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Liquidity Choice and Misallocation of Credit

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Abstract

This paper studies a novel type of misallocation of credit between investments of varying liquidity. One type of investment is more liquid, i.e., its return is more pledgeable, and the other is more productive. Low liquidities of both investment types imply that the allocation of credit is constrained inefficient and that there is overinvestment in the liquid type. Constrained inefficient equilibria feature non-positive, i.e., one less than or equal the economy's growth rate, and yet too high interest rate, too much investment and too little consumption. Financial development can reduce long-term welfare and output in a constrained inefficient equilibrium if it raises the liquidity of the liquid type. I show a maximum liquid asset ratio or a simple debt tax can achieve constrained efficiency. Introducing government bonds can make Pareto improvement whenever it does not raise the interest rate.

JEL Classification Numbers: D61, D62, E2, G2, G3, O16

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Keywords: Liquidity choice, borrowing constraint, constrained inefficiency, misallocation of credit, pecuniary externality, financial development

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1 Introduction

Financial frictions can distort the allocation of credit in the economy, resulting in low output and welfare. One type of credit misallocation is the overextending of credit to and overinvestment in liquid[1] but low-productivity assets. Previous academic and policy research suggest that this type of misallocation pertains to many different contexts. Examples include allocating too much credit to large firms with low productivity, firms in construction and real estate sectors and government-owned enterprises with especial access to credit. Despite low productivity, high pledgeability of investment is a potentially important factor contributing to overinvestment in these examples (Gopinath et al. (2017), Dell'Ariccia et al. (2019) and Zheng et al. (2011)).

This paper studies a novel type of constrained inefficiency in credit allocation in an economy consisting of high-return/low-liquidity and low-return/high-liquidity projects, which involves overinvestment in the latter. The allocation of credit in this environment is clearly not the first best but also not even the second best: a planner who faces the same liquidity constraints as the private agents can still improve upon the equilibrium allocation and raise welfare. Symptoms of this inefficiency are too-high interest rate and investment relative to the constrained optimum, i.e., the second best. As a result, constrained Pareto improving policies reduce the interest rate and investment. This is in stark contrast to the unconstrained inefficiency in the same model where the interest rate and investment are too low relative to the unconstrained optimum, i.e, the first best, and the unconstrained Pareto improvement raises the interest rate and investment.

This new inefficiency also has important implications for financial development. I show that financial development that is conventionally thought to bring economies with financial frictions closer to the first best may in fact reduce output and welfare because it can increase the extent of misallocation. Such adverse effects are present when the allocation of credit is constrained inefficient, and financial development raises the liquidity of the low-return/high-liquidity projects. These adverse effects can speak to the new evidence on the non-monotonic relationship between financial development, output, and welfare (Levine (2005)).

A brief description of the model is as follows. The economy consists of overlapping generations of entrepreneurs who live for three periods: young, middle-aged and old. There is a continuum of each generation present in each period, and there are no aggregate or idiosyncratic risks in the economy. When young, entrepreneurs receive a fixed endowment of perishable consumption goods (and nothing thereafter) that cannot be stored. Middle-aged entrepreneurs have an opportunity to invest in a portfolio of investments. There are two types of constant return to scale investment technologies. One type is more productive and has a higher return per unit of

[1] I use the term "liquidity" and "pledgeability" interchangeably throughout the paper. "Liquidity" or "pledgeability" refer to debt capacity or, using the terminology of Brunnermeier and Pedersen (2009), funding liquidity. Liquidity is the ability to transfer wealth across different time periods by pledging the returns to a real or financial investment.

investment (productive type). The other type of investment has a higher pledgeable return per unit of investment (liquid type). That is, a bigger part of its return can be credibly promised to be paid back. The middle-aged use their wealth which is the principal and interest on the loan they made when young to the middle-aged in the previous period (the old in the current period) plus funds they borrow from the young in the current period. There is a competitive credit market in every period in which borrowing by the middle-aged is constrained by the total amount of pledgeable return to their investment portfolio. Higher wealth, more liquid portfolio, and a lower interest rate allow the middle-aged to borrow more from the young. Hence a liquid portfolio allows for a bigger investment size while a productive one raises the average return per unit of investment. Finally, entrepreneurs consume only when they are old.

When liquidities of both types are low, entrepreneurs might end up investing too much in the liquid type of investment due to a pecuniary externality leading to inefficiently liquid equilibria. The constrained efficient allocation in such cases requires investing only in the productive type at the steady state. The portfolio choice of the middle-aged entrepreneurs at each date depends on the prevailing interest rate: all else equal, a higher interest rate implies lower leverage and less investment in the liquid type. An additional unit of investment in the liquid type by an entrepreneur bids up the interest rate and raises the debt payments for other middle-aged entrepreneurs. But low liquidity of both investment types implies that the initial wealth of the middle-aged will be low since a low fraction of the returns to investment in any period can be invested by future entrepreneurs. Hence, given the borrowing constraint, the additional increase in the interest rate will be small and cannot sufficiently discourage other entrepreneurs from investing in the liquid type. In fact, when liquidities are high enough, investment in the liquid type by entrepreneurs would bid up the interest rate so high that it would make them switch to the productive type; therefore, investment in the liquid type cannot be an equilibrium. Hence, when liquidities are low, the negative effect of an additional unit of investment in the liquid type on debt payments more than offset the reallocation away from the liquid type, leaving other agents worse off. Inefficiently liquid equilibria feature non-positive, i.e., lower or equal to the growth rate, and yet too-high interest rate, too much investment and too little consumption.

The constrained inefficiency introduced in this work is more likely to arise in financially underdeveloped economies, e.g, low-income and emerging markets, where most types of investment have low liquidity. It is applicable to a wide range of environments in which investment projects differ in both their productivity and liquidity. Examples include but are not limited to real estate versus non-real estate, small and medium enterprises (SME) versus big mature firms, public versus private firms, and capital-intensive versus labor-intensive production technologies. Inefficiently liquid equilibria in the context of these examples would feature credit and investment booms accompanied by overinvestment in large firms, firms with more tangible assets such

as land and physical capital, and state-owned firms.

This paper provides a new justification for public intervention in the financing of young firms and SME. A growing body of research studies the misallocation of credit especially in economies with underdeveloped financial markets. In contrast to this line of work, this paper argues that the allocation of credit to young firms and SME is not only not the first best, but that it even may not be the second best. Given that the second best allocation may be achieved using simple regulations such as a debt tax, this implies a stronger case for public support of young firms and SME financing than what was previously understood.

