageing Report, and combine it with the behavioural and feedback effects of our OLG model. At the same time, it should be noted that it is not possible to fully replicate all elements of the Ageing Report projections. In fact, some of the variables used in the Ageing Report cannot be easily translated into our model structure. For example, with respect to pension expenditures, the Ageing Report projects total pension expenditure, which includes other components such as early pension payments, minimum pensions or disability pensions. In our model, we only look at earnings-related pensions. To make the model results comparable with the Ageing Report, the values are therefore rescaled. Moreover, in the model we look at the effective retirement age, while the Ageing Report projects the effective exit rate from the labour market. We assume these two variables to be identical, while this is not necessarily the case as old-age workers can get unemployed.

[22] In Slovakia, the government decided in end-2018 to implicitly break the automatic link between changes in life expectancy and retirement age, by capping the retirement age at 64 years. This will be reached in 2045.

[23] When replicating the key pension system variables, we focus on the 2016 data provided in the 2018 Ageing Report (as 2016 is the starting year of the projections), to ensure comparability of the results.

To model the demographic structure, we use Eurostat's population data on the absolute size of the respective cohorts for Germany and Slovakia.[24] We translate the data such to obtain values for the three probabilities in the initial period for which the model is calibrated, i.e., for 2016. Concretely, in the initial period the probability to be born results from the number of people in the age cohort of the 19 year-olds, while the probability to pass away is based on the size of the very old-age cohort (i.e. of the people aged 85 years and older).[25] The retirement probability is derived from the effective exit age from the labour market, as reported by Eurostat.

Table 1: Calibrating the initial period

in % of GDP, unless stated otherwise

	Germany		**Slovakia**	
	calibrated value	*targeted value*	*calibrated value*	*targeted value*
Consumption (total)	60.7	54.6	60.8	60.4
Consumption (public)	19.1	19.1	16.9	16.9
Investment	20.1	20.0	22.3	20.2
Compensation of employees	59.7	56.3	54.9	46.0
Unemployment rate (%)	5.7	5.6	12.8	12.7
Unemployment benefit	1.5	1.8	0.7	0.7
Pension expenditure	7.2	7.8	6.9	6.7
Benefit ratio (%)	34.1	32.4	34.4	36.3
Old-age dependency ratio (%)	37.2	38.3	36.5	32.6
Public debt	75.4	75.4	53.3	53.3

Note: The targeted values are based on the 2018 Ageing Report and Eurostat data.

The model structure allows us to replicate Eurostat's projected path of the old-age dependency ratio until 2070 (see Figure 2, left-hand side panels). In fact, for Germany the old-age dependency ratio is projected to increase sharply until 2040 and more slowly thereafter, while for Slovakia the ratio will peak in 2060 before starting to decline thereafter. Moreover, the development of the old-age dependency ratio can be decomposed into the three driving forces, i.e., the probabilities to be born, to retire and to pass away. As shown in Figure 2 (upper right-hand side panel), for Germany, the dynamics of the old-age dependency ratio is mostly driven by the cohort effect, as the sizeable baby boom generation will enter retirement within the next one and a half decades. This is reflected in the strong contribution of the probability to retire until

[24] We use Eurostat data, in line with the Ageing Report. As a sensitivity analysis (available by the authors upon request) we replicated the analysis with UN population projections.

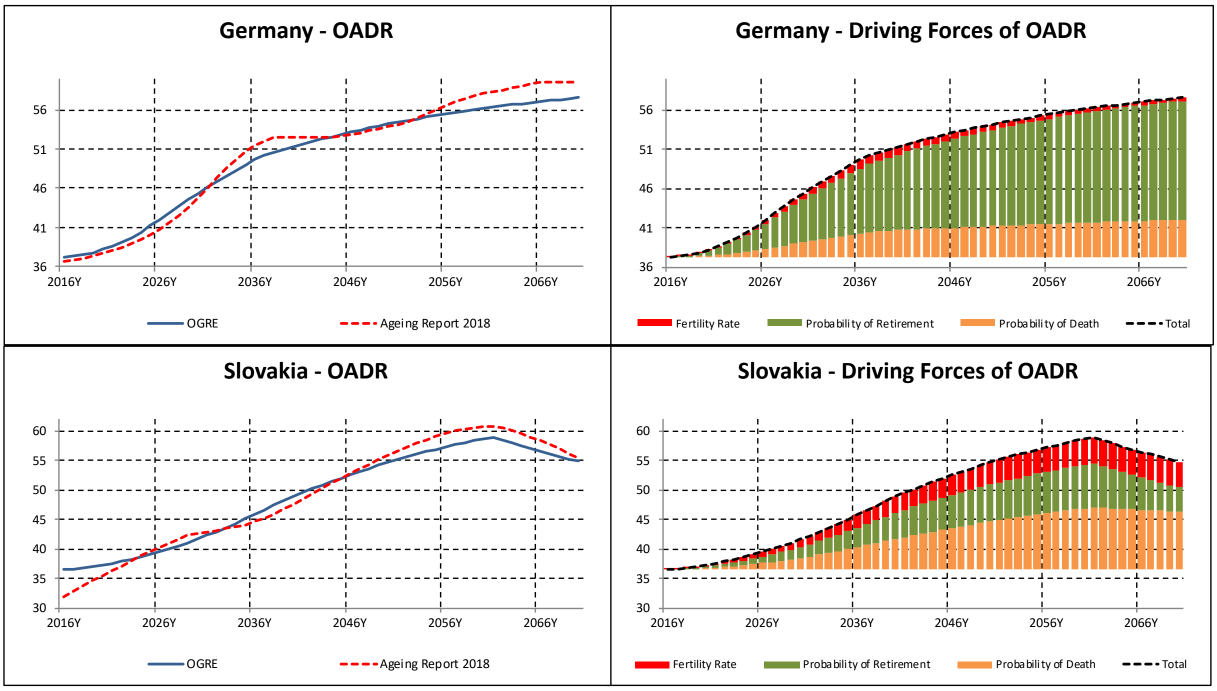
[25] While migration flows are not explicitly shown in the model, they are indirectly captured as they impact the absolute size of the cohorts.

2040. While improvements in life expectancy are also projected to matter, their impact on the dependency ratio will be much smaller.

In Slovakia, the expected change in the old-age dependency ratio is mostly due to the strong improvement in life expectancy, as shown by the probability of death (see Figure 2, lower right-hand side panel). The projected impact of the cohort effect, which is smaller compared to Germany, will gradually decline as of around 2060. To a lower extent, population ageing in Slovakia is also driven by its very low fertility rate.[26]

Figure 2: Old-age dependency ratio and driving forces

(retirees/workers, in %)



Sources: Eurostat and authors' calculations.

[26]Although Slovakia's fertility rate is assumed to improve in the long-run, the old-age dependency ratio in the long-run is expected to be impacted by today's very low fertility rate.

4 Results

4.1 The baseline scenario

Under the baseline scenario, we find strong support for the general view that population ageing has adverse macroeconomic and fiscal implications. Assuming no compensatory measures, the results show an increase in the public debt-to-GDP ratio by around 100 percentage points until 2070, compared to the initial period, for both Germany and Slovakia.[27] Moreover, real GDP per capita is projected to decline by almost 14% in Germany and 9% in Slovakia, compared to the initial period (see Tables 2 and 3).[28] The long-run model results can be decomposed into the impact driven by the demographic transition (see column "ageing" in Tables 2 and 3) and the impact that results from past pension reform measures (see the columns under "impact of adopted pension reforms" in Tables 2 and 3).

In the case of Germany, the model-induced decline in GDP per capita by almost 14% by 2070 largely results from population ageing (decline by 15.3%). Yet, previously adopted pension reforms can be expected to partly cushion or amplify the macroeconomic impact, depending on the respective instruments.[29] Specifically, the legislated gradual increase in the statutory retirement age until 2029 is expected to have a potentially growth-enhancing impact, and will thereby partly cushion the adverse growth impact of population ageing. In line with our model, the effective retirement age will increase from 64 years (in 2016) to 65.5 years by 2070. The baseline scenario captures also other legislative settings: in case of rising pension spending, this requires compensatory adjustments in form of higher contribution rates or a lower benefit ratio. Yet, higher contribution rates can be expected to further worsen the impact on GDP per capita (by 1.2%), while a less generous benefit ratio will only have a marginal positive growth impact.

[27]The results are qualitatively very similar to the results with the open-economy version of the model. Yet, for simplicity reasons we only report the results with the closed economy version here, while the results with the open-economy model can be shared upon request. Compared to the open-economy version, the results show an overall more pronounced adverse impact of population ageing in the closed-economy version, as in particular variables, such as consumption, seem to react more strongly to changes in the degree of openness.

[28]While the tables show the long-run results, compared to the initial period, the projected long-run developments over time are shown in the charts in Annex 4 (for Germany) and Annex 5 (for Slovakia).

[29]In Germany the pension system is prohibited to generate public debt. Instead, increases in pension spending need to be compensated by higher contribution rates and lower benefit ratios. Moreover, in 2007 Germany legislated a reform that foresees a gradual increase in the statutory retirement age from 65 to 67 years by 2029. Back then, it was also decided to reduce the pathways to early retirement, inter alia through higher penalty deductions, and to harmonise the parameters for calculating pension benefits in West and East Germany.

Table 2: Baseline scenario: long-run results for Germany

in % of GDP, unless stated otherwise

Germany	Baseline[1]	Ageing	Impact of adopted pension reforms:		
			retirement age	*benefit ratio*	*contribution rate*
	(1+2+3+4)	*(1)*	*(2)*	*(3)*	*(4)*
GDP per capita (%)	-13.9	-15.3	2.5	0.1	-1.2
Consumption: total	-1.5	-1.6	0.4	0.0	-0.2
Consumption: share of young (% of total)	-4.6	-7.8	1.1	1.7	0.4
Investment	-1.3	-1.4	0.1	0.0	0.0
Unemployment rate (%)	0.8	-0.3	0.0	0.0	1.1
Pension expenditure	1.9	3.8	-0.6	-1.3	0.0
Primary balance	-2.9	-6.5	1.1	1.1	1.4
Public debt	100.3	208.6	-33.9	-29.4	-44.9

[1]Deviation from the respective values in the initial steady state (see Table 1) until 2070

The baseline results for Germany show a decline of aggregate total consumption by 1.5 percentage points of GDP, compared to the initial period. This is largely driven by the steep fall in the workers' share of total aggregate consumption (by 4.6 percentage points), which can be explained by two factors. First, the cohort of the young will shrink relative to the cohort of the pensioners. Second, as agents are forward-looking, the young cohort tends to consume less and save more to prepare for increased longevity. By building up a higher stock of savings during working lives, future pensioners will be better prepared for consumption smoothing. The aging-induced decline in the young's share of total aggregate consumption is partly reduced by the reforms implying a higher retirement age (cohort effect) and a lower benefit ratio (lower disposable income of the pensioners).

Finally, the baseline results for Germany assume pension expenditure to increase by 1.9 percentage points of GDP in 2070, compared to the initial period. This figure is broadly comparable with the Ageing Report projections.[30] The ageing impact on pension expenditures will be almost halfed thanks to previously adopted reforms, namely the rise in the retirement age and the declining benefit ratio. As the German pension system is prohibited to accumulate debt, the projected public debt increase of 100 percentage points of GDP by 2070 only reflects the feedback effects via lower public revenues and a negative denominator effect. It abstracts from any others factors, like e.g. national fiscal rules, that could imply a different debt trajectory

[30]The 2018 Ageing Report projects pension expenditures in Germany to increase by 2.4 percentage points of GDP until 2070. See Table 1 in European Commission (2018).

besides the ageing impact.

While the results for Slovakia are qualitatively very similar to those for Germany, they reveal important differences between the two countries. In particular, differences relate to the main driving forces of population ageing, the structure of the economy as well as the impact of past pension reforms. In fact, as shown in Figure 2, population ageing in Slovakia is mainly driven by rising life expectancy. To address the ageing challenge, Slovakia adopted in 2012 an important pension reform that first automatically links the retirement age to changes in life expectancy and second reduces the generosity of the pension system through changes in the indexation rule.

The baseline results for Slovakia predict a decline in GDP per capita by 9% by 2070, compared to the initial period (see Table 3). Without the adopted pension reforms, namely the one adjusting the retirement in line with changes in life expectancy, the ageing impact would be more than twice as strong, given the projected strong increase in life expectancy.

Table 3: Baseline scenario: long-run results for Slovakia

in % of GDP, unless stated otherwise

Slovakia	**Baseline**[1]	**Ageing**	**Impact of adopted pension reforms:** *retirement age*	*benefit ratio*	*contribution rate*
	(1+2+3+4)	*(1)*	*(2)*	*(3)*	*(4)*
GDP per capita (%)	-9.0	-18.5	9.8	0.1	0.0
Consumption: total	-0.4	-1.3	0.9	0.0	0.0
Consumption: share of young (% of total)	-4.2	-9.5	4.0	1.3	0.0
Investment	-1.1	-2.0	0.8	0.0	0.0
Unemployment rate (%)	-1.0	-1.3	0.4	0.0	0.0
Pension expenditure	1.2	4.1	-2.2	-0.7	0.0
Primary balance	-2.4	-6.7	3.6	0.7	-0.1
Public debt	103.0	230.5	-105.3	-24.3	2.1

[1]Deviation from the respective values in the initial steady state (see Table 1) until 2070

The consumption share of the young in Slovakia is overall estimated to decline by 4.2 percentage points of total consumption, compared to the initial period. The previously adopted pension reforms can be seen as improving the intergenerational burden sharing as they help to cushion the adverse ageing impact for the young relative to those for the pensioners. In fact, the reforms leading to a higher retirement age and reducing the generosity of the pension system limit the decline in the young's consumption share by 4 and 1.3 percentage points, respectively.

The baseline results for Slovakia show an increase in pension expenditure by 1.2 percentage points of GDP by 2070, compared to the initial period. This is the same figure as projected in the Ageing Report. Without the impact of the adopted reforms the increase in pension spending would be three times as high. The public debt-to-GDP ratio is estimated to increase by 103 percentage points of GDP by 2070, but would be considerably stronger without the adopted reforms.

4.2 Policy scenarios with additional reform measures

The spending pressures arising from population ageing call for additional pension reforms which would need to go well beyond those already adopted. Therefore, as a next step, we examine the macroeconomic and fiscal impact of additional pension reforms. We report the macroeconomic impact of the additional pension policies in comparison to the baseline scenario, i.e. by how much would the additional policy measures change the results obtained under the baseline scenario.

We calibrate the additional pension reforms by the effort that would be required to fully compensate for the increase in the public-debt-to-ratio due to ageing. In other words, the additional pension reforms need to ensure that the public debt-to-GDP ratio is kept broadly constant at the initial level. We use the same modelling framework as in the previous section, as this allows us to explicitly account for the economic feedback effects of pension reforms. The assumption of perfect foresight continues to hold. Thus, households are expected to react immediately to policy changes by inter alia adjusting their savings behaviour. We consider the same types of pension reforms as discussed above. Concretely, we analyse by how much the three pension parameters, i.e., the retirement age, the benefit ratio and the contribution rate, would need to be changed in order to alleviate the public debt impact of population ageing, and assess their impact.

Taking again Germany and Slovakia as country cases, we find the following results in the additional reform scenarios: in order to fully compensate for the ageing-induced debt increase of around 100 percentage points of GDP, Germany would need to increase the effective retirement age by 3.5 years until 2070, compared to an increase in the effective retirement age to 65.5 years by 2070 under the baseline scenario (see Table 4). Similarly, Slovakia would need to increase its retirement age by 4.5 years, compared to an increase to more than 67 years by 2070 under the baseline scenario (see Table 5). Alternatively, if the compensatory measures were only to rely on reforming pension entitlements, the benefit ratio would need to be reduced by 26 percentage points in Germany and 21 percentage points in Slovakia, in order to keep the public-debt-to-ratio roughly constant at the initial level. For reforms targeting the financing side, pension contributions would need to be increased by almost 11 percentage points in Germany and 7 percentage points in Slovakia.

Table 4: Additional pension reforms: long-run results for Germany

in % of GDP, unless stated otherwise

Germany	**Baseline**[1]	**Impact of additional pension reforms**[2]			
		retirement age	*benefit ratio*	*contribution rate*	*mix*
		(1)	*(2)*	*(3)*	*(1)+(2)+(3)*
GDP per capita (%)	-13.9	6.1	0.2	-2.9	1.1
Consumption: total	-1.5	1.2	-0.1	-0.6	0.2
Consumption: share of young (% of total)	-4.6	2.5	5.8	0.9	3.1
Unemployment rate (%)	0.8	-0.1	-0.1	2.8	0.8
Pension expenditure	1.9	-1.3	-4.5	0.1	-1.9
Public debt	100.3	-108.6	-101.7	-97.1	-102.3
Policy instruments					
Retirement age (year)	65.5	3.5	-	-	1.2
Benefit ratio (%)	-8.0	-	-26.0	-	-8.7
Contribution rate (%)	4.4	-	-	10.6	3.5

[1]Deviation from the respective values in the initial steady state (see Table 1) until 2070, long-run results for the retirement age

[2]Deviation from the baseline

Furthermore, in line with the literature, we find evidence that the macroeconomic impact strongly differs depending on which reform measures have been adopted. For example, in the case of Germany an increase in the retirement age by 3.5 years, which would be necessary to keep the public-debt-to-ratio constant at the level in the initial period, would imply that GDP per capita would be 6.1% stronger, compared to the baseline (see Table 4). In the case of Slovakia, adjustments in the retirement age would allow to almost fully offset the adverse growth impact estimated in the baseline scenario. In fact, prolonging people's working lives helps to enlarge the cohort of workers compared to the baseline scenario.[31] This can be expected to boost aggregate consumption and investment, while public pension expenditure could be scaled down considerably. Cutting pension entitlements, in turn, would leave GDP per capita roughly unchanged, compared to the baseline scenario (see Table 4, the column "benefit ratio" for Germany and Slovakia). Furthermore, higher contribution rates would in the case of Germany aggravate the adverse impact on GDP per capita, compared to the baseline scenario, while for Slovakia the results would be unchanged compared to the baseline scenario.

Table 5: Additional pension reforms: long-run results for Slovakia

in % of GDP, unless stated otherwise

Slovakia	**Baseline**[1]	**Impact of additional pension reforms**[2]			
		retirement age	*benefit ratio*	*contribution rate*	*mix*
		(1)	*(2)*	*(3)*	*(1)+(2)+(3)*
GDP per capita (%)	-9.0	8.4	0.4	0.0	2.9
Consumption: total	-0.4	1.5	-0.1	-0.1	0.4
Consumption: share of young (% of total)	-4.2	2.8	5.4	1.4	3.2
Unemployment rate (%)	-1.0	0.0	-0.1	0.0	-0.1
Pension expenditure	1.2	-1.5	-2.9	0.0	-1.4
Public debt	103.0	-97.5	-101.7	-102.5	-100.9
Policy instruments					
Retirement age (year)	67.2	4.5	-	-	1.5
Benefit ratio (%)	-5.0	-	-21.0	-	-7.0
Contribution rate (%)	-0.2	-	-	6.8	2.3

[1]Deviation from the respective values in the initial steady state (see Table 1) until 2070, long-run results for the retirement age

[2]Deviation from the baseline

Relying at one reform measure at a time could, however, turn out to be politically challenging as the additional adjustment burden would be placed disproportionally on one generation. For example, by lifting the retirement age the adjustment costs would be fully born by the cohort of workers, as their working lives would be prolonged. Raising the contribution rate would only affect the cohort of workers, while lower pension entitlement would only affect the cohort of pensioners. Instead, reform mixes composed of various measures can be expected to help spreading

[31]In this paper we abstract from the discussion how plausible it is to assume healthy ageing. Yet, this is a relevant question with respect to the effectiveness of the policy instrument of postponing the retirement age. We also abstract from age-dependent productivity rates.

the adjustment burden more equally across generations. Therefore, we analyse the impact of a mixture of reforms. We assume that each reform measure would contribute by one-third to the adjustment effort that is needed to keep public debt constant at its initial level.

The estimation results under the reform mix scenario are encouraging in terms of intergenerational fairness and macroeconomic impact (see Tables 4 and 5, last column). Looking again at the examples of Germany and Slovakia, the necessary increase in the retirement age would be one-third of the adjustment coming only through changes in the retirement age. Likewise, pension entitlements and contribution rates would need to be adjusted by around one-third compared to a situation with only one single instrument. Overall, under the reform mix scenario the adverse ageing impact on GDP per capita would be contained, compared to the baseline scenario.

5 Quantifying the costs of pension reform reversals

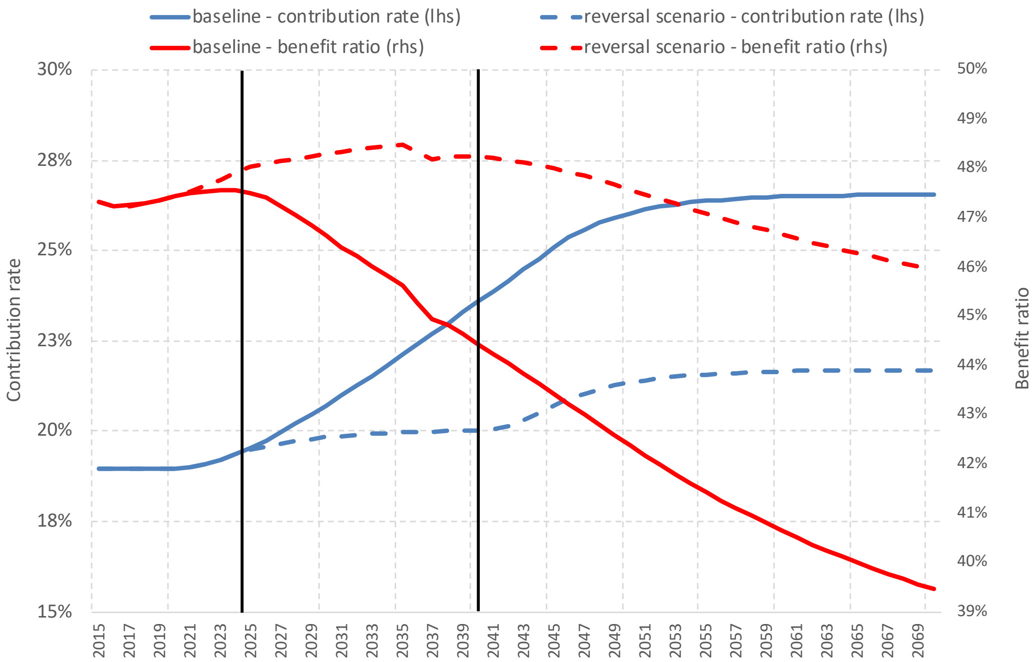
Several countries are currently discussing whether to reverse parts of their previously adopted pension reforms. Our framework allows us to evaluate the macroeconomic and fiscal costs of pension reform reversals. In the following, we study the impact of reform reversals by looking more closely at the cases of Germany and Slovakia. In both countries the reform reversals only emerged after the 2018 Ageing Report and therefore are not reflected in our baseline scenario.

In Germany, previously adopted reform measures have been lately reversed and further revisions are under discussion. Concretely, in August 2018 the German government agreed on a reform reversal, by setting the so-called "double threshold" until 2025. This "double threshold" foresees that (i) the pension contribution rate should not rise above 20% (compared to 18.9% in the initial period), and (ii) the benefit ratio should not fall below 48% until 2025.[32] On top of this, although not yet decided, it was proposed to prolong the "double threshold" until 2040. This latter proposal, however, can be expected to be considerably more costly, as the large cohort of baby boomers will retire between 2025 and 2040, and would therefore benefit from the additional

[32] It should be noted that the benefit ratio used in the model is not fully comparable to the national definition. The latter looks at the level of pensions in retirement relative to earnings for a standard pensioner who earns average income and pays contributions for 45 years. Therefore we re-scaled the benefit ratio in Figure 3 to align it with the policy discussion.

generosity foreseen under the "double threshold" scenario extended to 2040.

Figure 3: Pension reform reversal in Germany: "double threshold" until 2040



Sources: Börsch-Supan and Rausch (2018), and authors' calculations.

We define the "double threshold" until 2040 as our reform reversal scenario for Germany.[33] Thus, in the reform reversal scenario the pension contribution rate should not rise above 20% and the benefit ratio should not fall below 48% until 2040. After 2040, both the benefit ratio and the contribution rate are assumed to develop roughly in line with the Ageing Report projections.[34] This is shown in Figure 3, with the solid lines representing the baseline scenario and the dashed lines reflecting the reform reversal scenario. Like in the previous sections, the baseline scenario replicates the 2018 Ageing Report projections, including the already adopted reforms, according to which Germany's contribution rate will increase to around 27% and its benefit ratio will decline to below 40% by 2070 (see Figure 3, solid lines). Under the baseline scenario, both ratios are projected to start worsening rapidly from the mid-2020s onwards.[35]

[33] The reform reversal scenario for Germany includes the "double threshold" agreement until 2025, as this was only decided in mid-2018 and therefore is not included in the 2018 Ageing Report projections.

[34] Yet, the paths of the dotted and solid lines do not fully match after the year 2040 given that the two scenarios imply different feedback effects until 2040 which will have a lagged impact. This is particularly relevant for the benefit ratio.

[35] In the baseline scenario, the contribution rate is projected to increase to 19.5% and the benefit ratio to

In the reform reversal scenario, we quantify the costs of reform reversals by the adverse impact on the public debt ratio, in comparison to the debt impact under the baseline scenario. For Germany, we find that the reform reversal would imply sizeable costs (see Table 6, column "reform reversal"). Specifically, by 2070, the increase in the public debt-to-GDP ratio can be expected to be ceteris paribus almost 60 percentage points higher than under the baseline scenario, as a result of higher pension expenditures, adverse feedback effects and lower contribution rates. At the same time, the impact on GDP per capita would be only marginally better (decline by 13.3%) than under the baseline scenario.[36]

Table 6: Reform reversal scenario: long-run costs for Germany

in % of GDP, unless stated otherwise

Germany	**Baseline**[1]	**Reform reversal**[1]	**Compensatory measures**[2,3]
GDP per capita (%)	-13.9	-13.3	10.1
Consumption: total	-1.5	-1.3	2.0
Consumption: share of young (% of total)	-4.6	-6.5	4.2
Unemployment rate (%)	0.8	0.1	-0.2
Pension expenditure	1.9	3.1	-2.1
Public debt	100.3	157.5	-176.0
Policy instruments			
Retirement age (year)	65.5	65.5	5.5

[1]Deviation from the initial steady-state until 2070; long-run result for the retirement age
[2]Deviation from the reform reversal scenario
[3]Change in retirement age to compensate for the debt impact of the reform reversal

Moreover, we calculate the compensatory measures for Germany that would be needed in order to offset the adverse debt impact under the reform reversal scenario of almost 160 percentage points of GDP by 2070, compared to the initial period (see Table 6, column "compensatory measures").[37] We assume that the adjustment will be solely placed on increases in the retirement age. We find that the compensatory measure needed to offset the effects of the reform reversal would be very painful for future generations. In fact, the results suggest that Germany would

decrease to 47.6% by 2025. This implies that the "double threshold" will not be a binding constraint until 2025, except for a small gap arising for the benefit ratio towards the end of this period, while it will become considerably more constraining thereafter.

[36]It should be noted, however, that the results might reflect a potential upward bias, as the model does not explicitly account for behavioural changes of the labour supply in response to changes in the pension system. In fact, the labour force participation rate is kept constant.

[37]Thus, the estimated increase in the public debt-to GDP ratio by almost 160 percentage points under the reform reversal scenario reflects both the ageing-related debt impact, as derived under the baseline scenario, and the debt impact due to the reform reversal.

have to increase its effective retirement age by 5.5 years to 71 years by 2070 to be able to fully offset the increase in the public debt-to-GDP ratio.[38] Out of this suggested total increase in the retirement age, 2 years would be on account of the reform reversal, while the remaining 3.5 years reflect the compensatory measures needed to offset the ageing cost increase under the baseline scenario. Moreover, the compensatory measures in form of higher retirement age would help to largely alleviate the adverse macroeconomic and fiscal effects of the pension reform reversal.

Slovakia is another interesting example to study the impact of reform reversals. In end-2018, Slovakia decided to reverse one of its previous pension reforms, namely by capping the automatic link between the retirement age and changes in life expectancy at the age of 64 years. In the baseline scenario, the effective retirement age is expected to increase from its initial level of 61 years (2016) to above 67 years. In the reform reversal scenario we assume for Slovakia that the automatic link between the retirement age and increases in life expectancy will be capped in 2045, i.e., the year when the retirement age is expected to reach 64 years, according to the baseline scenario.

Like for Germany, we quantify the fiscal costs of the reform reversal in Slovakia by comparing the debt impact under the reform reversal scenario with that under the baseline scenario. Our results show that such a reform reversal would be very costly. In fact, the increase in the public debt-to-GDP ratio would be more than 50 percentage points higher than the estimated increase of around 100 percentage points of GDP under the baseline scenario (see Table 7). Moreover, the results under the reform reversal scenario suggest an even more adverse economic outlook than under the baseline scenario: GDP per capita can be expected to shrink by almost 15% by 2070, compared to the initial period. This is in line with earlier findings for Slovakia, namely that, as discussed in section 3, the increase in life expectancy is the most dominant factor for the expected rise in the old-age dependency ratio. Moreover, capping the retirement age at 64 years would lower further the young's total consumption share and increase pension expenditure even more, compared to the baseline.

[38] Other simulations obtain similar results. Börsch-Supan and Rausch (2018) calculate the fiscal costs of the "double threshold" assuming that it would be kept until 2060. These authors consider an alternative policy scenario, and ask by how much the VAT rate would need to be increased to finance the "double threshold" via tax subsidies to the pension system. They find that the VAT rate would need to increase by more than 6 percentage points by 2040 to offset the fiscal cost of the "double threshold".

Table 7: Reform reversal scenario: long-run costs for Slovakia

in % of GDP, unless stated otherwise

Slovakia	**Baseline**[1]	**Reform reversal**[1]	**Compensatory measures:**[2] *benefit ratio*	*contribution rate*	*mix*
			(1)	*(2)*	*(1)+(2)*
GDP per capita (%)	-9.0	-14.8	0.5	-0.1	0.2
Consumption: total	-0.4	-0.9	-0.2	-0.1	-0.2
Consumption: share of young (% of total)	-4.2	-6.7	7.8	2.2	5.0
Unemployment rate (%)	-1.0	-1.3	-0.1	0.2	0.0
Pension expenditure	1.2	2.6	-4.2	0.0	-2.1
Public debt	103.0	156.1	-145.4	-154.7	-150.5
Policy instruments					
Retirement age (year)	67.2	64.0	-	-	-
Benefit ratio (%)	-5.0	-5.0	-29.0	-	-14.5
Contribution rate (%)	-0.2	-0.2	-	12.2	6.1

[1]Deviation from the initial steady-state until 2070; long-run result for the retirement age
[2]Deviation from the reform reversal scenario

In addition, we analyse the hypothetical case that compensatory measures were adopted that would aim to offset the debt impact under the reform reversal in Slovakia, compared to the initial period. Concretely, we consider addressing the reform reversal costs through either lower benefit ratios, higher contribution rates, or a combination of the two (see Table 7, last three columns). We find that, if the retirement age were to be capped at 64 years, the benefit ratio would need to be lowered massively, by more than 29 percentage points, in order to fully compensate for the debt increase until 2070. Alternatively, the contribution rate would need to be increased by more than 12 percentage points to fully compensate for the debt impact of the reform reversal. Although the size of the compensatory measures could be halved when combining the two compensatory measures (see Table 7, column "mix"), they would not help to alleviate the adverse impact on GDP per capita.

6 Conclusions

The paper offers a framework to examine the measures needed to cope with the macroeconomic and fiscal effects of population ageing, as well as to quantify the costs of pension reform reversals. Concretely, we exploit the country-specific information contained in the 2018 Ageing Report which we integrate into a dynamic general equilibrium model with overlapping generations to capture feedback effects. In the baseline scenario we thereby account for the impact of already adopted pension reforms, as captured in the Ageing Report. We also examine the additional pension reforms that are needed to contain the public debt impact arising from the demographic transition, and assess their macroeconomic impact. We find evidence that mixed reforms which consist of different pension reform measures help to spread the adjustment burden more equally across generations. Intergenerational equity of public pension systems is an important consideration, also in view of raising political pressures to revert past pension reforms. We study the costs of pension reform reversals in two countries: Germany and Slovakia. We find that pension reform reversals in these two countries could generate substantial fiscal costs as measured by the additional public debt-to-GDP impact compared to the baseline scenario and potentially worsen the macroeconomic outlook. Addressing these costs through other policy instruments could be expected to require painful adjustments, thereby potentially worsening intergenerational fairness further.

Our paper contributes to the literature in two ways: first, to our knowledge this is the first paper which integrates the long-term dynamics of pension expenditures, as projected in the Ageing Report, into a dynamic general equilibrium model with overlapping generations. By this we are able to capture feedback effects in a general equilibrium framework, and to systematically account for important country-specific aspects and already adopted pension reforms which will become effective over time. Second, our framework allows us to quantify the expected macroeconomic and fiscal costs of reform reversals. While most European countries have adopted important pension reforms in the last two decades, more recently pension reform reversals are being discussed or have already been decided. So far pension reform reversals have hardly been reflected in the literature. We try to fill this gap with our framework that allows us to systematically quantify the costs of pension reform reversals.

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Annex 1 Technical description of the OGRE model

In this technical appendix, first, we focus on solving the optimizing problems of the young and old generations, we describe the pay-as-you-go pension system and the price and wage setting equations of the firms. At the end, we provide the normalized - detrended by population growth - equations and the steady-state calculation of the model. Regarding any other technical detail, further information is available from the authors upon request.