These results suggest that policies such as developing private bond markets, loan guarantees, and development of primary and secondary markets for asset securitization should be pursued with more caution in low-income and emerging market economies. Bond financing, for example, can mostly benefit large established firms that tend to have more liquid but low-productivity investment projects. Securitization and loan guarantees can be harmful if the underlying assets, e.g., residential mortgages, are more liquid relative to other investment opportunities in the economy. These policies may lower long-term output and welfare if they end up benefiting low-productivity but liquid investments in an already inefficiently liquid equilibrium. Policies that facilitate seizure of collateral can also be harmful to the long-term output and welfare when there is overinvestment in tangible assets in the economy. The reason is that these policies mainly increase the liquidity of investments with a high share of tangible assets. Examples of such policies include the creation of public property registries and the improvement of creditor rights in bankruptcy laws. In contrast, policies which raise the liquidity of the more productive but less liquid investment projects always enhance long-term output and welfare in an inefficient equilibrium.

Given an inefficient equilibrium, a planner can achieve efficiency by regulating the fraction of resources that is invested in the liquid type by entrepreneurs. This regulation may be implemented as a maximum liquid asset ratio, e.g., a cap on the ratio of real estate loans to total assets, within a perfectly competitive banking sector. Pareto efficiency can also be obtained by a simple debt tax. I study the welfare effects of government bonds, which are assumed to be fully liquid due to the ability of government to tax. Government bonds can make Pareto improvement only for inefficient equilibria where there is strictly positive investment in both types in the long run. In this case, government bonds crowd out the liquid type and crowd in the productive type so that the demand for funds and hence the interest rate remains unchanged in equilibrium. Entrepreneurs can substitute fully liquid government bonds for the liquid type to gain an extra amount of pledgeable return that can be used to borrow more funds that can be invested in the productive type. Since the interest rate does not increase, entrepreneurs' return goes up while their debt payment stays the same. This increases entrepreneurs' consumption and leads

to Pareto improvement.

I characterize the competitive equilibrium and the steady state, and do comparative statics with respect to liquidities and returns of the two types of investments. Contracting technology, contract enforcement, corporate governance, and bankruptcy laws are among the factors that can affect the liquidity of investment. I show that an increase in the liquidity or return of the productive type of investment leads to a lower steady-state interest rate. This interest rate effect can be understood as follows. An increase in the liquidity or return of the productive type has two effects. First, for any given investment portfolio, it increases the liquidity of that portfolio which, in turn, raises the investment demand and interest rate. Second, an increase in the liquidity or return of the productive type makes the productive type more attractive to investors. This leads the investors to substitute the liquid for productive but still less liquid type, which reduces the interest rate. It turns out that the second effect dominates the first one, and so the steady state interest rate falls as the liquidity or return of the productive type increases.

The above interest rate effect can be useful in understanding the patterns of capital outflows in emerging market economies such as China in the past few decades. Consider an open-economy version of this model with two countries, home and foreign. The liquid type represents investment in large mature firms with easy access to external finance, e.g., state-owned firms in China, while the productive type is investment in highly productive entrepreneurial firms in the private sector with limited access to external finance. In such an open-economy version, an increase in the return of the productive type at home, e.g., higher productivity of private entrepreneurial firms in China, results in outflows of funds to the foreign country. These outflows are accompanied by a reallocation of credit toward the productive type at home. Moreover, if the home country is not small, this reallocation of credit lowers the world interest rate. The above narrative resembles the one suggested by Zheng et al. (2011).

Finally, there are three differences worth noting between the constrained inefficiency discussed in this paper and the conventional dynamic inefficiency in models with overlapping generations, e.g. Diamond (1965). First, there exist inefficiently liquid equilibria with a zero interest rate in steady state, while an equilibrium with zero interest rate is efficient in the traditional models. Second, in contrast to traditional models, a negative[2] (lower than the growth rate) interest rate can be constrained efficient in this model. Third, a Pareto improvement in an inefficiently liquid equilibrium induces an even more negative interest rate. In traditional models, however, an interest rate below the growth rate of the output must be raised to make a Pareto improvement.

1.1 Related Literature

[2]All results of the paper survive a positive growth in endowments or population.

Farhi and Tirole (2012) is closely related to this paper and is used as a benchmark for the analysis. The model structure of my paper is similar to that in Farhi and Tirole (2012). The only difference is that in this paper, entrepreneurs have access to different types of investments with different liquidity. Heterogeneity in investment types leads to different results that are complementary to the ones obtained in Farhi and Tirole (2012). The way limited liquidity is modeled in this paper is similar to Farhi and Tirole (2009, 2010) and Matsuyama (2007) while it also has a close connection to Holmström and Tirole (1998) and Kiyotaki and Moore (2002, 2005, 2008).

Normative results in this paper are in contrast with those of Woodford (1990) and Holmström and Tirole (1998). Low liquidity generated by the private sector is at the heart of inefficiency in Woodford (1990) and Holmström and Tirole (1998). In my paper, too much liquidity is what makes the decentralized allocation inefficient. The nature of inefficiency also differs from that of Samuelson (1958) and Diamond (1965). A negative interest rate, i.e., one that is less than economy's growth rate, in Samuelson (1958) and Diamond (1965) implies that the interest rate is too low. In contrast, a non-positive interest rate in an inefficient equilibrium in this paper indicates that the interest rate is in fact too high.

Similar to this paper, Kehoe and Levine (1993) and Lorenzoni (2008) feature pecuniary externality as the source of inefficiency in the credit market. Pecuniary externality in these models arises because asset prices, spot prices, or the interest rate appear in constraints other than the budget constraint. The inefficient sale of productive assets in an environment with aggregate uncertainty is the key in Lorenzoni (2008) that leads to an externality, while here, it is the demand for investible resources in an economy without any uncertainty that entails an inefficient outcome.

Matsuyama (2007) studies a model with heterogenous assets of different liquidity. Matsuyama (2007) focuses on the dynamics of aggregate credit and capital stock when investment composition plays an important role for given returns and liquidities of investment. The goal of this paper, however, is to study the credit misallocation resulting from heterogeneity in investment liquidity as well as the effects of financial development, i.e., exogenous changes in liquidity of investment, on the economy. There are also two different types of assets with different liquidity in Giglio and Severo (2012), namely tangible and intangible capital. Besides having a different focus, Giglio and Severo (2012) does not feature any portfolio choice between liquid and illiquid investments. There is a high degree of complementarity between liquid and illiquid capitals in Giglio and Severo (2012) due to the Cobb-Douglas production technology. In this paper, however, the liquid and illiquid types are perfect substitutes. Hence Giglio and Severo (2012) is closer to an economy with one type of asset, e.g., Farhi and Tirole (2012), than this paper.