A.1.1 Demography and overlapping generations

Demography

Total population (N_t) is equal to the sum of the number of old (retired) (N_t^O) and young (worker) people (N_t^Y):

$$\begin{aligned} N_t &= N_t^O + N_t^Y \\ N_t^Y &= (1-\omega_{t-1}^Y)N_{t-1}^Y + n_t N_{t-1}^Y \\ N_t^O &= (1-\omega_{t-1}^O)N_{t-1}^O + \omega_{t-1}^Y N_{t-1}^Y \end{aligned}$$

s_t denotes the ratio of the number of old and young people, while s_t^Y denotes the share of young people in the whole population:

$$\begin{aligned} s_t &= \frac{N_t^O}{N_t^Y} = \frac{(1-\omega_{t-1}^O)N_{t-1}^O + \omega_{t-1}^Y N_{t-1}^Y}{N_t^Y} = (1-\omega_{t-1}^O)\frac{N_{t-1}^O}{N_{t-1}^Y}\frac{N_{t-1}^Y}{N_t^Y} + \omega_{t-1}^Y \frac{N_{t-1}^Y}{N_t^Y} \\ &= \frac{(1-\omega_{t-1}^O)}{(1-\omega_{t-1}^Y+n_t)}s_{t-1} + \frac{\omega_{t-1}^Y}{(1-\omega_{t-1}^Y+n_t)} \\ s_t^Y &= \frac{N_t^Y}{N_t} = \frac{N_t^Y}{N_t^Y+N_t^O} = \frac{1}{1+\frac{N_t^O}{N_t^Y}} = \frac{1}{1+s_t} \end{aligned}$$

Then, we can express the growth rate of each cohort:

$$\begin{aligned} 1+g_t^{N,Y} &= \frac{N_t^Y}{N_{t-1}^Y} = \frac{(1-\omega_{t-1}^Y)N_{t-1}^Y + n_t N_{t-1}^Y}{N_{t-1}^Y} = 1-\omega_{t-1}^Y + n_t \\ 1+g_t^{N,O} &= \frac{N_t^O}{N_{t-1}^O} = \frac{(1-\omega_{t-1}^O)N_{t-1}^O + \omega_{t-1}^Y N_{t-1}^Y}{N_{t-1}^O} = (1-\omega_{t-1}^O) + \frac{\omega_{t-1}^Y}{s_{t-1}} \end{aligned}$$

Finally, population (and the BGP) growth follows as:

$$
\begin{aligned}
1+g_t &= 1+g_t^N = \frac{N_t^Y + N_t^O}{N_{t-1}^Y + N_{t-1}^O} = \frac{\frac{N_t^Y}{N_{t-1}^Y} + \frac{N_t^O}{N_{t-1}^Y}}{\frac{N_{t-1}^Y}{N_{t-1}^Y} + \frac{N_{t-1}^O}{N_{t-1}^Y}} = \frac{1+g_t^{N,Y} + \frac{N_t^O}{N_{t-1}^Y}}{1+s_{t-1}} = \\
&= \frac{1+g_t^{N,Y} + \frac{N_t^O}{N_{t-1}^Y}\frac{N_t^Y}{N_t^Y}}{1+s_{t-1}} = \frac{1+g_t^{N,Y} + s_t(1+g_t^{N,Y})}{1+s_{t-1}} = (1+g_t^{N,Y})\frac{1+s_t}{1+s_{t-1}}
\end{aligned}
$$

Retired generation

First-order conditions of a retired agent

'Retired' agent i of retired cohort a is one individual who retired a years ago. Each agent maximises the following Bellman-equation:

$$
V^O(B_{a-1,t-1}^O(i)) = \max\left\{\frac{1}{1-\gamma}C_{a,t}^O(i)^{1-\gamma} + \beta E_t(1-\omega_t^O)V^O(B_{a,t}^O(i))\right\}
$$

subject to this budget constraint:

$$
(1+\tau_t^C)p_t^O C_{a,t}^O(i) + (1-\omega_t^O)B_{a,t}^O(i) = (1+r_{t-1})B_{a-1,t-1}^O(i) + TR_{a,t}^{YO}(i)
$$

where O denotes the retired cohort, the $TR_{a,t}^{YO}(i)$ is the pension benefit that was set by the government a years ago. Here, we assume that $TR_{n,t+n}^{YO}(i) = TR_{0,t}^{YO}(i)\ \forall n > 0$, i.e. the government in the pay-as-you-go pension system sets the real value of the benefit at the time of retirement, and provides the same real amount until the pensioner passes away. $C_{a,t}^O(i)$ is the level of individual consumption, $B_{a,t}^O(i)$ is the individual risk-free bond, p_t^O is the retailer price of the pensioners' consumption basket, and τ_t^C is the consumption tax rate.
The first-order conditions are as follows:

$$
\begin{aligned}
C_{a,t}^O(i) &: \quad C_{a,t}^O(i)^{-\gamma} + \lambda_{a,t}^O(1+\tau_t^C)p_t^O = 0 \\
B_{a,t}^O(i) &: \quad \beta E_t(1-\omega_t^O)V_{B_{a,t}^O(i)}^O + E_t(1-\omega_t^O)\lambda_{a,t}^O = 0
\end{aligned}
$$

The one-period-ahead Envelope-theorem is:

$$
E_t V_{B_{a,t}^O(i)} = -E_t\lambda_{a+1,t+1}^O(1+r_t)
$$

The first-order conditions imply the Euler equation:

$$\beta E_t \frac{C^O_{a,t}(i)^\gamma}{C^O_{a+1,t+1}(i)^\gamma}(1+r_t)\frac{(1+\tau^C_t)p^O_t}{(1+\tau^C_{t+1})p^O_{t+1}} = 1$$

which can be rearranged:

$$E_t C^O_{a+1,t+1}(i) = C^O_{a,t}(i)\beta^{\frac{1}{\gamma}}(1+r_t)^{\frac{1}{\gamma}}\Lambda^O_t$$

where

$$\Lambda^O_{t,t+1} = E_t\left[\frac{(1+\tau^C_t)p^O_t}{(1+\tau^C_{t+1})p^O_{t+1}}\right]^{\frac{1}{\gamma}}$$

Based on the Euler-equation, all future individual retired consumptions follow:

$$E_t C^O_{a+n,t+n}(i) = C^O_{a,t}(i)\beta^{\frac{n}{\gamma}} E_t \prod_{k=1}^{n}(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda^O_{t,t+k}$$

Individual consumption of a retired agent

First, we derive the intertemporal budget constraint from the one-period budget constraint:

$$E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+\tau^C_{t+n})p^O_{t+n}C^O_{a+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} = E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^O_{t+k-1})TR^{YO}_{a+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} + \\ +(1+r_{t-1})B^O_{a-1,t-1}(i)$$

if $k > n$ and $r_{t+k} = 0$.

We can use the Euler equation for future consumptions:

$$E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+\tau^C_{t+n})p^O_{t+n}C^O_{a,t}(i)\beta^{\frac{n}{\gamma}}\prod_{k=1}^{n}(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda^O_{t,t+k}}{\prod_{k=1}^{n}(1+r_{t+k-1})} = \\ = E_t \sum_{n=0}^{\infty} \frac{(1-\omega^O_{t+k-1})^n TR^{YO}_{a+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} + +(1+r_{t-1})B^O_{a-1,t-1}(i)$$

If we rearrange, we get consumption of agent i of cohort a at time t as a function of the present value of pension benefits, other income and initial wealth:

$$C^O_{a,t}(i) = \frac{E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^O_{t+k-1})TR^{YO}_{a+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})}}{E_t \sum_{n=0}^{\infty}(1+\tau^C_{t+n})p^O_{t+n}\beta^{\frac{n}{\gamma}}\prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1}\Lambda^O_{t,t+k}} +$$

$$+\frac{(1+r_{t-1})B^O_{a-1,t-1}(i)}{E_t\sum_{n=0}^{\infty}(1+\tau^C_{t+n})p^O_{t+n}\beta^{\frac{n}{\gamma}}\prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1}\Lambda^O_{t,t+k}}$$

Finally, using the assumption that $TR^{YO}_{n,t+n}(i) = TR^{YO}_{0,t}(i)\ \forall n > 0$:

$$C^O_{a,t}(i) = TR^{YO}_{a,t}(i)\frac{E_t\sum_{n=0}^{\infty}\frac{\prod_{k=1}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})}}{E_t\sum_{n=0}^{\infty}(1+\tau^C_{t+n})p^O_{t+n}\beta^{\frac{n}{\gamma}}\prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1}\Lambda^O_{t,t+k}} +$$
$$+\frac{(1+r_{t-1})B^O_{a-1,t-1}(i)}{E_t\sum_{n=0}^{\infty}(1+\tau^C_{t+n})p^O_{t+n}\beta^{\frac{n}{\gamma}}\prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1}\Lambda^O_{t,t+k}}$$

The denominators are the function of non-individual variables, then we can denote it as an aggregate variable:

$$\begin{aligned}
&\frac{1}{MPC^O_t} = E_t\sum_{n=0}^{\infty}(1+\tau^C_{t+n})p^O_{t+n}\beta^{\frac{n}{\gamma}}\prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1}\Lambda^O_{t,t+k}\\
&= (1+\tau^C_t)p^O_t + E_t\sum_{n=1}^{\infty}(1+\tau^C_{t+n})p^O_{t+n}\beta^{\frac{n}{\gamma}}\prod_{k=1}^{n}(1-\omega^O_{t+k-1})(1+r_{t+k-1})^{\frac{1}{\gamma}-1}\Lambda^O_{t,t+k}\\
&= (1+\tau^C_t)p^O_t +\\
&+\beta^{\frac{1}{\gamma}}(1-\omega^O_t)(1+r_t)^{\frac{1}{\gamma}-1}\Lambda^O_{t,t+1}E_t\sum_{n=0}^{\infty}(1+\tau^C_{t+n+1})p^O_{t+n+1}\beta^{\frac{n}{\gamma}}\prod_{k=1}^{n}(1-\omega^O_{t+k})(1+r_{t+k})^{\frac{1}{\gamma}-1}\Lambda^O_{t+1,t+k}\\
&= (1+\tau^C_t)p^O_t + \beta^{\frac{1}{\gamma}}(1-\omega^O_t)(1+r_t)^{\frac{1}{\gamma}-1}\Lambda^O_{t,t+1}E_t\frac{1}{MPC^O_{t+1}}
\end{aligned}$$

Using the same recursive substitution for the future expected pension benefit, the consumption function of agent i of cohort a at time t is:

$$C^O_{a,t}(i) \quad = \quad MPC^O_t\Omega^O_t TR^{YO}_{a,t}(i) + MPC^O_t(1+r_{t-1})B^O_{a-1,t-1}(i)$$

where Ω^O is the discount factor of the retired households:

$$\Omega^O_t \quad = \quad 1 + E_t\frac{1-\omega^O_t}{1+r_t}\Omega^O_{t+1}$$

Aggregate consumption of the retired cohort

Aggregate consumption is equal to the sum of pension benefits, other income and initial wealth:

$$\sum_{a=0}^{\infty} N^O_{a,t}(i)C^O_{a,t}(i) = MPC^O_t \Omega^O_t \sum_{a=0}^{\infty} N^O_{a,t}(i)TR^{YO}_{a,t}(i) + + MPC^O_t(1+r_{t-1})\sum_{a=0}^{\infty} N^O_{a,t}(i)B^O_{a-1,t-1}(i)$$

First, the number of old people declines over time:

$$\begin{aligned} N^O_{a+1,t} &= (1-\omega^O_{t-1})N^O_{a,t-1} \\ N^O_{a+2,t} &= (1-\omega^O_{t-1})(1-\omega^O_{t-2})N^O_{a,t-2} \\ &\vdots \end{aligned}$$

and

$$N^O_t = \sum_{a=0}^{\infty} N^O_{a,t}$$

Now, we can express aggregate pension income in period t of those who get retired at period t, one period before, etc.:

$$\begin{aligned} N^O_{0,t}TR^{YO}_{0,t}(i) &= TR^{YO}_t \\ N^O_{1,t}TR^{YO}_{1,t}(i) &= (1-\omega^O_{t-1})N^O_{0,t-1}TR^{YO}_{0,t-1}(i) = (1-\omega^O_{t-1})TR^{YO}_{t-1} \\ N^O_{2,t}TR^{YO}_{2,t}(i) &= (1-\omega^O_{t-1})(1-\omega^O_{t-2})N^O_{0,t-2}TR^{YO}_{0,t-2}(i) = (1-\omega^O_{t-1})(1-\omega^O_{t-2})TR^{YO}_{t-2} \\ &\vdots \end{aligned}$$

using $TR^{YO}_{n,t+n}(i) = TR^{YO}_{0,t}(i)\ \forall n > 0$ again.

Then, adding up all pensions implies:

$$\begin{aligned} TR_t &\equiv \sum_{a=0}^{\infty} N^O_{a,t}(i)TR^{YO}_{a,t}(i) = TR^{YO}_t + (1-\omega^O_{t-1})TR^{YO}_{t-1} + ... \\ &= TR^{YO}_t + (1-\omega^O_{t-1})TR_{t-1} \end{aligned}$$

Now, aggregate consumption of the retired cohort cohort is defined as:

$$C_t^O = \sum_{a=0}^{\infty} N_{a,t}^O C_{a,t}^O(i)$$

while total savings of the retired is:

$$\sum_{a=0}^{\infty} N_{a,t}^O B_{a,t-1}^O(i) = N_{0,t}^O B_{0,t-1}^O(i) + \sum_{a=1}^{\infty} N_{a,t}^O B_{a,t-1}^O(i)$$

Here, we need to be careful with the just-retired agents: they were young one period before without knowing about their next period retirement. We can use the law of large numbers to get the following expression: $N_{0,t}^O = \omega_{t-1}^Y N_{t-1}^Y$:

$$N_{0,t}^O B_{0,t-1}^O(i) = N_{0,t}^O \sum_{b=1}^{\infty} B_{b,t-1}^{Y,last}(i) \simeq \omega_{t-1}^Y N_{t-1}^Y \frac{B_{t-1}^Y}{N_{t-1}^Y}$$

where the *last* refers to the fact that those who get retired today spent their last year in the young cohort in the previous year.

Then, from $t-1$ to t it is easy to see that: $\sum_{a=1}^{\infty} N_{a,t}^O = \sum_{a=1}^{\infty}(1-\omega_{t-1}^O)N_{a-1,t-1}^O$ which implies that

$$\sum_{a=0}^{\infty} N_{a,t}^O B_{a,t-1}^O(i) = \omega_{t-1}^Y B_{t-1}^Y + \sum_{a=1}^{\infty}(1-\omega_{t-1}^O)N_{a-1,t-1}^O B_{a,t-1}^O(i)$$

Here, the second term means that only those retired agents cumulate savings who expect to survive the next period. Hence, the amount of aggregate old-age savings from the previous period is $B_{t-1}^O = \sum_{a=1}^{\infty}(1-\omega_{t-1}^O)N_{a-1,t-1}^O B_{a,t-1}^O(i)$. Then, overall savings of the retired cohort in period t can be expressed easily by adding just-retired savings from the previous period's young cohorts:

$$\sum_{a=0}^{\infty} N_{a,t}^O B_{a,t-1}^O(i) = \omega_{t-1}^Y B_{t-1}^Y + B_{t-1}^O$$

As a last step, we put together all parts of the equation, so, aggregate consumption of formal goods of the retired cohort is:

$$C_t^O = MPC_t^O \Omega_t^O TR_t + (1+r_{t-1})MPC_t^O(\omega_{t-1}^Y B_{t-1}^Y + B_{t-1}^O).$$

Young generation

First-order conditions of a young agent

'Young' agent i of young cohort b is one individual of its cohort who started to work (was born) b years ago. The Bellman-equation of a young individual is:

$$V_t^Y(B_{b-1,t-1}^Y(i)) = \max\left\{\frac{1}{1-\gamma}C_{b,t}^Y(i)^{1-\gamma} + E_t\beta\left((1-\omega_t^Y)V_{t+1}^Y(B_{b,t}^Y(i)) + \omega_t^Y V_{t+1}^O(B_{b,t}^{YO}(i))\right)\right\}$$

while the budget constraint is:

$$\begin{aligned}
&(1+\tau_t^C)p_t^Y C_{b,t}^Y(i) + (1-\omega_t^Y)B_{b,t}^Y(i) + \omega_t^Y B_{b,t}^{YO}(i) = \\
&= (1+r_{t-1})B_{b-1,t-1}^Y(i) + (1-\tau_t^L)w_t L_{b,t}(i) + w_t^U U_{b,t}(i) + Profit_{b,t}(i) - LTax_{b,t}(i)
\end{aligned}$$

where $C_{b,t}^Y(i)$ denotes the young agent's individual consumption, p_t^Y is the retailer price of the young consumption basket, $B_{b,t}^Y(i)$ is young households risk-free bond, $B_{b,t}^{YO}(i)$ is young households' state-contigent risk-free bond, $L_{b,t}(i)$ is individuals' working hours, w_t is the real wage, $Profit_{b,t}(i)$ denotes the profits from the firms, and $LTax_{b,t}(i)$ is the other lump-sum taxes. If the given household is unemployed (U_t), she receives unemployment benefit (w_t^U) from the government, but this benefit will not be taken into account at the time of retirement.
The first-order conditions are as follows:

$$\begin{aligned}
C_{b,t}^Y(i) &: \quad C_{b,t}^Y(i)^{-\gamma} + \lambda_{b,t}^Y(1+\tau_t^C)p_t^Y = 0 \\
B_{b,t}^Y(i) &: \quad \beta E_t(1-\omega_t^Y)V_{B_{b,t}^Y}^Y + E_t(1-\omega_t^Y)\lambda_{b,t}^Y = 0 \\
B_{b,t}^{YO}(i) &: \quad \beta E_t\omega_t^Y V_{B_{b,t}^{YO}}^Y + E_t\omega_t^Y\lambda_{b,t}^Y = 0
\end{aligned}$$