Similar to Bianchi (2011), in any inefficient equilibria a debt tax can restore efficiency in this model. The inefficiency in Bianchi (2011) is due to distortions in the relative price of non-tradable

to tradable goods in an small open economy with a fixed interest rate. In contrast, the pecuniary externality in this paper works through the interest rate.

On the empirical side, Gopinath et al. (2017) show that the declining interest rate on borrowing has led to a reallocation of credit toward larger firms with less binding borrowing constraint in Spain. Dell'Ariccia et al. (2019) show that construction sector grows significantly more than other sectors during credit booms and especially so during the bad booms, i.e., those that end in crisis or subpar growth. Reis (2013) argues that the reallocation of credit toward low productivity firms in the nontradable sector has been the main cause of the economic stagnation and subsequent slump in Portugal between 2000 and 2012. In Reis (2013), and in contrast to my model, too many resources are invested in the less liquid nontradable sector and especially in its less productive firms at the expense of the firms in the tradable sector.

The paper proceeds as follows. Section 2 describes the model and characterizes competitive equilibria and steady states. Section 3 discusses properties of equilibria and their interpretations and applications. Section 4 studies the efficiency of competitive equilibria and how a planner can Pareto improve the competitive equilibrium allocation when it is inefficient. In Section 5, I introduce government bonds and analyze their welfare implications, and in Section 6 I conclude.

2 Model

2.1 Agents, Preferences and Technology

The model economy is comprised of overlapping generations of entrepreneurs with no uncertainty. Each individual lives for three periods, and there is a unit measure of young, middle-aged, and old cohorts in each period. Entrepreneurs receive a fixed endowment $e > 0$ of non-storable and homogenous consumption goods when young and no endowment thereafter, and consume only when they are old.

The choice of overlapping generations is mainly for simplicity and tractability. One can think of the agents in this economy as firms in the real or financial sectors facing alternating investment opportunities and borrowing constraints. The main feature of the model is the ability of these firms to choose a portfolio of investment projects while pledging the return to their portfolio to outside investors, i.e., non investing firms with otherwise idle resources. In this sense, this model is similar to Kiyotaki and Moore (2002) and Kiyotaki and Moore (2005) and has a close connection with Woodford (1990).

In any period, the middle-aged have the opportunity to invest in two types of projects which pay off in the next period. Projects differ in their return and liquidity. A project of type $j \in \{1, 2\}$ has a constant return to scale R_j where $\theta_j R_j$ can be pledged to the outside investors. Limited

liquidity of return can arise in many contexts and for a number of different reasons, including asymmetric information, moral hazard, and limited commitment. Following Kiyotaki and Moore (2002, 2005, 2008) and Farhi and Tirole (2009, 2012, 2010), I summarize all these frictions in the variable $\theta_j, j \in \{1, 2\}$. Given (R_1, R_2), I refer to $\theta_j, j \in \{1, 2\}$ as well as $\theta_j R_j, j \in \{1, 2\}$ as liquidity of type j. I make the following assumption about the return and liquidity of projects:

Assumption 1. $R_1 > R_2 > 1$ *and* $\theta_1 R_1 < \theta_2 R_2 < 1$.

Assumption 1 captures the trade-off between liquidity and return across the two types of projects; type 1 is more productive (productive type) while type 2 is more liquid (liquid type). This type of trade-off between liquidity and return can be observed in both real and financial sectors. Large and more mature firms in the real sector tend to have lower cost of external financing, that is, their investment is more liquid, than SME and young firms. This may be due to the availability of extensive records and accounts, their reputation, or the higher value of their collateral. State-owned firms in many emerging market economies with easy access to finance tend to be less productive than the financially constrained entrepreneurial firms. Investment in tangible assets is more liquid than investment in intangibles, which are typically hard to liquidate in the event of bankruptcy. Liquid assets such as real estate versus other types of assets with lower liquidity but higher return, e.g., machinery and human capital, provide another example. Financial securities with different haircuts[3] can serve as yet another example in the financial sector.

It is helpful for future analysis to define a *benchmark economy* in which there is no trade-off between the two types:

Definition 1. *The* ***Benchmark Economy*** *is an economy where* $R_1 > R_2 > 1$ *and* $1 > \theta_1 R_1 > \theta_2 R_2$.

Note that in the benchmark economy, type 1 projects dominate type 2 projects in terms of both liquidity and return. This implies that entrepreneurs never invest in type 2 in equilibrium, such that the economy collapses to one with a single type of investment project, similar to Farhi and Tirole (2012). The benchmark economy is used throughout the current paper to provide better understanding of the results.

2.2 The Problem of Middle-Aged Entrepreneurs

In each period a competitive credit market opens up in which young and middle-aged entrepreneurs can lend and borrow. The young born in period $t > 0$ inelastically supply all their endowments in the capital market. The middle-aged entrepreneur at time t, who has transferred funds from

[3]The supply of assets have to be elastic enough so that the constant return assumption is a good approximation.

period $t-1$ by investing in the projects of the middle-aged at $t-1$, can borrow additional funds from the young. But this borrowing by the middle-aged entrepreneur is constrained by the limited liquidity of her investment portfolio. She chooses her optimal investment portfolio given the ongoing interest rate r_t and resources that have been transferred from period $t-1$ to period t.

Let x_{1t} and x_{2t} denote investments in types 1 and 2 and let i_t denote the new funds raised by the middle-aged entrepreneur at t using the resources of the young entrepreneurs in period t. Given the interest rate r_t, a middle-aged entrepreneur at t solves the following problem:

$$
\begin{aligned}
c^o_{t+1} \equiv \max_{i_t, x_{1t}, x_{2t} \geq 0} \quad & R_1 x_{1t} + R_2 x_{2t} - (1 + r_t) i_t \qquad \text{(I)} \\
s.t. \quad & x_{1t} + x_{2t} \leq (1 + r_{t-1}) e + i_t \,, \\
& (1 + r_t) i_t \leq \theta_1 R_1 x_{1t} + \theta_2 R_2 x_{2t} \,.
\end{aligned}
$$

The first constraint in the maximization above is the resource constraint of the middle-aged entrepreneur. $(1 + r_{t-1})e$ is the wealth transferred from period $t-1$ to t by the middle-aged entrepreneur through investing her endowment e, in the projects of middle-aged entrepreneurs in period $t-1$. The second term, i_t, is the total external funds that the middle-aged entrepreneurs borrow from the young entrepreneurs at t. The second constraint is the manifestation of the limited liquidity of the investments; the middle-aged entrepreneur cannot borrow more than what she can credibly commit to pay in period $t+1$. For type $j \in \{1, 2\}$, the maximum that can be credibly promised to the lenders is $\theta_j R_j x_{jt}$, and so the total amount of pledgeable return is given by the right-hand side of the second constraint. Finally, c^o_{t+1} denotes the consumption of the old entrepreneur in period $t+1$.

The implicit assumption in Problem I, that the middle-aged can cross-pledge, i.e., pledge the return to one type of project to invest in the other, is not essential. There are at least two other variations which produce the same results as in this setup. In one variation, the middle-aged entrepreneurs have to decide first how much of their initial wealth they want to invest in each type of project (which cannot be altered later), and then they can pledge the return of each type only for investment in that type. In the second alternative, each middle-aged entrepreneur can invest in only one type of project. One can show that both of these alternatives lead to results similar to this model and that what matters is how the aggregate investment portfolio is determined and not whether firms, i.e., entrepreneurs, actually hold any portfolios.

The resource constraint is always binding in Problem I. If the interest rate is not too high, the borrowing constraint has to be binding as well. In this case one can eliminate x_{1t} and x_{2t} in the above problem and reach the following reduced form:

LEMMA 1. *In any competitive equilibrium where $1 + r_t < R_1$ for all t, the borrowing constraint*

of the middle-aged entrepreneur binds in every period. Moreover, the problem of the middle-aged entrepreneur can be written in the following form:

$$\begin{aligned} \max_{i_t} \quad & \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t) i_t + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1}) e \\ s.t. \quad & \left(\frac{\theta_1 R_1 (1 + r_{t-1})}{1 + r_t - \theta_1 R_1}\right) e \leq i_t \leq \left(\frac{\theta_2 R_2 (1 + r_{t-1})}{1 + r_t - \theta_2 R_2}\right) e \,, \end{aligned} \tag{II}$$

where,

$$\begin{aligned} \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t) &\equiv \left(\frac{(\theta_2 - \theta_1) R_1 R_2}{\theta_2 R_2 - \theta_1 R_1}\right) - \left(\frac{(1 - \theta_1) R_1 - (1 - \theta_2) R_2}{\theta_2 R_2 - \theta_1 R_1}\right) (1 + r_t) \,, \\ \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1}) &\equiv \left(\frac{(\theta_2 - \theta_1) R_1 R_2}{\theta_2 R_2 - \theta_1 R_1}\right) (1 + r_{t-1}) \,. \end{aligned}$$

The bold symbols $(\boldsymbol{\theta}, \boldsymbol{R})$ are the vector of liquidities and returns of the two types of investments, i.e., $(\theta_1, \theta_2, R_1, R_2)$.

In Lemma 1 the term Λ is the net marginal (and average) return of external funds i_t when borrowing constraint is binding. The two bounds in the constraint of Problem II corresponds to the two limits; when i_t hits the lower (upper) bound, the entrepreneur invests only in the productive (liquid) type, depending on the sign of Λ. Define $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ as the interest rate in period t that makes the entrepreneurs indifferent between the two types:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \frac{(\theta_2 - \theta_1) R_1 R_2}{(1 - \theta_1) R_1 - (1 - \theta_2) R_2} \,. \tag{1}$$

Then the entrepreneurs' optimal demand for funds is characterized as follows:

$$\begin{cases} i_t = \left(\frac{\theta_2 R_2 (1 + r_{t-1})}{1 + r_t - \theta_2 R_2}\right) e \,, & \text{if } r_t < r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \,, \\ i_t \in \left[\left(\frac{\theta_1 R_1 (1 + r_{t-1})}{1 + r_t - \theta_1 R_1}\right) e, \left(\frac{\theta_2 R_2 (1 + r_{t-1})}{1 + r_t - \theta_2 R_2}\right) e\right] \,, & \text{if } r_t = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \,, \\ i_t = \left(\frac{\theta_1 R_1 (1 + r_{t-1})}{1 + r_t - \theta_1 R_1}\right) e \,, & \text{if } r_t > r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \,. \end{cases} \tag{2}$$

Figure 1 is an illustration of the middle-aged demand for funds given by 2 and the inelastic supply of funds by young entrepreneurs at time t.

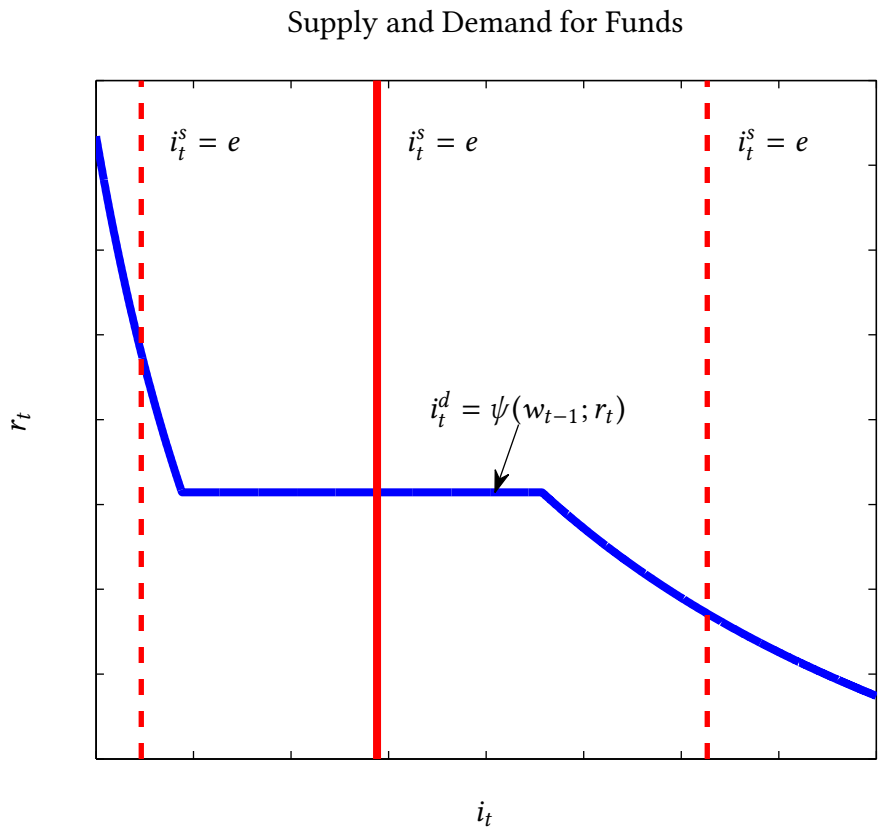

Figure 1: Supply (red) and demand (blue) for funds at any period t as a function of the interest rate. w_{t-1} denotes the wealth of the middle-aged, i.e. $(1 + r_{t-1})e$. The two arms on the demand curve correspond to investing only in type 1 or 2. The flat segment in between corresponds to $r_t = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, where entrepreneurs mix. A higher period $t-1$ interest rate, i.e. higher w_{t-1}, makes the two arms of the demand curve shift to the right but has no effect on the demand curve's flat segment.