The one-period-ahead Envelope-theorem is:

$$E_t V_{B_{b,t}^Y} = -E_t\lambda_{b+1,t+1}^Y(1+r_t)$$

Also, from the retired agent's optimization we know that:

$$E_t V_{B_{b,t}^{YO}} = -E_t\lambda_{0,t+1}^O(1+r_t) = -E_t\lambda_{b+1,t+1}^O(1+r_t)$$

where $E_t\lambda^O_{0,t+1} = E_t\lambda^O_{b+1,t+1}$ because someone who was young in t gets retired in $t+1$. Thus, the Euler equations of the young individual are:

$$\beta E_t \frac{C^Y_{b+1,t+1}(i)^{-\gamma}}{C^Y_{b,t}(i)^{-\gamma}}(1+r_t)\frac{(1+\tau^C_t)p^Y_t}{(1+\tau^C_{t+1})p^Y_{t+1}} = 1$$

$$\beta E_t \frac{\left(C^O_{b+1,t+1}(i)\right)^{-\gamma}}{C^Y_{b,t}(i)^{-\gamma}}(1+r_t)\frac{(1+\tau^C_t)p^Y_t}{(1+\tau^C_{t+1})p^O_{t+1}} = 1$$

Rearranging results in the following:

$$E_t C^Y_{b+1,t+1}(i) = C^Y_{b,t}(i)\beta^{\frac{1}{\gamma}}(1+r_t)^{\frac{1}{\gamma}}\Lambda^Y_{t,t+1}$$

$$E_t C^O_{0,t+1}(i) = C^Y_{b,t}(i)\beta^{\frac{1}{\gamma}}(1+r_t)^{\frac{1}{\gamma}}\Lambda^{YO}_{t,t+1}$$

where

$$\Lambda^Y_{t,t+1} = E_t\left[\frac{(1+\tau^C_t)p^Y_t}{(1+\tau^C_{t+1})p^Y_{t+1}}\right]^{\frac{1}{\gamma}}$$

$$\Lambda^{YO}_{t,t+1} = E_t\left[\frac{(1+\tau^C_t)p^Y_t}{(1+\tau^C_{t+1})p^O_{t+1}}\right]^{\frac{1}{\gamma}}$$

Also, we can express each period's consumption as a function of period-t consumption and the discount rate:

$$E_t C^Y_{b+n,t+n}(i) = C^Y_{b,t}(i)\beta^{\frac{n}{\gamma}}E_t\prod_{k=1}^{n}(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda^Y_{t,t+k}$$

Individual consumption of a young agent

First of all, we would like to stress that one needs to be careful when deriving the young agent's individual consumption because old-age incomes and expenditures must be taken into account, too. Moreover, the young agents also consider the probability of retirement, for instance, in period t the probability that a young agent becomes retired in period $t+1$ is ω^Y_t, while the probability that the same agent becomes retired in period $t+2$ is $(1-\omega^Y_t)\omega^Y_{t+1}$. So, the first term of the left-hand side of this equation shows the stream of lifetime consumption if the agent stays young, then, from the second term onwards she retires with some probability in each period:

$$E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1})(1+\tau^C_{t+n})p^Y_{t+n}C^Y_{b+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+E_t\omega^Y_t \left(\sum_{n=1}^{\infty} \frac{\prod_{k=2}^{n}(1-\omega^O_{t+k-1})(1+\tau^C_{t+n})p^O_{t+n}C^O_{n-1,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) +$$
$$+E_t(1-\omega^Y_t)\omega^Y_{t+1} \left(\sum_{n=2}^{\infty} \frac{\prod_{k=3}^{n}(1-\omega^O_{t+k-1})(1+\tau^C_{t+n})p^O_{t+n}C^O_{n-2,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + ...$$
$$= E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1}) \left[(1-\tau^L_{t+n})w_{t+n}L_{b+n,t+n}(i) + w^U_{t+n}U_{b+n,t+n}(i) + Profit_{b+n,t+n}(i) - LTax_{b+n,t+n}(i)\right]}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+(1+r_{t-1})B^Y_{b-1,t-1}(i) +$$
$$+E_t\omega^Y_t \sum_{n=1}^{\infty} TR^{YO}_{n-1,t+n}(i) \frac{\prod_{k=2}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+E_t(1-\omega^Y_t)\omega^Y_{t+1} \sum_{n=2}^{\infty} TR^{YO}_{n-2,t+n}(i) \frac{\prod_{k=3}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+E_t(1-\omega^Y_t)(1-\omega^Y_{t+1})\omega^Y_{t+2} \sum_{n=3}^{\infty} TR^{YO}_{n-3,t+n}(i) \frac{\prod_{k=4}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} + ...$$

Based on the Euler-equations, we can express expected future consumptions. Let's consider an agent who is young in period t, then her consumption functions in the next periods after getting retired are:

$$E_t C^O_{n,t+n+1}(i) = E_t C^O_{0,t+1}(i)\beta^{\frac{n}{\gamma}} \prod_{k=2}^{n}(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda^O_{t+1,t+k}$$

On the other hand, if the agent stays young in period $t+1$ and gets retired after that, her future old-age consumptions look like:

$$E_t C^O_{n,t+n+2}(i) = E_t C^O_{0,t+2}(i)\beta^{\frac{n}{\gamma}} \prod_{k=3}^{n}(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda^O_{t+2,t+k}$$

Now, we plug them in the intertemporal budget constraint:

$$E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1})(1+\tau^C_{t+n})p^Y_{t+n}C^Y_{b+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+E_t\omega^Y_t \left(\sum_{n=1}^{\infty} \frac{\beta^{\frac{n}{\gamma}}\prod_{k=2}^{n}(1-\omega^O_{t+k-1})(1+\tau^C_{t+n})p^O_{t+n}C^O_{0,t+1}(i)(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda^O_{t+1,t+k}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) +$$
$$+E_t(1-\omega^Y_t)\omega^Y_{t+1} \left(\sum_{n=2}^{\infty} \frac{\beta^{\frac{n}{\gamma}}\prod_{k=3}^{n}(1-\omega^O_{t+k-1})(1+\tau^C_{t+n})p^O_{t+n}C^O_{0,t+2}(i)(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda^O_{t+2,t+k}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + ... =$$

$$= E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1}) \left[(1-\tau^L_{t+n}) w_{b+n,t+n} L_{b+n,t+n}(i) + w^U_{t+n} U_{b+n,t+n}(i) + Profit_{b+n,t+n}(i) - LTax_{b+n,t+n}(i)\right]}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+(1+r_{t-1}) B^Y_{b-1,t-1}(i) +$$
$$+E_t \omega^Y_t \sum_{n=1}^{\infty} TR^{YO}_{n-1,t+n}(i) \frac{\prod_{k=2}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+E_t (1-\omega^Y_t)\omega^Y_{t+1} \sum_{n=2}^{\infty} TR^{YO}_{n-2,t+n}(i) \frac{\prod_{k=3}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} + ...$$

After that, we use the other Euler equation (the one that shows the substitution between period-t young and period-$t+1$ old-age consumption):

$$E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1})(1+\tau^C_{t+n}) p^Y_{t+n} C^Y_{b+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+E_t \omega^Y_t \left(\sum_{n=1}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=2}^{n}(1-\omega^O_{t+k-1})(1+\tau^C_{t+n}) p^O_{t+n} C^Y_{b,t}(i)(1+r_t)^{\frac{1}{\gamma}} \Lambda^{YO}_{t,t+1} (1+r_{t+k-1})^{\frac{1}{\gamma}} \Lambda^O_{t+1,t+k}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) +$$
$$+E_t (1-\omega^Y_t)\omega^Y_{t+1} \left(\sum_{n=2}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=3}^{n}(1-\omega^O_{t+k-1})(1+\tau^C_{t+n}) p^O_{t+n} C^Y_{b+1,t+1}(i)(1+r_{t+1})^{\frac{1}{\gamma}} \Lambda^{YO}_{t+1,t+2} (1+r_{t+k-1})^{\frac{1}{\gamma}} \Lambda^O_{t+2,t+k}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + ...$$
$$= E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1}) \left[(1-\tau^L_{t+n}) w_{b+n,t+n} L_{b+n,t+n}(i) + w^U_{t+n} U_{b+n,t+n}(i) + Profit_{b+n,t+n}(i) - LTax_{b+n,t+n}(i)\right]}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+(1+r_{t-1}) B^Y_{b-1,t-1}(i) +$$
$$+E_t \omega^Y_t \sum_{n=1}^{\infty} TR^{YO}_{n-1,t+n}(i) \frac{\prod_{k=2}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+E_t (1-\omega^Y_t)\omega^Y_{t+1} \sum_{n=2}^{\infty} TR^{YO}_{n-2,t+n}(i) \frac{\prod_{k=3}^{n}(1-\omega^O_{t+k-1})}{\prod_{k=1}^{n}(1+r_{t+k-1})} + ...$$

Concentrating on consumptions:

$$E_t \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-\omega^Y_{t+k-1})(1+\tau^C_{t+n}) p^Y_{t+n} C^Y_{b+n,t+n}(i)}{\prod_{k=1}^{n}(1+r_{t+k-1})} +$$
$$+E_t \omega^Y_t \left(\sum_{n=1}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=2}^{n}(1-\omega^O_{t+k-1})(1+\tau^C_{t+n}) p^O_{t+n} C^Y_{b,t}(i)(1+r_t)^{\frac{1}{\gamma}} \Lambda^{YO}_{t,t+1} (1+r_{t+k-1})^{\frac{1}{\gamma}} \Lambda^O_{t+1,t+k}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) +$$
$$+E_t (1-\omega^Y_t)\omega^Y_{t+1} \left(\sum_{n=2}^{\infty} \frac{\beta^{\frac{n}{\gamma}} \prod_{k=3}^{n}(1-\omega^O_{t+k-1})(1+\tau^C_{t+n}) p^O_{t+n} C^Y_{b+1,t+1}(i)(1+r_{t+1})^{\frac{1}{\gamma}} \Lambda^{YO}_{t+1,t+2} (1+r_{t+k-1})^{\frac{1}{\gamma}} \Lambda^O_{t+2,t+k}}{\prod_{k=1}^{n}(1+r_{t+k-1})} \right) + ...$$

We rearrange and get the following:

$$
\begin{aligned}
&(1+\tau_t^C)p_t^Y C_{b,t}^Y(i) + \\
&+C_{b,t}^Y(i)E_t\omega_t^Y\left(\sum_{n=1}^{\infty}\frac{(1+\tau_{t+n}^C)p_{t+n}^O\beta^{\frac{n}{\gamma}}\prod_{k=2}^n(1-\omega_{t+k-1}^O)(1+r_t)^{\frac{1}{\gamma}}\Lambda_{t,t+1}^{YO}(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda_{t+1,t+k}^O}{\prod_{k=1}^n(1+r_{t+k-1})}\right)+ \\
&+(1+\tau_{t+1}^C)p_{t+1}^Y C_{b+1,t+1}^Y(i)\frac{(1-\omega_t^Y)}{(1+r_t)} + \\
&+E_tC_{b+1,t+1}^Y(i)(1-\omega_t^Y)\omega_{t+1}^Y\left(\sum_{n=2}^{\infty}\frac{(1+\tau_{t+n}^C)p_{t+n}^O\beta^{\frac{n}{\gamma}}\prod_{k=3}^n(1-\omega_{t+k-1}^O)(1+r_{t+1})^{\frac{1}{\gamma}}\Lambda_{t+1,t+2}^{YO}(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda_{t+2,t+k}^O}{\prod_{k=1}^n(1+r_{t+k-1})}\right)+...
\end{aligned}
$$

Simplifying before recursive substitution gives us:

$$
\begin{aligned}
&(1+\tau_t^C)p_t^Y C_{b,t}^Y(i) + \\
&+C_{b,t}^Y(i)E_t\frac{\omega_t^Y(1+r_t)^{\frac{1}{\gamma}}\Lambda_{t,t+1}^{YO}\beta^{\frac{1}{\gamma}}}{1+r_t}\left(\sum_{n=1}^{\infty}\frac{(1+\tau_{t+n}^C)p_{t+n}^O\beta^{\frac{n}{\gamma}}\prod_{k=2}^n(1-\omega_{t+k-1}^O)(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda_{t+1,t+k}^O}{\prod_{k=2}^n(1+r_{t+k-1})}\right)+ \\
&+E_t(1+\tau_{t+1}^C)p_{t+1}^Y C_{b+1,t+1}^Y(i)\frac{(1-\omega_t^Y)}{(1+r_t)} + \\
&+E_tC_{b+1,t+1}^Y(i)(1-\omega_t^Y)\frac{\omega_{t+1}^Y(1+r_{t+1})^{\frac{1}{\gamma}}\Lambda_{t+1,t+2}^{YO}\beta^{\frac{1}{\gamma}}}{(1+r_t)(1+r_{t+1})}\left(\sum_{n=2}^{\infty}\frac{(1+\tau_{t+n}^C)p_{t+n}^O\beta^{\frac{n}{\gamma}}\prod_{k=3}^n(1-\omega_{t+k-1}^O)(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda_{t+2,t+k}^O}{\prod_{k=3}^n(1+r_{t+k-1})}\right)+...
\end{aligned}
$$

Now, we can use $\frac{1}{MPC_{t+1}^O}$ from the retired agents' optimization:

$$
\begin{aligned}
&C_{b,t}^Y(i)\left[(1+\tau_t^C)p_t^Y + E_t\beta^{\frac{1}{\gamma}}\omega_t^Y(1+r_t)^{\frac{1}{\gamma}-1}\Lambda_{t,t+1}^{YO}\frac{1}{MPC_{t+1}^O}\right]+ \\
&+E_tC_{b+1,t+1}^Y(i)\frac{(1-\omega_t^Y)}{(1+r_t)}\left[(1+\tau_{t+1}^C)p_{t+1}^Y + \beta^{\frac{1}{\gamma}}\omega_{t+1}^Y(1+r_{t+1})^{\frac{1}{\gamma}-1}\Lambda_{t+1,t+2}^{YO}\frac{1}{MPC_{t+2}^O}\right]+...
\end{aligned}
$$

And, using the Euler-equation again (to have period-t consumption only):

$$
E_tC_{b+n,t+n}^Y(i) = C_{b,t}^Y(i)\beta^{\frac{n}{\gamma}}E_t\prod_{k=1}^n(1+r_{t+k-1})^{\frac{1}{\gamma}}\Lambda_{t,t+k}^Y
$$

Finally, we get:

$$
\begin{aligned}
&C_{b,t}^Y(i)\frac{1}{MPC_t^Y} = C_{b,t}^Y(i)\left[(1+\tau_t^C)p_t^Y + E_t\omega_t^Y(1+r_t)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\Lambda_{t,t+1}^{YO}\frac{1}{MPC_{t+1}^O}\right]+ \\
&C_{b,t}^Y(i)E_t(1+r_t)^{\frac{1}{\gamma}}\beta^{\frac{1}{\gamma}}\Lambda_{t,t+1}^Y\frac{(1-\omega_t^Y)}{(1+r_t)}\left[(1+\tau_{t+1}^C)p_{t+1}^Y + \omega_{t+1}^Y(1+r_{t+1})^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\Lambda_{t+1,t+2}^{YO}\frac{1}{MPC_{t+2}^O}\right]+...
\end{aligned}
$$

where

$$\frac{1}{MPC_t^Y} = (1+\tau_t^C)p_t^Y + E_t\beta^{\frac{1}{\gamma}}(1+r_t)^{\frac{1}{\gamma}-1}\left[(1-\omega_t^Y)\Lambda_{t,t+1}^Y\frac{1}{MPC_{t+1}^Y} + \omega_t^Y\Lambda_{t,t+1}^{YO}\frac{1}{MPC_{t+1}^O}\right]$$

Similarly, the young agent's budget constraint contains old-age income items, i.e., expected revenues from the pension fund.