2.3 Competitive Equilibrium

In each period there is a fixed supply of funds e. Market clearing condition dictates:

$$i_t = e, \quad \forall t \geq 0\,. \tag{3}$$

Combining 2 and 3 yields the equilibrium path of the interest rates:

$$r_t = \begin{cases} \theta_2 R_2(2 + r_{t-1}) - 1 & \text{if } \theta_2 R_2(2 + r_{t-1}) - 1 < r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}), \\ \theta_1 R_1(2 + r_{t-1}) - 1 & \text{if } \theta_1 R_1(2 + r_{t-1}) - 1 > r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}), \\ r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) & \text{otherwise} \end{cases} \tag{4}$$

Given 2, the dynamic upper and lower bounds on the interest rate are:

$$\theta_1 R_1(2 + r_{t-1}) - 1 \leq r_t \leq \theta_2 R_2(2 + r_{t-1}) - 1 . \tag{5}$$

I can now define a competitive equilibrium as follows:

Definition 2. *A competitive equilibrium is a sequence* $\{i_t, x_{1t}, x_{2t}, r_t\}_{t=0}^{\infty}$ *of investments and interest rates and an initial value of* r_{-1} *that satisfy conditions 1 to 5, in which* x_{1t} *and* x_{2t} *solve problem I and* $1 + r_t < R_1$ *for all* $t > 0$.[4]

Using 2 and 3, the aggregate investment portfolio at any date is:

$$\begin{cases} x_{1t} = 0, x_{2t} = (2 + r_{t-1})e & \text{if } r_t < r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}), \\ x_{1t} = \left(\frac{\theta_2 R_2(2+r_{t-1}) - (1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}))}{\theta_2 R_2 - \theta_1 R_1}\right) e, & \\ x_{2t} = \left(\frac{(1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})) - \theta_1 R_1(2+r_{t-1})}{\theta_2 R_2 - \theta_1 R_1}\right) e & \text{if } r_t = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}), \\ x_{1t} = (2 + r_{t-1})e, x_{2t} = 0 & \text{if } r_t > r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}). \end{cases} \tag{6}$$

Entrepreneurs specialize in the productive (liquid) type when the interest rate is relatively high (low). To characterize competitive equilibrium, it is useful to define the following three regions in the parameter space:

Definition 3. *Define* F *as the set of* $(\boldsymbol{\theta}, \boldsymbol{R})$ *that satisfies Assumption 1 and also* $\frac{\theta_1 R_1}{1-\theta_1 R_1} < R_1$. *Then the three regions of* F *are defined as follows:*

The Liquid Region *is defined as* $F_\ell = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F | \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) < \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right) \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\}$.

The Mixed Region *is defined as* $F_m = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F | \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right)\}$.

[4] I impose $1 + r_t < R_1$ to focus on equilibria in which borrowing constraint is binding at all dates.

The Illiquid Region *is defined as* $F_i = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F | 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) < \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right)\}$.

Notice that all three regions, F_ℓ, F_m, and F_i, have nonempty interiors. I require $\frac{\theta_1 R_1}{1-\theta_1 R_1} < R_1$ to ensure that the borrowing constraint is binding in the steady state.[5]

Lemma 2. *Each of the three regions in Definition 3 has a unique and stable steady state equilibrium. More specifically:*

$$\begin{cases} r_\ell^{ss} = \left(\frac{\theta_2 R_2}{1-\theta_2 R_2}\right) - 1 & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \,. \\ r_m^{ss} = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_m \,. \\ r_i^{ss} = \left(\frac{\theta_1 R_1}{1-\theta_1 R_1}\right) - 1 & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_i \,. \end{cases}$$

Moreover, at the steady state, the entrepreneurs specialize in the liquid and productive type of investments in regions F_ℓ and F_i, respectively. Entrepreneurs invest in both types in F_m, where the amounts of each type are given by 6.

Using Lemma 2, the following proposition establishes the existence and uniqueness of competitive equilibrium:

Proposition 1. *Given any $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and an initial condition $1 + r_{-1} < R_1$, there exists a unique competitive equilibrium that converges to the steady state corresponding to $(\boldsymbol{\theta}, \boldsymbol{R})$, given by Lemma 2.*

3 Properties of Equilibria

In this subsection I analyze the three regions in Lemma 2 and do comparative statics with respect to $\boldsymbol{\theta} = (\theta_1, \theta_2)$ for a given vector of returns $\boldsymbol{R} = (R_1, R_2)$. Given $\boldsymbol{R}$, values of $\boldsymbol{\theta}$ correspond to different liquidities of the two investment types, which reflect different institutional environments, i.e., contract enforcement, contracting technology, bankruptcy laws, corporate governance, etc. The following lemma summarizes general properties of the three regions for any given vector of returns.

Proposition 2. *For a given vector of returns $\boldsymbol{R}$ the following are correct:*

[5] When $\frac{\theta_1 R_1}{1-\theta_1 R_1} \geq R_1$, a steady state equilibrium exists in the illiquid region where the borrowing constraint does not bind. In this steady state equilibrium $1 + r_i^{ss} = R_1$.

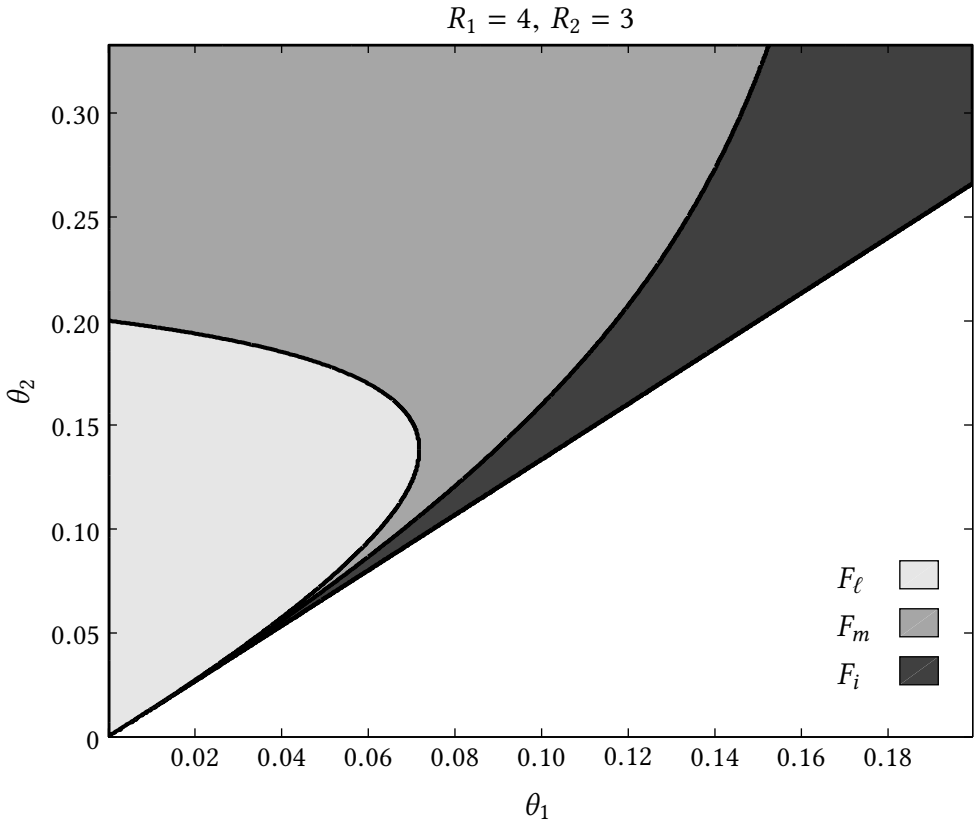

Figure 2: Image of F_ℓ, F_m and F_i for $R_1 = 4$ and $R_2 = 3$ over the space of (θ_1, θ_2). The white area below the positively sloped straight line where Assumption 1 is violated corresponds to the *benchmark economy* in Definition 1.

1- When $\boldsymbol{\theta}$ is small enough (close to the origin), one can have all three types of steady-state equilibria.

2- For any $\boldsymbol{\theta}$ in the liquid region, $\boldsymbol{\theta} \leq (\frac{1}{1+R_1}, \frac{1}{1+R_1})$.

3- For any value of θ_1, the values of θ_2 for which (θ_1, θ_2) belongs to the liquid region lies strictly above the respective values of θ_2 for which (θ_1, θ_2) belongs to the illiquid region.

4- The boundary of the liquid region is a non-monotonic curve cutting the $\theta_1 = 0$ line twice: once at the origin and again at $\boldsymbol{\theta} = (0, \frac{1}{1+R_1})$.

5- The inner boundary of the illiquid region is a strictly increasing and convex function of θ_1 which reaches the maximum possible of $\theta_2 = \frac{1}{R_2}$.

6- The top right corner of F in the space of liquidities, that is, $\boldsymbol{\theta} = (\frac{1}{1+R_1}, \frac{1}{R_2})$, belongs to the illiquid region.

Figure 2 suggests that the allocation of credit is non-monotonic in θ_2. The following lemma establishes the non-monotonicity of credit allocation and interest rate with respect to changes in $\boldsymbol{\theta}$.

LEMMA 3. *In the mixed region, the steady state interest rate is strictly decreasing in θ_1 and R_1 while it is strictly increasing in θ_2 and R_2. Moreover, the fraction of total funds invested in the liquid type at the steady state, i.e., $\frac{x_2^{ss}}{x_1^{ss}+x_2^{ss}}$, is non monotonic in θ_2 and has an interior maximum for relatively low values of θ_1. In contrast, this ratio is always weakly decreasing in θ_1 and strictly decreasing in θ_1 and R_1 in F_m.*

Suppose that θ_2 increases while θ_1 is held constant. On the one hand, this increase encourages middle-aged entrepreneurs to invest more in the liquid type at any given interest rate. On the other hand, this increase in liquidity of the liquid type increases the demand for funds and raises the interest rate at the steady state, which discourages the entrepreneurs from investing in the liquid type. Following Lemma 3, the second effect dominates the first one for high enough values of θ_2. In this case, any further increase in θ_2 bids up the interest rate so much that entrepreneurs are forced to lower the share of the liquid type in their portfolios.

An increase in θ_2 or R_2 raises the average liquidity of any portfolio and so raises the demand for funds at a given interest rate. This tends to bid up the steady-state equilibrium interest rate. In contrast, an increase in θ_1 or R_1 makes the productive type more attractive for a given interest rate. This, in turn, encourages the entrepreneurs to substitute the productive for the liquid type. Since the productive type is still the less liquid project, this substitution lowers the interest rate. It turns out that the second effect dominates the first one in the mixed region, which results in a strictly lower interest rate.

In a more general case, where financial development affects both liquidities or both returns, the direction of change in the interest rate and allocation of funds depends only on the relative change in liquidities or returns, e.g., if $\frac{\Delta\theta_2}{\Delta\theta_1}$ is less (more) than the slope of the isoline in Figure 3, the interest rate decreases (increases).

Lemma 3 can be useful in thinking about capital outflows in emerging market economies such as China. These outflows have been puzzling because in a neoclassical world, capital has to flow to countries with the highest marginal product of capital, which seems to be a feature of fast-growing emerging market economies. In particular, consider an open-economy version of this model with two countries, home and foreign. The middle-aged entrepreneurs at home and abroad can borrow from young entrepreneurs both at home and abroad. Suppose that the liquid type at home represents large mature firms with easy access to external finance, e.g., state-owned firms in the case of China, while the productive type represents more productive entrepreneurial firms