$$\mathcal{I}_{b,t}^{YO}(i) = E_t\omega_t^Y\Omega_{t+1}^O TR_{0,t+1}^{YO}(i) + E_t\frac{(1-\omega_t^Y)\omega_{t+1}^Y}{(1+r_{t+1})}\Omega_{t+2}^O TR_{0,t+2}^{YO}(i) +$$
$$+E_t\frac{(1-\omega_t^Y)(1-\omega_{t+1}^Y)\omega_{t+2}^Y}{(1+r_{t+1})(1+r_{t+2})}\Omega_{t+3}^O TR_{0,t+3}^{PG,YO}(i) + ...$$

Again, we use that $TR_{n,t+n}^{PG,YO}(i) = TR_{0,t}^{PG,YO}(i)\ \forall n > 0$. In a recursive way it looks as:

$$\mathcal{I}_{b,t}^{YO}(i) = E_t\omega_t^Y TR_{0,t+1}^{YO}(i)\Omega_{t+1} + E_t\frac{(1-\omega_t^Y)}{(1+r_{t+1})}\mathcal{I}_{b+1,t+1}^{YO}(i)$$

Furthermore, young-age income is:

$$\begin{aligned}\mathcal{I}_{b,t}^{Y}(i) &= E_t\sum_{n=0}^{\infty}\frac{\prod_{k=1}^{\infty}(1-\omega_{t+k-1}^Y)^n\left[(1-\tau_{t+n}^L)w_{t+n}L_{b+n,t+n}(i) + w_{t+n}^U U_{b+n,t+n}(i) + Profit_{b+n,t+n}(i) - Tax_{b+n,t+n}(i)\right]}{\prod_{k=1}^{n}(1+r_{t+k-1})} = \\ &= (1-\tau_t^L)w_tL_{b,t}(i) + w_t^U U_{b,t}(i) + Profit_{b,t}(i) - LTax_{b,t}(i) + E_t\frac{1-\omega_t^Y}{1+r_t}\mathcal{I}_{b+1,t+1}^Y(i)\end{aligned}$$

If we add the present value of young income and expected pension benefits, we can introduce a new variable:

$$\begin{aligned}&Inc_{b,t}^Y(i) = \mathcal{I}_{b,t}^Y(i) + \frac{\mathcal{I}_{b,t}^{YO}(i)}{1+r_t} \\ = \ &(1-\tau_t^L)w_tL_{b,t}(i) + w_t^U U_{b,t}(i) + Profit_{b,t}(i) - LTax_{b,t}(i) + E_t\frac{\omega_t^Y}{1+r_t}TR_{0,t+1}^{YO}(i)\Omega_{t+1} + E_t\frac{1-\omega_t^Y}{1+r_t}Inc_{b+1,t+1}^Y(i)\end{aligned}$$

Thus, the individual consumption function of agent i of cohort b in period t is

$$C_{b,t}^Y(i) = MPC_t^Y Inc_{b,t}^Y(i) + (1+r_{t-1})MPC_t^Y B_{b-1,t-1}^Y(i)$$

Introducing a new variable for life-time income, and using marginal propensity to consume:

$$\begin{aligned}
C^Y_{b,t}(i) &= MPC^Y_t Inc^Y_{b,t}(i) + MPC^Y_t(1+r_{t-1})B^Y_{b-1,t-1}(i) \\
Inc^Y_{b,t}(i) &= (1-\tau^L_t)w_t L_{b,t}(i) + w^U_t U_{b,t}(i) + Profit_{b,t}(i) - LTax_{b,t}(i) + E_t\frac{\omega^Y_t}{1+r_t}TR^{YO}_{0,t+1}(i)\Omega_{t+1} + \\
&\quad + E_t\frac{1-\omega^Y_t}{1+r_t}Inc^Y_{b+1,t+1}(i) \\
\frac{1}{MPC^Y_t} &= (1+\tau^C_t)p^Y_t + E_t(1+r_t)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\left[(1-\omega^Y_t)\Lambda^Y_{t,t+1}\frac{1}{MPC^Y_{t+1}} + \omega^Y_t\Lambda^{YO}_{t,t+1}\frac{1}{MPC^O_{t+1}}\right]
\end{aligned}$$

Aggregate consumption of the young cohort

As a first step we need to express the total number of young people. If $N^Y_{b,t}$ is the number of b-year old workers, the total number of workers is

$$N^Y_t = \sum_{b=0}^{\infty} N^Y_{b,t}$$

Following the previous idea, we sum up all consumptions, incomes and savings:

$$\sum_{b=0}^{\infty} N^Y_{b,t}C^Y_{b,t}(i) = MPC^Y_t \sum_{b=0}^{\infty} N^Y_{b,t}Inc^Y_{b,t}(i) + (1+r_{t-1})MPC^Y_t \sum_{b=0}^{\infty} N^Y_{b,t}B^Y_{b-1,t-1}(i)$$

where we note that the new young workers in time t have zero savings from the previous period.

$$\sum_{b=0}^{\infty} N^Y_{b,t}C^Y_{b,t}(i) = MPC^Y_t \sum_{b=0}^{\infty} N^Y_{b,t}Inc^Y_{b,t}(i) + (1+r_{t-1})MPC^Y_t \sum_{b=1}^{\infty} N^Y_{b,t}\frac{N^Y_{b-1,t-1}}{N^Y_{b-1,t-1}}B^Y_{b-1,t-1}(i)$$

Rearranging gives us:

$$\sum_{b=0}^{\infty} N^Y_{b,t}C^Y_{b,t}(i) = MPC^Y_t \sum_{b=0}^{\infty} N^Y_{b,t}Inc^Y_{b,t}(i) + (1+r_{t-1})MPC^Y_t(1-\omega^Y_{t-1}) \sum_{b=1}^{\infty} N^Y_{b-1,t-1}B^Y_{b-1,t-1}(i)$$

Aggregate values are defined as:

$$\begin{aligned}
C^Y_t &\equiv \sum_{b=0}^{\infty} N^Y_{b,t}C^Y_{b,t}(i) \\
B^Y_{t-1} &\equiv \sum_{b=1}^{\infty} N^Y_{b-1,t-1}B^Y_{b-1,t-1}(i) \\
Inc^Y_t &\equiv \sum_{b=0}^{\infty} N^Y_{b,t}Inc^Y_{b,t}(i)
\end{aligned}$$

It is important to note that in each period, independently from the survival probabilities, each young agent saves for the next period, hence, the overall savings $B^Y_{t-1} = \sum_{b=1}^{\infty} N^Y_{b-1,t-1} B^Y_{b-1,t-1}(i)$ are divided among those who remain young and get retired.
As a result, the aggregate consumption functions are:

$$C^Y_t = MPC^Y_t Inc^Y_t + (1 + r_{t-1}) MPC^Y_t (1 - \omega^Y_{t-1}) B^Y_{t-1}$$

Now we need to aggregate the supporting variables as well. First of all, we rename individual contemporary income as follows:

$$Inc^Y_{b,t}(i) = \mathcal{I}^Y_{b,t}(i) + \mathcal{I}^{YO}_{b,t}(i)$$

Aggregating gives us:

$$Inc^Y_t = \mathcal{I}^Y_t + \mathcal{I}^{YO}_t$$

After aggregating and rearranging we get:

$$\sum_{b=0}^{\infty} N^Y_{b,t} \mathcal{I}^Y_{b,t}(i) = \sum_{b=0}^{\infty} N^Y_{b,t} \left((1 - \tau^L_t) w_t L_{b,t}(i) + Profit_{b,t}(i) - Tax_{b+n,t+n}(i) \right) +$$
$$+ E_t \frac{(1 - \omega^Y_t)}{1 + r_t} \sum_{b=0}^{\infty} N^Y_{b,t} \mathcal{I}^Y_{b+1,t+1}(i)$$
$$= \sum_{b=0}^{\infty} N^Y_{b,t} \left((1 - \tau^L_t) w_t L_{b,t}(i) + Profit_{b,t}(i) - LTax_{b,t}(i) \right) + E_t \frac{1}{1 + r_t} \sum_{b=0}^{\infty} N^Y_{b+1,t+1} \mathcal{I}^Y_{b+1,t+1}(i)$$

Because $\mathcal{I}^Y_{t+1}$ contains the income of the new-born people as well, the last term can be rearranged, using the law of large numbers as follows:

$$E_t \sum_{b=0}^{\infty} N^Y_{b+1,t+1} \mathcal{I}^Y_{b+1,t+1}(i) = E_t \mathcal{I}^Y_{t+1} - E_t N^Y_{b,t+1} \mathcal{I}^Y_{b,t+1}(i) =$$
$$= E_t \mathcal{I}^Y_{t+1} \left(1 - \frac{N^Y_{b,t+1}}{N^Y_{t+1}} \right) = E_t \mathcal{I}^Y_{t+1} \left(1 - \frac{n_t N^Y_t}{N^Y_{t+1}} \right)$$

Then, total young income is:

$$\mathcal{I}^Y_t = (1 - \tau^L_t) w_t L_t + Profit_t - LTax_t + E_t \frac{1 - \omega^Y_t}{(1 + r_t)(1 + g^{N,Y}_{t+1})} \mathcal{I}^Y_{t+1}$$

A similar exercise can be done for pension benefits. First, we define $\mathcal{I}_t^{YO}$ which can be rearranged as:

$$\mathcal{I}_t^{YO} = \sum_{b=0}^{\infty} N_{b,t}^Y \mathcal{I}_{b,t}^{YO}(i) = E_t \omega_t^Y \sum_{b=0}^{\infty} N_{b,t}^Y E_t TR_{0,t+1}^{YO}(i) \Omega_{t+1}^O +$$
$$+ \frac{(1-\omega_t^Y)}{(1+r_{t+1})} \sum_{b=0}^{\infty} N_{b,t}^Y \mathcal{I}_{b+1,t+1}^{YO}(i) = E_t N_{0,t+1}^O TR_{0,t+1}^{YO}(i) \Omega_{t+1}^O +$$
$$+ E_t \frac{1}{(1+r_{t+1})} \sum_{b=0}^{\infty} N_{b+1,t+1}^Y \mathcal{I}_{b+1,t+1}^{YO}(i)$$

Now, similarly to total young income, the last term can be expressed as:

$$E_t \sum_{b=0}^{\infty} N_{b+1,t+1}^Y \mathcal{I}_{b+1,t+1}^{YO}(i) = E_t \frac{1-\omega_t^Y}{1+g_{t+1}^{N,Y}} \mathcal{I}_{t+1}^{YO}$$

We also know that the following expression holds:

$$E_t N_{0,t+1}^O TR_{0,t+1}^{YO}(i) \Omega_{t+1}^O = E_t TR_{t+1}^{YO} \Omega_{t+1}^O$$

Finally, the expected income of the young after getting retired is

$$\mathcal{I}_t^{YO} = E_t TR_{t+1}^{YO} \Omega_{t+1}^O + E_t \frac{1-\omega_t^Y}{(1+r_{t+1})(1+g_{t+1}^{N,Y})} \mathcal{I}_{t+1}^{YO}$$

Based on the derivation above, we can express the aggregate version of the young household's income as:

$$Inc_t^Y = (1-\tau_t^L) w_t L_t + w_t^U U_t + Profit_t - LTax_t + E_t \frac{TR_{t+1}^{YO}}{1+r_t} \Omega_{t+1}^O + E_t \frac{1-\omega_t^Y}{(1+r_t)(1+g_{t+1}^{N,Y})} Inc_{t+1}^Y$$

where aggregate unemployment is as follows:

$$U_t = N_t^Y - L_t$$

Aggregating the young households' budget constraints

The individual budget constraint of a young agent is as follows:

$$(1+\tau_t^C) p_t^Y C_{b,t}^Y(i) + (1-\omega_t^Y) B_{b,t}^Y(i) + \omega_t^Y B_{b,t}^{Y*}(i) =$$

$$= (1-\tau_t^L)w_t L_{b,t}(i) + w_t^U U_{b,t}(i) + Profit_{b,t}(i) - LTax_{b,t}(i) + (1+r_{t-1})B_{b-1,t-1}^Y(i)$$

Aggregating implies the following:

$$\sum_{b=0}^{\infty} N_{b,t}^Y (1+\tau_t^C) p_t^Y C_{b,t}^Y(i) + \sum_{b=0}^{\infty} N_{b,t}^Y (1-\omega_t^Y) B_{b,t}^Y(i) + \sum_{b=0}^{\infty} N_{b,t}^Y \omega_t^Y B_{b,t}^{Y*}(i) =$$
$$= \sum_{b=0}^{\infty} N_{b,t}^Y \Big((1-\tau_t^L) w_t L_{b,t}(i) + w_t^U U_{b,t}(i) + Profit_{b,t}(i) - LTax_{b,t}(i) \Big) + (1+r_{t-1}) \sum_{b=1}^{\infty} N_{b,t}^Y B_{b-1,t-1}^Y(i)$$

where the definition of aggregate savings is:

$$\sum_{b=1}^{\infty} N_{b,t}^Y B_{b-1,t-1}^Y(i) = \sum_{b=1}^{\infty} (1-\omega_{t-1}^Y) N_{b-1,t-1}^Y B_{b-1,t-1}^Y(i)$$

After aggregation, there is no difference between the B_t^Y and B_t^{Y*}. So, we can easily express aggregate budget constraint:

$$(1+\tau_t^C) p_t^Y C_t^Y + B_t^Y = (1-\tau_t^L) w_t L_t + w_t^U U_t + Profit_t - LTax_t + (1+r_{t-1})(1-\omega_{t-1}^Y) B_{t-1}^Y$$

A.1.2 Fiscal policy and pay-as-you-go pension plan

Pension system

To account for the overall expenditure of the pension system, we need to count the number of just-retired and retired agents. The number of just-retired agents, (those who were young one period before) is:

$$N_{0,t}^O = \sum_{b=1}^{\infty} \omega_{t-1}^Y N_{b-1,t-1}^Y$$

and the total number of retired agents in period t is the just-retired agents plus those who survived the previous periods:

$$N_t^O = N_{0,t}^O + (1-\omega_t^O) N_{1,t-1}^O + (1-\omega_t^O)(1-\omega_{t-1}^O) N_{2,t-2}^O + ...$$

Individual (i)'s pension in the year of retirement t is based on replacement rate ν_t and the previous period income:

$$TR_{0,t}^{YO}(i) = \nu_t w_{t-1} L_{b-1,t-1}(i)$$

We need to use the following expressions to aggregate:

$$\begin{aligned} N_{0,t}^{O} TR_{0,t}^{YO}(i) &= \nu_t N_{0,t}^{O} w_{t-1} L_{b-1,t-1}(i) = \nu_t \omega_{t-1}^{Y} \sum_{b=1}^{\infty} N_{b-1,t-1}^{Y} w_{t-1} L_{b-1,t-1}(i) \\ TR_t^{YO} &= \nu_t \omega_{t-1}^{Y} w_{t-1} L_{t-1} \end{aligned}$$

Furthermore, total pension expenditure of all retired people is as follows:

$$TR_t = TR_t^{YO} + (1-\omega_{t-1}^{O}) TR_{t-1}^{YO} + (1-\omega_{t-1}^{O})(1-\omega_{t-2}^{O}) TR_{t-2}^{YO} + ...$$

which can be rewritten as:

$$TR_t = TR_t^{YO} + (1-\omega_{t-1}^{O}) TR_{t-1}$$

Rest of the fiscal sector

The government budget constraint is as follows:

$$\begin{aligned} Debt_t + LTax_t + Tax_t &= Gov_t + TR_t + w_t^U U_t + (1+r_{t-1}) Debt_{t-1} \\ Tax_t &= \tau_t^C (p_t^O C_t^O + p_t^Y C_t^Y) + (\tau_t^{SSC} + \tau_t^L) w_t L_t \end{aligned}$$

where *Gov* denotes the other - exogenous - current expenditures, *Debt* is the level of public debt. The government wants to stabilizes the public debt-to-GDP ratio by adjusting the level of lump-sum taxes:

$$LTax_t = Gov_t + TR_t + w_t^U U_t + (1+r_{t-1}) Debt_{t-1} - \left\{ \frac{Debt}{Y} \right\}^{Target} Y_t - Tax_t$$

The households finance government debt and the bond market equilibrium is the following:

$$Debt_t = B_t^Y + B_t^O$$

A.1.3 Firms' optimization

The young households own the firms and the labor union, hence, in their optimization we take into account their survival probability.