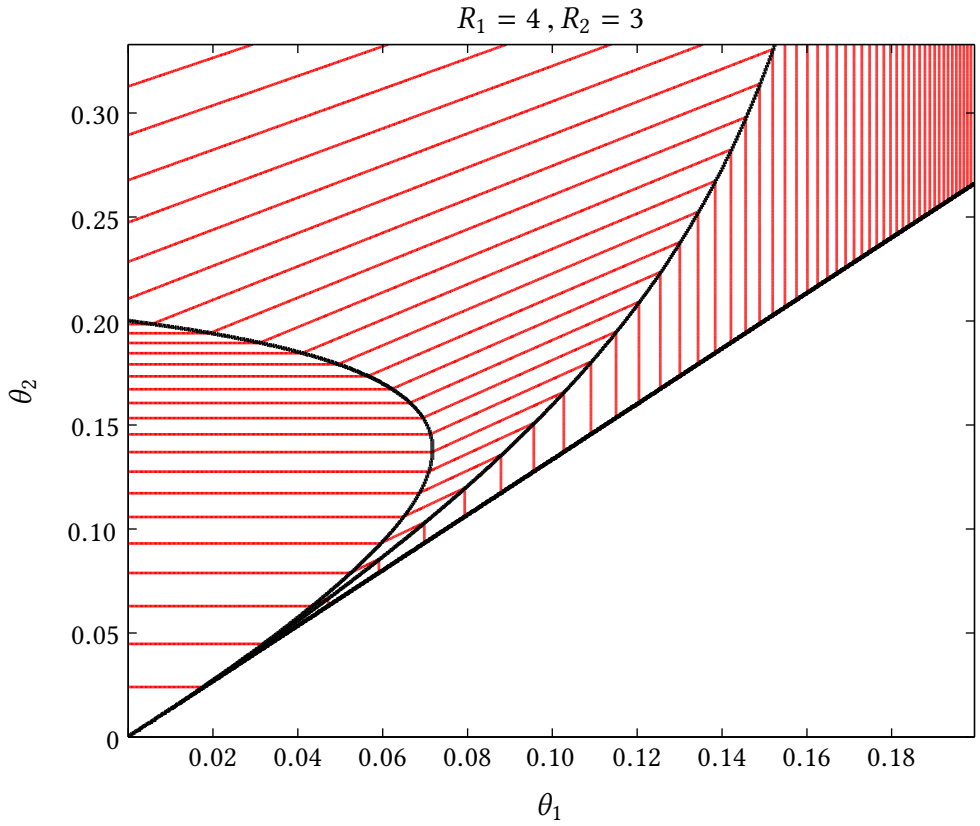

Figure 3: Contour plot of the steady-state interest rates (red lines) for the three regions. Interest rates are highest at the top left corner.

with limited access to external finance. To simplify the exposition, assume that initially home and foreign have the same liquidities and returns but different endowments e and e^* where $e < e^*$, so that the net flows of funds is zero. In autarky an increase in the productivity of the productive type R_1 at home leads to a lower interest rate and a higher fraction of resources invested in the productive type. This implies that in the two-country version, higher R_1 at home induces the funds to flow out of the economy. The reallocation of credit toward the more productive type is still present in the two-country case but its magnitude is somewhat dampened relative to the autarky. If the home economy is large enough, i.e., $\frac{e}{e+e^*}$ is not small, the outflow of funds lowers the equilibrium world interest rate.

The above narrative is similar to the one suggested by Zheng et al. (2011). They build an OLG model to reconcile high growth and high return to capital with a growing foreign surplus in China over the past three decades. The reallocation of labor from the less productive state-owned firms to the more productive but financially constrained private firms makes the economy look like an AK model during the transition. The constant returns to investments in state-owned and

private firms during the transition are reminiscent of R_1 and R_2 in this model.[6]

4 Welfare and Efficiency

In this section I first study the efficiency of competitive equilibria and investigate policies that can Pareto improve upon the inefficient allocations. Next, I examine the effects of financial development on long-term welfare and discuss implications for measurement and policy.

4.1 Efficiency of Competitive Equilibria

I start by defining the notion of efficiency that I use throughout:

Definition 4. *An allocation in the overlapping generations economy is called* ***constrained Pareto efficient*** *if a social planner cannot reallocate the resources to make at least one entrepreneur strictly better off while keeping all others at least as well off and if the reallocation respects the liquidity*[7] *constraint in I. More formally, an allocation* $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ *is constrained Pareto efficient if it is feasible, i.e., it satisfies the following series of constraints for all* $t \geq 0$*:*

$$\begin{cases} c_t + x_{1t} + x_{2t} \leq R_1 x_{1t-1} + R_2 x_{2t-1} + e\,, \\ \\ x_{1t} + x_{2t} \leq \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} + e \end{cases} \tag{7}$$

and there does not exist any feasible allocation $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}$ *such that* $c_t \geq c_t^*$ *for all* $t \geq 0$ *with at least one strict inequality, given initials* $x_{j,-1} = x_{j,-1}^*$ *for* $j \in \{1, 2\}$*.*

The following proposition about the benchmark economy can help the reader understand the results more clearly:

Proposition 3. *Any competitive equilibrium in the benchmark economy is constrained Pareto efficient.*

Consider steady-state equilibria where investment in the liquid type is strictly positive. Suppose the planner reduces the aggregate debt payments of all middle-aged entrepreneurs in every generation to the young by an amount of $\delta > 0$ by substituting the productive for the liquid type. Let the increase in the productive type be $\epsilon > 0$; then the resource constraint implies that investment in the liquid type has to be reduced by $\epsilon + \delta$. Given δ, the maximum possible ϵ is

[6] Wage in Zheng et al. (2011) is similar to endowment e in this model with the difference that wage in their model is growing due to exogenous productivity growth.

[7] I use the term "liquidity constraint" instead of "borrowing constraint" in the analysis of the social planner problem.