Production and nominal price setting

The firms produce differentiated products and due to their monopolistic power they are able to set optimal prices. However, nominal price setting is costly. Hence their optimal price settings conditional on the current a future cost of the price settings (Rotemberg, 1982). The cost function can be given by a quadratic convex function:

$$R\left(\frac{P_t^Y(i)}{P_{t-1}^Y(i)}\right) = \frac{\phi_P}{2}\left(\frac{P_t^Y(i)}{P_{t-1}^Y(i)\left(1+\pi_{t-1}\right)^{\vartheta}} - 1\right)^2$$

where $P_t(i)$ is the optimal individual price in period t, π denotes the inflation. If the firms adjust their prices by the previous period inflation is cost-less, debends on the size of ϑ.
Total cost of production follows as:

$$TC_t(i) = P_t Inv_t(i) + (1+\tau_t^{SSC})W_t L_t(i) + P_t(i)Y_t R\left(\frac{P_t(i)}{P_{t-1}(i)}\right) + HC_t H_t$$

Law motion of employment can be described as the function of previous level of employment, firing probability and the actual hiring (that also imposed by convex adjustment cost):

$$L_t(i) = (1 - pr_t^F)L_{t-1}(i) + H_t(i)\left(1 - S\left(\frac{H_t(i)}{H_{t-1}(i)}\right)\right)$$

Hiring cost is the positive function of the hiring-probability:

$$\begin{aligned} HC_t &= \kappa pr_t^{H^{\alpha_{HC}}} \\ pr_t^H &= \frac{H_t}{U_{t-1} + pr_t^F L_{t-1}} \end{aligned}$$

The firms are responsible for capital accumulation:

$$K_t(i) = (1-\delta)K_{t-1}(i) + Inv_t(i)\left(1 - S\left(\frac{Inv_t(i)}{Inv_{t-1}(i)}\right)\right)$$

where $S(\cdot)$ function denotes the investment adjustment cost, any changes in investment which differ from the balanced growth path is costly. The cost functions are the following:

$$S\left(\frac{Inv_t(i)}{Inv_{t-1}(i)}\right) = \frac{\phi_{Inv}}{2}\left(\frac{1}{1+g_t}\frac{Inv_t(i)}{Inv_{t-1}(i)} - 1\right)^2$$
$$S\left(\frac{H_t(i)}{H_{t-1}(i)}\right) = \frac{\phi_H}{2}\left(\frac{1}{1+g_t^N}\frac{H_t(i)}{H_{t-1}(i)} - 1\right)^2$$

We can write up the Bellman-equation:

$$\begin{aligned}
&V(P_{t-1}(i), L_{t-1}(i), H_{t-1}(i), K_{t-1}(i), Inv_{t-1}(i)) = P_t(i)Y_t(i) - TC_t(i) + \\
&+E_t(1-\omega_t^Y)\frac{V(P_t(i), L_t(i), H_t(i), K_t(i), Inv_t(i))}{1+i_t} \\
&+MC_t\left(\left[\alpha^{\frac{1}{\Theta}}K_{t-1}(i)^{\frac{\theta-1}{\theta}} + (1-\alpha)^{\frac{1}{\theta}}(A_tL_t(i))^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} - Y_t(i)\right) + \\
&+\lambda_t^H\left((1-pr_t^F)L_{t-1}(i) + H_t(i)\left(1 - S\left(\frac{H_t(i)}{H_{t-1}(i)}\right)\right) - L_t(i)\right) \\
&+Q_t\left((1-\delta)K_{t-1}(i) + Inv_t(i)\left(1 - S\left(\frac{Inv_t(i)}{Inv_{t-1}(i)}\right)\right) - K_t(i)\right)
\end{aligned}$$

where Q is the nominal Tobin-Q.

We can plug the demand for individual product, that can be derived from a Dixit-Stiglitz aggregator function:

$$\begin{aligned}
V(...) &= P_t(i)\left(\frac{P_t(i)}{P_t}\right)^{-\varphi} Y_t - TC_t(i) + E_t(1-\omega_t^Y)\frac{V(...)}{1+i_t} \\
&+ MC_t\left(\left[\alpha^{\frac{1}{\theta}}K_{t-1}(i)^{\frac{\theta-1}{\theta}} + (1-\alpha)^{\frac{1}{\theta}}(A_tL_t(i))^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} - \left(\frac{P_t(i)}{P_t}\right)^{-\varphi} Y_t\right) + \\
&+ \lambda_t^H\left((1-pr_t^F)L_{t-1}(i) + H_t(i)\left(1 - S\left(\frac{H_t(i)}{H_{t-1}(i)}\right)\right) - L_t(i)\right) \\
&+ Q_t\left((1-\delta)K_{t-1}(i) + Inv_t(i)\left(1 - S\left(\frac{Inv_t(i)}{Inv_{t-1}(i)}\right)\right) - K_t(i)\right)
\end{aligned}$$

Substitute out total cost:

$$\begin{aligned}
V(...) &= P_t(i)\left(\frac{P_t(i)}{P_t}\right)^{-\varphi} Y_t - P_tInv_t(i) - (1+\tau_t^{SSC})W_tL_t(i) - \\
&- P_t(i)Y_tR\left(\frac{P_t(i)}{P_{t-1}(i)}\right) - HC_tH_t(i) + E_t(1-\omega_t^Y)\frac{V(...)}{1+i_t} \\
&+ MC_t\left(\left[\alpha^{\frac{1}{\theta}}K_{t-1}(i)^{\frac{\theta-1}{\theta}} + (1-\alpha)^{\frac{1}{\theta}}(A_tL_t(i))^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} - \left(\frac{P_t(i)}{P_t}\right)^{-\varphi} Y_t\right) +
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \Lambda_t^H \left((1 - pr_t^F) L_{t-1}(i) + H_t(i) \left(1 - S\left(\frac{H_t(i)}{H_{t-1}(i)} \right) \right) - L_t(i) \right) \\
&+ \quad Q_t \left((1-\delta) K_{t-1}(i) + Inv_t(i) \left(1 - S\left(\frac{Inv_t(i)}{Inv_{t-1}(i)} \right) \right) - K_t(i) \right)
\end{aligned}
$$

The first-order conditions are as follows:

$$
\begin{aligned}
\frac{\partial V_t}{\partial P_t(i)} &= \left(\frac{P_t(i)}{P_t} \right)^{-\varphi} Y_t - \varphi P_t(i) \left(\frac{P_t(i)}{P_t} \right)^{-\varphi-1} \frac{1}{P_t} Y_t - Y_t R \left(\frac{P_t(i)}{P_{t-1}(i)} \right) - \\
&- \quad P_t(i) Y_t R' \left(\frac{P_t(i)}{P_{t-1}(i)} \right) \frac{1}{P_{t-1}(i) \left(1 + \pi_{t-1}\right)^{\vartheta}} + \\
&+ \quad E_t (1 - \omega_t^Y) \frac{V_{P_t(i)}}{1 + i_t} + MC_t \varphi \left(\frac{P_t(i)}{P_t} \right)^{-\varphi-1} \frac{1}{P_t} Y_t = 0 \\
\frac{\partial V_t}{\partial K_t(i)} &= E_t \frac{1 - \omega_t^Y}{1 + i_t} V_{K_t(i)} - Q_t = 0 \\
\frac{\partial V_t}{\partial H_t(i)} &= -HC_t + E_t \frac{1 - \omega_t^Y}{1 + i_t} V_{H_t(i)} + \Lambda_t^H \left(1 - S\left(\frac{H_t(i)}{H_{t-1}(i)} \right) - S'\left(\frac{H_t(i)}{H_{t-1}(i)} \right) \frac{1}{1 + g_t} \frac{H_t(i)}{H_{t-1}(i)} \right) = 0 \\
\frac{\partial V_t}{\partial L_t(i)} &= -(1 + \tau_t^{SSC}) W_t + MC_t (1 - \alpha)^{\frac{1}{\theta}} \frac{Y_t(i)^{\frac{1}{\Theta}}}{(A_t L_t(i))^{\frac{1}{\theta}}} A_t - \Lambda_t^H + E_t \frac{1 - \omega_t^Y}{1 + i_t} V_{L_t(i)} = 0 \\
\frac{\partial V_t}{\partial Inv_t(i)} &= -P_t + E_t \frac{1 - \omega_t^Y}{1 + i_t} V_{Inv_t(i)} + Q_t \left(1 - S(\cdot) - S'(\cdot) \frac{1}{1 + g_t} \frac{Inv_t(i)}{Inv_{t-1}(i)} \right)
\end{aligned}
$$

Using the Envelope-theorem we get:

$$
\begin{aligned}
V_{P_{t-1}(i)} &= P_t(i) Y_t R' \left(\frac{P_t(i)}{P_{t-1}(i)} \right) \frac{P_t(i)}{P_{t-1}(i)^2 \left(1 + \pi_{t-1}\right)^{\vartheta}} \\
V_{L_{t-1}(i)} &= (1 - pr_t^F) \Lambda_t^H \\
V_{H_{t-1}(i)} &= \Lambda_t^H S' \left(\frac{H_t(i)}{H_{t-1}(i)} \right) \frac{1}{1 + g_t^N} \left(\frac{H_t(i)}{H_{t-1}(i)} \right)^2 \\
V_{K_{t-1}(i)} &= \alpha^{\frac{1}{\theta}} MC_t \frac{Y_t(i)^{\frac{1}{\theta}}}{K_{t-1}(i)^{\frac{1}{\theta}}} + Q_t (1 - \delta) \\
V_{Inv_{t-1}(i)} &= Q_t S'(\cdot) \frac{1}{1 + g_t} \left(\frac{Inv_t(i)}{Inv_{t-1}(i)} \right)^2
\end{aligned}
$$

Stepping one period ahead gives us:

$$
V_{P_t(i)} = P_{t+1}(i) Y_{t+1} R' \left(\frac{P_{t+1}(i)}{P_t(i)} \right) \frac{P_{t+1}(i)}{P_t(i)^2 \left(1 + \pi_t\right)^{\vartheta}}
$$

$$
\begin{aligned}
V_{L_t(i)} &= (1 - pr^F_{t+1})\Lambda^H_{t+1} \\
V_{H_t(i)} &= \Lambda^H_{t+1} S'\left(\frac{H_{t+1}(i)}{H_t(i)}\right)\frac{1}{1+g^N_{t+1}}\left(\frac{H_{t+1}(i)}{H_t(i)}\right)^2 \\
V_{K_t(i)} &= \alpha^{\frac{1}{\theta}} MC_{t+1}\frac{Y_{t+1}(i)^{\frac{1}{\theta}}}{K_t(i)^{\frac{1}{\theta}}} + Q_{t+1}(1-\delta) \\
V_{Inv_t(i)} &= Q_{t+1}S'(\cdot)\frac{1}{1+g_{t+1}}\left(\frac{Inv_{t+1}(i)}{Inv_t(i)}\right)^2
\end{aligned}
$$

As a simplification, we can introduce a new variable for the marginal product of capital which modifies the no-arbitrage condition:

$$
\begin{aligned}
r^K_t &= \alpha^{\frac{1}{\theta}} mc_t \frac{Y_t(i)^{\frac{1}{\theta}}}{K_{t-1}(i)^{\frac{1}{\theta}}} \\
q_t &= \frac{1-\omega^Y_t}{1+r_t}\left(r^K_{t+1} + q_{t+1}(1-\delta)\right)
\end{aligned}
$$

The effective real wage can be given as: labor demand can be expressed as:

$$
\bar{w}_t = (1+\tau^{SSC}_t)w_t + \lambda^H_t - E_t\frac{1-\omega^Y_t}{1+i_t}(1-pr^F_{t+1})\lambda^H_{t+1}
$$

And the labor demand has the following form:

$$
\bar{w}_t = mc_t(1-\alpha)^{\frac{1}{\theta}}\frac{Y_t(i)^{\frac{1}{\Theta}}}{(A_tL_t(i))^{\frac{1}{\theta}}}A_t
$$

If we substitute out the capital and labor from the production function we can get marginal cost function:

$$
mc_t = \left[\alpha {r^K_t}^{1-\theta} + (1-\alpha)\left(\frac{\bar{w}_t}{A_t}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}
$$

The firms' hiring decision can be given as:

$$
\begin{aligned}
hc_t &= E_t\frac{1-\omega^Y_t}{1+r_t}\lambda^H_{t+1}\frac{1}{1+g^N_{t+1}}S'\left(\frac{H_{t+1}(i)}{H_t(i)}\right)\left(\frac{H_{t+1}(i)}{H_t(i)}\right)^2 + \\
&+ \lambda^H_t\left(1 - S\left(\frac{H_t(i)}{H_{t-1}(i)}\right) - S'\left(\frac{H_t(i)}{H_{t-1}(i)}\right)\frac{1}{1+g^N_t}\frac{H_t(i)}{H_{t-1}(i)}\right)
\end{aligned}
$$

The investment decision is as follows:

$$1 = E_t \frac{1-\omega_t^Y}{1+r_t} \frac{q_{t+1}}{1+g_{t+1}} S'(\cdot) \left(\frac{Inv_{t+1}(i)}{Inv_t(i)} \right)^2 + q_t \left(1 - S(\cdot) - S'(\cdot) \frac{1}{1+g_t} \frac{Inv_t(i)}{Inv_{t-1}(i)} \right)$$

The price-setting curve can be expressed as:

$$\begin{aligned} & \left(\frac{P_t(i)}{P_t} \right)^{-\varphi} Y_t - \varphi P_t(i) \left(\frac{P_t(i)}{P_t} \right)^{-\varphi-1} \frac{1}{P_t} Y_t - Y_t R \left(\frac{P_t(i)}{P_{t-1}(i)} \right) - \\ - \quad & P_t(i) Y_t R' \left(\frac{P_t(i)}{P_{t-1}(i)} \right) \frac{1}{P_{t-1}(i) \left(1+\pi_{t-1}\right)^{\vartheta}} + \\ + \quad & E_t \frac{(1-\omega_t^Y)}{1+i_t} P_{t+1}(i) Y_{t+1} R' \left(\frac{P_{t+1}(i)}{P_t(i)} \right) \frac{P_{t+1}(i)}{P_t(i)^2 \left(1+\pi_{t-1}\right)^{\vartheta}} + MC_t \varphi \left(\frac{P_t(i)}{P_t} \right)^{-\varphi-1} \frac{1}{P_t} Y_t = 0 \end{aligned}$$

We can assume that all firms follows the same price setting behaviour, it means the individual and the aggregate prices are the identical, and we can introduce the inflation $1+\pi_t = \frac{P_t}{P_{t-1}}$

$$\begin{aligned} 1 + \frac{1}{\varphi-1} R \left(\frac{1+\pi_t}{(1+\pi_{t-1})^{\gamma}} \right) \quad & + \quad \frac{1}{\varphi-1} R' \left(\frac{1+\pi_t}{(1+\pi_{t-1})^{\gamma}} \right) \frac{1+\pi_t}{(1+\pi_{t-1})^{\gamma}} - \\ & - \quad E_t \frac{1-\omega_t^Y}{\varphi-1} \frac{Y_{t+1}}{Y_t} \frac{R' \left(\frac{1+\pi_{t+1}}{(1+\pi_t)^{\gamma}} \right) \left(\frac{1+\pi_{t+1}}{(1+\pi_t)^{\gamma}} \right)}{1+r_t} = \frac{\varphi}{\varphi-1} mc_t \end{aligned}$$

Retailers

We can assume that in short run the households can not adjust their consumptions immediately. There is a retailer sector that sets the cohort specific relative prices for the consumption baskets in a way to smooth out the short run cyclical adjustment of the households consumption.

$$\sum_{n=0}^{\infty} \prod_{k=1}^{n} \frac{1-\omega_{t+k-1}^j}{1+r_{t+k-1}} \left\{ p_{t+n}^j C_{t+n}^j - C_{t+n}^j \left(1 + G^{C,j} \left(\frac{C_{t+n}^j}{C_{t+n-1}^j} \right) \right) \right\}$$

Taking the first-order condition of consumption:

$$p_t^j \quad = \quad 1 + G^{C,j} \left(\frac{C_t^j}{C_{t-1}^j} \right) + G^{C,j'} \left(\frac{C_t^j}{C_{t-1}^j} \right) \frac{1}{1+g_t} \frac{C_t^j}{C_{t-1}^j} - \frac{1-\omega_t^j}{1+r_t} G^{C,j'} \left(\frac{C_{t+1}^j}{C_t^j} \right) \frac{1}{1+g_{t+1}} \left(\frac{C_{t+1}^j}{C_t^j} \right)^2$$

where the adjustment cost is a usual convex function:

$$G^{C,j}\left(\frac{C_t^j}{C_{t-1}^j}\right) = \frac{\phi^{C,j}}{2}\left(\frac{1}{1+g_t}\frac{C_t^j}{C_{t-1}^j} - 1\right)^2$$

Wage bargaining

The value function of the workers is as follows:

$$V_t^w = (1-\tau_t^L)w_t + E_t\frac{1}{1+r_t}\left[(1-pr_{t+1}^F + pr_{t+1}^F pr_{t+1}^H)V_{t+1}^w + pr_{t+1}^F(1-pr_{t+1}^H)V_{t+1}^U\right]$$

The value function of the unemployed is:

$$V_t^U = w_t^U + E_t\frac{1}{1+r_t}\left[(1-pr_{t+1}^H)V_{t+1}^U + pr_{t+1}^H V_{t+1}^w\right]$$

Wage-bargaining is as follows:

$$\max_{w_t}(V_t^w - V_t^U)^\sigma {hc_t}^{1-\sigma}$$

The first-order conditions are:

$$\begin{aligned}
&\sigma(V_t^w - V_t^U)^{\sigma-1}{hc_t}^{1-\sigma}(1-\tau_t^L) - (1-\sigma)(V_t^w - V_t^U)^\sigma {hc_t}^{-\sigma}(1+\tau_t^{SSC}) = 0\\
&\sigma(V_t^w - V_t^U)^{-1}hc_t(1-\tau_t^L) = (1-\sigma)(1+\tau_t^{SSC})\\
&\frac{\sigma}{1-\sigma}hc_t\frac{1-\tau_t^L}{1+\tau_t^{SSC}} = V_t^w - V_t^U
\end{aligned}$$

We can express the differences between the two value functions:

$$\begin{aligned}
V_t^w - V_t^{Y,U} &= (1-\tau_t^L)w_t - w_t^U + E_t\frac{1}{1+r_t}\Bigg[(1-pr_{t+1}^F + pr_{t+1}^F pr_{t+1}^H)V_{t+1}^w +\\
&+pr_{t+1}^F(1-pr_{t+1}^H)V_{t+1}^{Y,U}\Bigg] - E_t\frac{1}{1+r_t}\Bigg[(1-pr_{t+1}^H)V_{t+1}^U + pr_{t+1}^H V_{t+1}^w\Bigg] =\\
&= (1-\tau_t^L)w_t - w_t^U + E_t\frac{1}{1+r_t}\Bigg[(1-pr_{t+1}^{F,F})(1-pr_{t+1}^H)\left(V_{t+1}^w - V_{t+1}^U\right)\Bigg]
\end{aligned}$$

Finally, we can plug in the first-order conditions:

$$\begin{aligned}\frac{\sigma}{1-\sigma}hc_t\frac{1-\tau_t^L}{1+\tau_t^{SSC}} &= (1-\tau_t^L)w_t - w_t^U + \\ &+ E_t\frac{1}{1+r_t}(1-pr_{t+1}^F)(1-pr_{t+1}^H)\left(\frac{\sigma}{1-\sigma}hc_{t+1}\frac{1-\tau_{t+1}^L}{1+\tau_{t+1}^{SSC}}\right)\end{aligned}$$

A.1.4 Monetary policy

The behaviour of the central bank can be discribed by the Taylor-type monetary policy rule:

$$1+i_t = (1+i_{t-1})^{\rho_i}\left((1+r_t^n)(1+\pi_t)^{\phi_\pi}\right)^{1-\rho_i} e^{\varepsilon_t^i}$$

where ρ_i is interest smoothing, ϕ_π denotes the inflationary reaction, and ε_t^i assigns the monetary policy shock. Once the inflation is reaches the target again, the central should set the interest rate at its flexible price equilibrium level.

The real interest rate is defined by the Fisher-identity:

$$1+i_t = E_t(1+r_t)(1+\pi_{t+1})$$

A.1.5 Normalized model equations

Each variable must be detrended: individual variables are normalized by population (N_t) because there is only population growth in the model. This section lists all the final equations of the model: detrended variables are denoted by $\tilde{x}_t$.

Demography:

$$\begin{aligned}s_t &= \frac{(1-\omega_{t-1}^O)}{(1-\omega_{t-1}^Y+n_t)}s_{t-1} + \frac{\omega_{t-1}^Y}{(1-\omega_{t-1}^Y+n_t)} \\ s_t^Y &= \frac{1}{1+s_t} \\ 1+g_t^Y &= 1-\omega_{t-1}^Y+n_t \\ 1+g_t^O &= (1-\omega_{t-1}^O)+\frac{\omega_{t-1}^Y}{s_{t-1}} \\ 1+g_t &= 1+g_t^N = (1+g_t^{N,Y})\frac{1+s_t}{1+s_{t-1}}\end{aligned}$$

Households:

$$
\begin{aligned}
\tilde{C}_t^O &= MPC_t^O \tilde{TR}_t \Omega_t^O + MPC_t^O \frac{1+r_{t-1}}{1+g_t}\left[\omega_{t-1}^Y \tilde{B}_{t-1}^Y + \tilde{B}_{t-1}^O\right] \\
\frac{1}{MPC_t^O} &= (1+\tau_t^C)p_t^O + \beta^{\frac{1}{\gamma}}(1-\omega_t^O)(1+r_t)^{\frac{1}{\gamma}-1}\Lambda_t^O E_t \frac{1}{MPC_{t+1}^O} \\
\Omega_t^O &= 1 + E_t \frac{1-\omega_t^O}{1+r_t}\Omega_{t+1}^O \\
\tilde{C}_t^Y &= MPC_t^Y \tilde{Inc}_t^Y + MPC_t^Y \frac{1+r_{t-1}}{1+g_t}(1-\omega_{t-1}^Y)\tilde{B}_{t-1}^Y \\
\tilde{Inc}_t^Y &= (1-\tau_t^L)\tilde{w}_t \tilde{L}_t + \tilde{w}_t^U \tilde{U}_t + \tilde{Profit}_t - \tilde{LTax}_t + E_t(1+g_{t+1})\frac{\tilde{TR}_{t+1}^{YO}}{1+r_t}\Omega_{t+1}^O + E_t \frac{1-\omega_t^Y}{1+r_t}\frac{1+s_{t+1}}{1+s_t}\tilde{Inc}_{t+1}^Y \\
\frac{1}{MPC_t^Y} &= (1+\tau_t^C)p_t^Y + \beta^{\frac{1}{\gamma}}(1+r_t)^{\frac{1}{\gamma}-1}E_t\left((1-\omega_t^Y)\Lambda_t^Y \frac{1}{MPC_{t+1}^Y} + \omega_t^Y \Lambda_t^{YO}\frac{1}{MPC_{t+1}^O}\right) \\
\Lambda_t^O &= E_t\left(\frac{(1+\tau_t^C)p_t^O}{(1+\tau_{t+1}^C)p_{t+1}^O}\right)^{\frac{1}{\gamma}} \\
\Lambda_t^Y &= E_t\left(\frac{(1+\tau_t^C)p_t^Y}{(1+\tau_{t+1}^C)p_{t+1}^Y}\right)^{\frac{1}{\gamma}} \\
\Lambda_t^{YO} &= E_t\left(\frac{(1+\tau_t^C)p_t^Y}{(1+\tau_{t+1}^C)p_{t+1}^O}\right)^{\frac{1}{\gamma}} \\
\tilde{B}_t^Y &= (1-\tau_t^L)\tilde{w}_t \tilde{L}_t + \tilde{w}_t^U \tilde{U}_t + \tilde{Profit}_t - \tilde{LTax}_t + \frac{(1+r_{t-1})(1-\omega_{t-1}^Y)}{1+g_t}\tilde{B}_{t-1}^Y - (1+\tau_t^C)p_t^Y \tilde{C}_t^Y
\end{aligned}
$$

Labor market:

$$
\begin{aligned}
\frac{\sigma}{1-\sigma}\tilde{hc}_t \frac{1-\tau_t^L}{1+\tau_t^{SSC}} &= (1-\tau_t^L)\tilde{w}_t - \tilde{w}_t^U + \\
&+ E_t \frac{1+g_{t+1}^A}{1+r_t}(1-pr_{t+1}^F)(1-pr_{t+1}^H)\left(\frac{\sigma}{1-\sigma}\tilde{hc}_{t+1}\frac{1-\tau_{t+1}^L}{1+\tau_{t+1}^{SSC}}\right) \\
\tilde{U}_t &= s_t^Y - \tilde{L}_t \\
\tilde{hc}_t &= \kappa {pr_t^H}^{\alpha_{HC}} \\
pr_t^H &= (1+g_t^N)\frac{\tilde{H}_t}{\tilde{U}_{t-1} + pr_t^F \tilde{L}_{t-1}}
\end{aligned}
$$

Producing firms:

$$
\begin{aligned}
\frac{1}{1+g_t}\tilde{K}_{t-1} &= \alpha\left(\frac{r_t^K}{mc_t}\right)^{-\theta}\tilde{Y}_t \\
q_t &= \frac{1-\omega_t^Y}{1+r_t}\left(r_{t+1}^K + q_{t+1}(1-\delta)\right) \\
\tilde{\tilde{w}}_t &= (1+\tau_t^{SSC})\tilde{w}_t + \tilde{\lambda}_t^H - E_t \frac{1-\omega_t^Y}{1+r_t}(1-pr_{t+1}^F)(1+g_{t+1}^A)\tilde{\lambda}_{t+1}^H
\end{aligned}
$$

$$
\begin{aligned}
\tilde{L}_t &= (1-\alpha)\left(\frac{\tilde{\tilde{w}}_t}{\tilde{A}_t mc_t}\right)^{-\theta}\frac{\tilde{Y}_t}{\tilde{A}_t} \\
mc_t &= \left[\alpha {r_t^K}^{1-\theta} + (1-\alpha)\left(\frac{\tilde{\tilde{w}}_t}{\tilde{A}_t}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}} \\
\tilde{hc}_t &= E_t(1+g_{t+1})\frac{1-\omega_t^Y}{1+r_t}\tilde{\lambda}_{t+1}^H S'\left(\frac{\tilde{H}_{t+1}}{\tilde{H}_t}\right)\left(\frac{\tilde{H}_{t+1}}{\tilde{H}_t}\right)^2 + \\
&+ \tilde{\lambda}_t^H\left(1 - S\left(\frac{\tilde{H}_t}{\tilde{H}_{t-1}}\right) - S'\left(\frac{\tilde{H}_t}{\tilde{H}_{t-1}}\right)\frac{\tilde{H}_t}{\tilde{H}_{t-1}}\right) \\
1 &= E_t(1+g_{t+1})\frac{1-\omega_t^Y}{1+r_t}q_{t+1}S'(\cdot)\left(\frac{\tilde{Inv}_{t+1}}{\tilde{Inv}_t}\right)^2 + q_t\left(1 - S(\cdot) - S'(\cdot)\frac{\tilde{Inv}_t}{\tilde{Inv}_{t-1}}\right) \\
\frac{\varphi}{\varphi-1}mc_t &= 1 + \frac{1}{\varphi-1}R\left(\frac{1+\pi_t}{(1+\pi_{t-1})^\gamma}\right) + \frac{1}{\varphi-1}R'\left(\frac{1+\pi_t}{(1+\pi_{t-1})^\gamma}\right)\frac{1+\pi_t}{(1+\pi_{t-1})^\gamma} - \\
&- E_t\frac{1-\omega_t^Y}{\varphi-1}(1+g_{t+1})\frac{\tilde{Y}_{t+1}}{\tilde{Y}_t}\frac{R'\left(\frac{1+\pi_{t+1}}{(1+\pi_t)^\gamma}\right)\left(\frac{1+\pi_{t+1}}{(1+\pi_t)^\gamma}\right)}{1+r_t} \\
\tilde{L}_t &= \frac{1-pr_t^F}{1+g_t}\tilde{L}_{t-1} + \tilde{H}_t\left(1 - S\left(\frac{\tilde{H}_t}{\tilde{H}_{t-1}}\right)\right) \\
\tilde{K}_t &= \frac{1-\delta}{1+g_t}\tilde{K}_{t-1} + \tilde{Inv}_t\left(1 - S\left(\frac{\tilde{Inv}_t}{\tilde{Inv}_{t-1}}\right)\right) \\
\tilde{Profit}_t &= \tilde{Y}_t - \tilde{Inv}_t - (1+\tau_t^{SSC})\tilde{w}_t\tilde{L}_t - R\left(\frac{1+\pi_t}{(1+\pi_{t-1})^\gamma}\right)\tilde{Y}_t - \tilde{hc}_t\tilde{H}_t
\end{aligned}
$$

Retailers:

$$
\begin{aligned}
p_t^Y &= 1 + G^{C,Y}\left(\frac{\tilde{C}_t^Y}{\tilde{C}_{t-1}^Y}\right) + G^{C,Y'}\left(\frac{\tilde{C}_t^Y}{\tilde{C}_{t-1}^Y}\right)\frac{\tilde{C}_t^Y}{\tilde{C}_{t-1}^Y} - \frac{1-\omega_t^Y}{1+r_t}G^{C,Y'}\left(\frac{\tilde{C}_{t+1}^Y}{\tilde{C}_t^Y}\right)(1+g_{t+1})\left(\frac{\tilde{C}_{t+1}^Y}{\tilde{C}_t^Y}\right)^2 \\
p_t^O &= 1 + G^{C,O}\left(\frac{\tilde{C}_t^O}{\tilde{C}_{t-1}^O}\right) + G^{C,O'}\left(\frac{\tilde{C}_t^O}{\tilde{C}_{t-1}^O}\right)\frac{\tilde{C}_t^O}{\tilde{C}_{t-1}^O} - \frac{1-\omega_t^Y}{1+r_t}G^{C,O'}\left(\frac{\tilde{C}_{t+1}^O}{\tilde{C}_t^Y}\right)(1+g_{t+1})\left(\frac{\tilde{C}_{t+1}^O}{\tilde{C}_t^O}\right)^2
\end{aligned}
$$

Fiscal policy and the pension system:

$$
\begin{aligned}
\tilde{TR}_t^{YO} &= \nu_t\frac{\omega_{t-1}^Y}{1+g_t}\tilde{w}_{t-1}\tilde{L}_{t-1} \\
\tilde{TR}_t &= \tilde{TR}_t^{YO} + \frac{1-\omega_{t-1}^O}{1+g_t}\tilde{TR}_{t-1} \\
\tilde{Debt}_t + \tilde{LTax}_t + \tilde{Tax}_t &= \tilde{Gov}_t + \tilde{TR}_t + \frac{1+r_{t-1}}{1+g_t}\tilde{Debt}_{t-1} \\
\tilde{Tax}_t &= \tau_t^C(p_t^O\tilde{C}_t^O + p_t^Y\tilde{C}_t^Y) + (\tau_t^{SSC} + \tau_t^L)\tilde{w}_t\tilde{L}_t
\end{aligned}
$$

$$
\begin{aligned}
L\tilde{T}ax_t &= G\tilde{o}v_t + T\tilde{R}_t + \tilde{w}_t^U \tilde{U}_t + \frac{1+r_{t-1}}{1+g_t} D\tilde{e}bt_{t-1} - T\tilde{a}x_t - \left\{\frac{Debt}{Y}\right\}^{Target} \tilde{Y}_t \\
D\tilde{e}bt_t &= \tilde{B}_t^Y + \tilde{B}_t^O
\end{aligned}
$$

Monetary policy:

$$
\begin{aligned}
&1+i_t = (1+i_{t-1})^{\rho_i} \left((1+r_t^n)\,(1+\pi_t)^{\phi_\pi}\right)^{1-\rho_i} e^{\varepsilon_t^i} \\
&1+i_t = E_t(1+r_t)(1+\pi_{t+1})
\end{aligned}
$$

Market clearing:

$$
\tilde{Y}_t = \tilde{C}_t^Y + \tilde{C}_t^O + I\tilde{n}v_t + G\tilde{o}v_t + R\left(\frac{1+\pi_t}{(1+\pi_{t-1})^\gamma}\right)\tilde{Y}_t + \tilde{hc}_t \tilde{H}_t
$$

A.1.6 Steady state of the model

To be able to calculate the steady-state solution we need to specify initial guess for r and we calibrate the hiring cost to gross wage ratio. Then the rest of the variables and equations can be solved numerically. As a function of the initial guess, we can determine the variables of production, labor market and those of the government and pension system. Finally, we turn to the consumption and savings functions. At the end, using the market clearing equations we can check whether our initial guesses are correct.
First, the demographic equations are:

$$
\begin{aligned}
s &= \frac{\omega^Y}{(1-\omega^Y+n)}\left(1-\frac{(1-\omega^O)}{(1-\omega^Y+n)}\right)^{-1} \\
s^Y &= \frac{1}{1+s} \\
1+g^{N,O} &= 1-\omega^O + \frac{\omega^Y}{s} \\
1+g &= 1+g^N = 1+g^{N,Y} = 1-\omega^Y+n
\end{aligned}
$$

Then, we need to guess an initial value for r which is verified by the Newton-Raphson algorithm. Assuming $\pi = 0$ in the steady state implies:

$$
i = r
$$

The Tobin-Q (q) is one in the steady-state equilibrium, so the initial assumption for r and the no-arbitrage condition imply the steady-state value of the marginal product of capital:

$$r^K = \frac{1+r}{1-\omega^Y} - 1 + \delta$$

The firms' supply curve in the steady state gives us the marginal cost as the inverse of the markup:

$$mc = \frac{\varphi - 1}{\varphi}$$

Based on the marginal cost function we can calculate the real wage:

$$\tilde{\tilde{w}} = \tilde{A}\left\{\frac{mc^{1-\theta} - \alpha {r^K}^{1-\theta}}{1-\alpha}\right\}^{\frac{1}{1-\theta}}$$

We can calculate the capital and labor per production ratios from the input demand functions of the firms:

$$\frac{\tilde{K}}{\tilde{Y}} = (1+g)\alpha\left(\frac{r^K}{mc}\right)^{-\theta}$$

$$\frac{\tilde{L}}{\tilde{Y}} = (1-\alpha)\left(\frac{\tilde{\tilde{w}}}{\tilde{A}mc}\right)^{-\theta}\frac{1}{\tilde{A}}$$

$\frac{\tilde{K}}{\tilde{Y}}$ also implies $\frac{\tilde{Inv}}{\tilde{Y}}$:

$$\frac{\tilde{Inv}}{\tilde{Y}} = \frac{\tilde{K}}{\tilde{Y}}\left(1 - \frac{1-\delta}{1+g}\right)$$

$\frac{\tilde{L}}{\tilde{Y}}$ also implies $\frac{\tilde{H}}{\tilde{Y}}$:

$$\frac{\tilde{H}}{\tilde{Y}} = \frac{\tilde{L}}{\tilde{Y}}\left(1 - \frac{1-pr^F}{1+g}\right)$$

In the steady state the wage setting equations imply the real wage of the households:

$$\tilde{\lambda}^H = \tilde{hc}$$

And we can use our assumption about the hiring cost gross wage ratio:

$$WR = \frac{\tilde{hc}}{\tilde{w}}$$

We can express the real wage from the firm equation:

$$\begin{aligned}
\tilde{\tilde{w}} &= (1+\tau_t^{SSC})\tilde{w} + \tilde{hc} - \frac{1-\omega^Y}{1+r}(1-pr^F)(1+g^A)\tilde{hc} \\
\tilde{w} &= \frac{\tilde{\tilde{w}}}{(1+\tau_t^{SSC}) + WR - \frac{1-\omega^Y}{1+r}(1-pr^F)(1+g^A)WR}
\end{aligned}$$

The unemployment benefit is exogenous, can be also calibrated in terms of real wage:

$$WUR = \frac{\tilde{w}^U}{\tilde{w}}$$

These assumption can be used for the wage bargaining equation and we can express the pr^H:

$$\begin{aligned}
\frac{\sigma}{1-\sigma}\tilde{hc}\frac{1-\tau_t^L}{1+\tau_t^{SSC}} &= (1-\tau_t^L)\tilde{w} - \tilde{w}^U + \\
&+ \frac{1+g^A}{1+r}(1-pr^F)(1-pr^H)\left(\frac{\sigma}{1-\sigma}\tilde{hc}\frac{1-\tau^L}{1+\tau^{SSC}}\right) \\
\frac{\sigma}{1-\sigma}WR\tilde{w}\frac{1-\tau_t^L}{1+\tau_t^{SSC}} &= (1-\tau_t^L)\tilde{w} - WUR\tilde{w} + \\
&+ \frac{1+g^A}{1+r}(1-pr^F)(1-pr^H)\left(\frac{\sigma}{1-\sigma}WR\tilde{w}\frac{1-\tau^L}{1+\tau^{SSC}}\right) \\
pr^H &= 1 - \frac{\frac{\sigma}{1-\sigma}WR\tilde{w}\frac{1-\tau_t^L}{1+\tau_t^{SSC}} - (1-\tau_t^L)\tilde{w} + WUR\tilde{w}}{\frac{1+g^A}{1+r}(1-pr^F)\left(\frac{\sigma}{1-\sigma}WR\tilde{w}\frac{1-\tau^L}{1+\tau^{SSC}}\right)}
\end{aligned}$$

Based on pr^H we can calculate κ that consistent with our wage cost ratio:

$$\kappa = \frac{\tilde{hc}}{pr^{H^{\alpha_{HC}}}}$$

And we can express the unemployment-to-GDP ratio:

$$\frac{\tilde{U}}{\tilde{Y}} = \frac{1+g^N}{pr^H}\frac{\tilde{H}}{\tilde{Y}} - pr^F\frac{\tilde{L}}{\tilde{Y}}$$

We can also add the profit-to-GDP ratio:

$$\frac{\tilde{Profit}}{\tilde{Y}} = 1 - \frac{\tilde{Inv}}{\tilde{Y}} - (1+\tau_t^{SSC})\tilde{w}\frac{\tilde{L}}{\tilde{Y}} - \tilde{hc}\frac{\tilde{H}}{\tilde{Y}}$$

We can use the unemployment-to-GDP ratio to express the level of GDP from the unemployment definition since we now the labor-to-GDP ratio and the share of young population in the steady state:

$$\tilde{Y} = \frac{s^Y}{\frac{\tilde{U}}{\tilde{Y}} + \frac{\tilde{L}}{\tilde{Y}}}$$

Based on the GDP identity we can express the total consumption:

$$\frac{\tilde{C}}{\tilde{Y}} = 1 - \frac{\tilde{Inv}}{\tilde{Y}} - \frac{\tilde{Gov}}{\tilde{Y}} - \tilde{hc}\frac{\tilde{H}}{\tilde{Y}}$$

The retailers relative prices are one in the steady state, since we know all tax bases we can also express the sum of distortionary tax revenue of the government:

$$\frac{\tilde{Tax}}{\tilde{Y}} = \tau_t^C\frac{\tilde{C}}{\tilde{Y}} + (\tau^{SSC} + \tau^L)\tilde{w}\frac{\tilde{L}}{\tilde{Y}}$$

Using the assumption of replacement ratio and labor market variables we can calculate the steady-state pension expenditures. Using the pension expenditures, the assumptions for public debt-to-GDP ratio, and government expenditure to GDP ratios we can calculate the equilibrium level of the tax burden:

$$\frac{\tilde{TR}^{YO}}{\tilde{Y}} = \nu\frac{\omega^Y}{1+g}\tilde{w}\frac{\tilde{L}}{\tilde{Y}}$$
$$\frac{\tilde{TR}}{\tilde{Y}} = \frac{\tilde{TR}^{YO}}{\tilde{Y}}\left(1 - \frac{1-\omega^O}{1+g}\right)^{-1}$$

The level of the public debt-to-GDP ratio is given, based on the expenditures and distortionary tax revenues we can calculate the lump-sum tax-to-GDP ratio:

$$\frac{\tilde{LTax}}{\tilde{Y}} = \frac{\tilde{Gov}}{\tilde{Y}} + \frac{\tilde{TR}}{\tilde{Y}} + \tilde{w}^U\frac{\tilde{U}}{\tilde{Y}} + \left(\frac{1+r}{1+g} - 1\right)\frac{\tilde{Debt}}{\tilde{Y}} - \frac{\tilde{Tax}}{\tilde{Y}}$$

Λ^Y, Λ^O, and Λ^{YO} are one in the steady-state, we can use them to calculate the MPC^O and

MPC^Y:

$$MPC^O = \frac{1-(1-\omega^O)(1+r)^{\frac{1}{\gamma}-1}\beta^{\frac{1}{\gamma}}\Lambda^O}{(1+\tau^C)p^O}$$
$$MPC^Y = \left(1-\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}(1-\omega^Y)\Lambda^Y\right)\left((1+\tau^C)p^Y+\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}\omega^Y\Lambda^{YO}\frac{1}{MPC^O}\right)^{-1}$$

The pensioners' discount factor in the steady state is the following:

$$\Omega^O = \left(1-\frac{1-\omega^O}{1+r}\right)^{-1}$$

We can express the young households' expected lifetime-income-to-GDP ratio by using the initial guess of $\tilde{Y}$:

$$\frac{\tilde{Inc}^Y}{\tilde{Y}} = \left(1-\frac{1-\omega^Y}{1+r}\right)^{-1}\left((1-\tau^L)\tilde{w}\frac{\tilde{L}}{\tilde{Y}}+\frac{\tilde{Profit}}{\tilde{Y}}-\frac{\tilde{LTax}}{\tilde{Y}}+(1+g)\frac{\tilde{TR}^{YO}}{\tilde{Y}}\frac{\Omega^O}{1+r}\right)$$

Based on the young consumption function one can substitute out the young consumption-to-GDP ratio in the budget constraint, and express the young bond-to-GDP ratio:

$$\frac{\tilde{B}^Y}{\tilde{Y}} = \frac{(1-\tau^L)\tilde{w}\frac{\tilde{L}}{\tilde{Y}}+\frac{\tilde{Profit}}{\tilde{Y}}-\frac{\tilde{LTax}}{\tilde{Y}}-(1+\tau^C)p^C MPC^Y\frac{\tilde{Inc}^Y}{\tilde{Y}}}{1+(MPC^Y(1+\tau^C)p^C-1)\frac{1+r}{1+g}(1-\omega^Y)}$$

Now, we can express the old households' bond-to-GDP ratio from the bond market equilibrium:

$$\frac{\tilde{B}^O}{\tilde{Y}} = \frac{\tilde{Debt}}{\tilde{Y}}-\frac{\tilde{B}^Y}{\tilde{Y}}$$

And based on the consumption functions we can calculate the consumption-to-GDP ratios:

$$\frac{\tilde{C}^O}{\tilde{Y}} = MPC^O\frac{\tilde{TR}}{\tilde{Y}}\Omega^O+MPC^O\frac{1+r}{1+g}\left[\omega^Y\frac{\tilde{B}^Y}{\tilde{Y}}+\frac{\tilde{B}^O}{\tilde{Y}}\right]$$
$$\frac{\tilde{C}^Y}{\tilde{Y}} = MPC^Y\frac{\tilde{Inc}^Y}{\tilde{Y}}+MPC^Y\frac{1+r}{1+g}(1-\omega^Y)\frac{\tilde{B}^Y}{\tilde{Y}}$$

Finally we need to check if the initial assumptions for r is correct. It means that we need to check if the total consumption is equal with the sum of the young and retired consumption.

Otherwise the algorithm should choose another initial value until the condition is satisfied.

$$\frac{\tilde{C}}{\tilde{Y}} \stackrel{?}{=} \frac{\tilde{C^Y}}{\tilde{Y}} + \frac{\tilde{C^O}}{\tilde{Y}}$$

If we have the right initial values, we can calcuate the levels of all the normalized variables, and run the simulations.

Annex 2 The Ageing Report's accounting framework[39] and the calibration results with OGRE

$$\begin{aligned}\frac{pension\ expenditure}{GDP} &= \frac{population(65+)}{population(20-64)} \times \frac{retirees}{population(65+)} \times \frac{average\ pension\ income}{\frac{GDP}{hours\ worked}} \times \frac{population(20-64)}{hours\ worked} \\ &= \text{(dependency ratio)} \times \text{(coverage ratio)} \times \text{(benefit ratio)} \times \text{(labour market effect)}\end{aligned}$$

Changes in pension expenditures can be explained by the following factors:

- **Dependency ratio effect:** quantifies the impact of changes in the old-age dependcy ratio on pension expenditure;
- **Coverage ratio effect:** looks at the number of pensioners relative to the population older than 64 years. The ratio can capture how developments of the effective exit age and the share of the population covered by the pension system influence pension expenditure;
- **Benefit ratio effect:** indicates how average public pension spending develops relative to the average wage. The ratio assesses how changes to the legal framework of pension systems (concerning pension calculations and indexation rules) affect pension expenditure; and
- **Labour market effect:** describes the effect labour market behaviour/reforms have on pension expenditure.

[39] For further information see European Commission (2018).

Table 8: Comparison of the driving factors of pension expenditures in OGRE and the Ageing Report

in % of GDP, unless stated otherwise

	Germany		Slovakia	
	OGRE	*AR*[1]	*OGRE*	*AR*[1]
Dependency Ratio Effect	4.1	5.1	5.2	6.9
Coverage Ratio Effect	-0.4	-1.0	-1.0	-3.2
Benefit Ratio Effect	-1.6	-1.9	-1.5	-1.2
Labor Market Effect	-0.2	-0.2	-1.5	-0.9
Residual	0.0	-0.2	0.0	-0.6
Total	**1.9**	**1.8**	**1.2**	**1.0**

[1] *The Ageing Report's pension expenditures are rescaled to earnings-related pension expenditures.*

Annex 3 Calibration of the parameters

Table 9: Structural parameters

Name	Sign	Values	
		Germany	*Slovakia*
Discount factor	β	0.999	
Physical capital depreciation rate	δ	0.1	
Physical capital income	α	0.385	0.32
Elasticity of production technology	θ	0.8	0.95
Price markup	μ	1.2	
Gross pension-wage replacement rate	ν	0.43	0.52
Ratio of hiring cost to wage	WR	0.1675	0.575
Bargaining power	σ	0.75	
Elasticity of hiring cost	α_{HC}	0.5	
Technology growth rate (%)	g_A	1.5	3
Inverse of intertemporal substitution	γ	1	
Adjustment cost of physical capital investment	ϕ_{Inv}	2.5	
Consumption habit	ϕ_C	4	
Rotember price adjustment cost	ϕ_P	80	
Price indexation	γ_P	0.5	
Interest rate smoothing	ρ_i	0.5	
Reaction to inflation in Taylor rule	ϕ_π	2.5	

Table 10: Detailed calibration table

	Germany			Slovakia		
	calibrated value	*targeted value*	*source*	*calibrated value*	*targeted value*	*source*
Consumption (total)	60.7	54.6	*Eurostat: National*	60.8	60.4	*Eurostat: National*
Consumption (public)	19.1	19.1	*Accounts, 2011-2016*	16.9	16.9	*Accounts, 2011-2016*
Investment	20.1	20.0		22.3	20.2	
Compensation of employees	59.7	56.3		54.9	46.0	
Unemployment rate (%)	5.7	5.6	*Eurostat:*	12.8	12.7	*Eurostat:*
Unemployment benefit expenditure	1.5	1.8	*2009-2016*	0.7	0.7	*2011-2016*
Firing probability (%)	13.9	13.9	*Eurostat: 2001-2015*	11.1	11.1	*Eurostat: 2001-2015*
Pension expenditure*	7.2	7.8	*Ageing Report (2018): 2016*	6.9	6.7	*Ageing Report (2018): 2016*
Benefit ratio (%)*	34.1	32.4		34.4	36.3	
Old-age dependency ratio** (%)	37.2	38.3		36.5	32.6	
Fertility rate** (%)	1.9	1.9	*Eurostat:*	2.6	2.6	*Eurostat:*
Probability of death** (%)	4.2	4.2	*2006-2016*	5.3	5.3	*2002-2016*
Probability of retirement** (%)	1.7	1.7		2.1	2.1	
Value Added Tax Revenue	7.0	7.0		6.5	6.5	
Labor Income Tax Revenue	8.7	8.7	*Eurostat: 1995-2016*	2.9	2.9	*Eurostat: 2009-2016*
Social Sec. Contr. Revenue: Employer	6.5	6.5		7.3	7.3	
Social Sec. Contr. Revenue: Employee	6.2	6.2		3.0	3.0	
Public debt	75.4	75.4	*Eurostat: 2009-2016*	53.3	53.3	*Eurostat: 2009-2016*

**Earnings related pension expenditures*

***Implied from population data and effective retirement rate*

Annex 4 Baseline scenario

Figure 4: Baseline scenario - Germany

(deviation from initial steady-state)

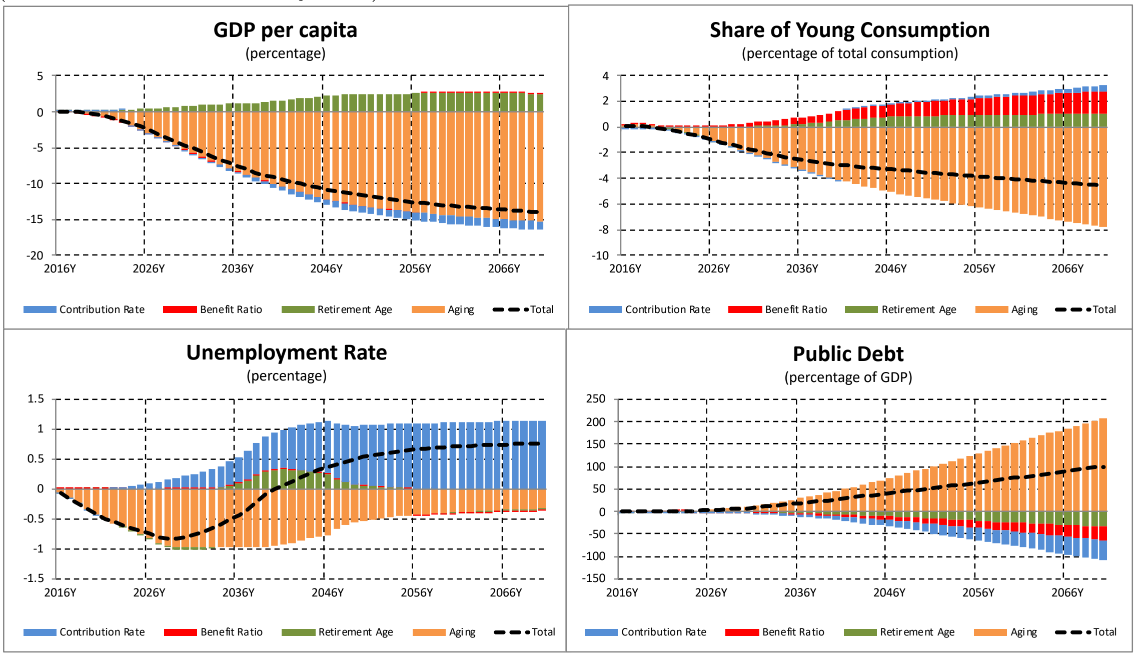

Figure 5: Baseline scenario - Slovakia

(deviation from initial steady-state)