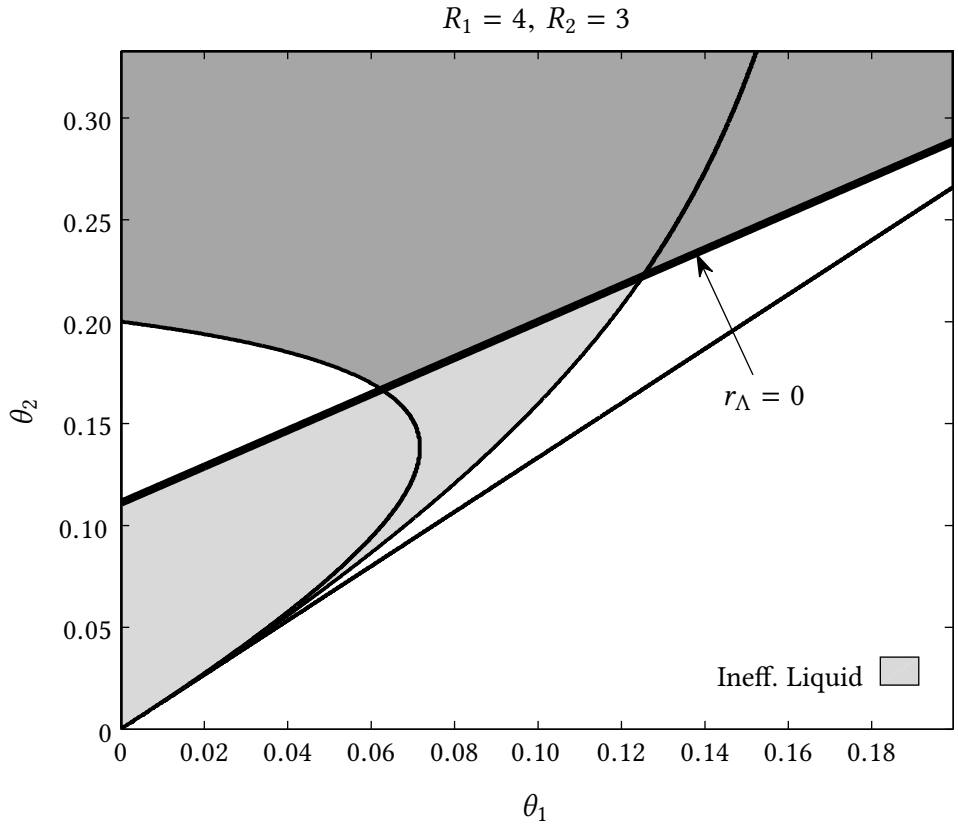

Figure 4: An illustration of inefficiently liquid equilibria (light gray region) and the line corresponding to $r_\Lambda = 0$ for $R_1 = 4$ and $R_2 = 3$.

determined when the borrowing constraint binds. The change in the consumption level of the old at $t \geq 1$ is $\Delta V^{ss} = R_1\epsilon - (\epsilon + \delta)R_2 + \delta$. Note that the initial middle-aged entrepreneur is strictly better off because the planner reduces her debt payments to the young while, in contrast with the future middle-aged, her receipts from the initial old do not change.[8] Therefore if $\Delta V^{ss} \geq 0$, the steady-state allocation is constrained inefficient. The above reallocation can also work outside the steady state. The following proposition characterizes the constrained inefficient equilibria:

Proposition 4. *Consider any competitive equilibrium with liquidities and returns given by* $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m$. *If* $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$, *the competitive equilibrium is constrained inefficient. Moreover, the equilibrium interest rate at the steady state is strictly negative when* $(\boldsymbol{\theta}, \boldsymbol{R})$ *lies in the interior of the inefficient region, i.e.,* $\{(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m \mid r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0\}$, *and zero on part of its boundary that lies in* F_m. [9]

[8] Note that the initial old pay off their debts to the middle-aged at $t = 0$–which are given as an initial condition to the middle-aged and are not changed by the planner–consume, and die.

[9] The social planner can also do the opposite by increasing the aggregate debt payments by $\delta > 0$ when there is strictly positive investment in the productive type along the equilibrium path. When $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$ this reallocation

Comparing Proposition 4 and Proposition 3 implies that entrepreneurs' portfolio choices are the key feature causing inefficiency. Inefficiency arises because middle aged entrepreneurs' portfolio choices entail a pecuniary externality. In order to better understand this pecuniary externality, the following lemma summarizes some of the properties of the constrained inefficient equilibria:

LEMMA 4. *For any $\boldsymbol{R}$ satisfying Assumption 1, the following are correct. The sets of inefficiently liquid competitive equilibria in F are nonempty with a strictly positive measure. There are inefficiently liquid equilibria in any arbitrarily small neighborhood of the origin. At $\theta_1 = 0$, the maximum value of θ_2 that results in constrained inefficient equilibria is increasing in R_1. The set of inefficiently liquid equilibria in F_ℓ is a proper subset of F_ℓ if and only if $\frac{R_1 - R_2}{R_2 - 1} \leq 1$. Finally, let S_i denote the unique intersection of $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ with the boundary of F_i. Then all $(\boldsymbol{\theta}, \boldsymbol{R})$ which correspond to inefficiently liquid equilibria have liquidities, i.e., $\boldsymbol{\theta}$, less than S_i.*

As Figure 4 and Lemma 4 suggest, the economy becomes inefficiently liquid when liquidities of both types are relatively low.[10] The portfolio choice of middle-aged entrepreneurs at each date depends on the prevailing interest rate: all else equal, higher interest rate implies lower leverage and less investment in the liquid type. An additional unit of investment in the liquid type by an entrepreneur bids up the interest rate and raises the debt payments for other middle-aged entrepreneurs. But low liquidity of both investment types implies that the initial wealth of the middle-aged will be low since a small fraction of the returns to investment in any period can be invested by future entrepreneurs. Given the borrowing constraint, the additional increase in the interest rate will be small and cannot sufficiently discourage other entrepreneurs from investing in the liquid type. In fact, when liquidities are high enough, investment in the liquid type by entrepreneurs would bid up the interest rate so high that it would make them switch to the productive type; therefore, investment in the liquid type cannot be an equilibrium. Hence, when liquidities are low the negative effect of an additional unit of investment in the liquid type on debt payments more than offsets the reallocation away from the liquid type, leaving other agents worse off. Low liquidities of both investment types are also the reason the inefficiently

increases the utility of every generation starting from $t = 1$ but is not a Pareto improvement because the initial middle-aged would suffer. Consider the following reinterpretation. There are three types of infinitely-lived individuals where type $j \in \{0, 1, 2\}$ receives endowment $e > 0$ in periods $(j + 1 \bmod 3) + 3k$, produces in $(j + 2 \bmod 3) + 3k$ and consumes in periods $(j \bmod 3) + 3k$ for all $k \in \{0, 1, 2, \ldots\}$ with preferences $U_j = \sum_{k=0}^{\infty} \beta^k c_{j+3k}$. This economy is similar, though not isomorphic, to that in Kiyotaki and Moore (2002) and Woodford (1990). With this reinterpretation the proposed reallocation above would be a Pareto improvement for a high enough β, and the set of Pareto inefficient equilibria is the region $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$ in Figure 4 shown in dark gray color. In contrast with this model, the inefficiency under the above reinterpretation is similar to the one in Woodford (1990) in that there is insufficient liquidity and the interest rate is too low while being positive.

[10] Note that in one sense, liquidities, i.e., $\boldsymbol{\theta}$, are low for all $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$, which is why the borrowing constraint is binding. Hence "high" and "low" should be understood in relative terms inside F.

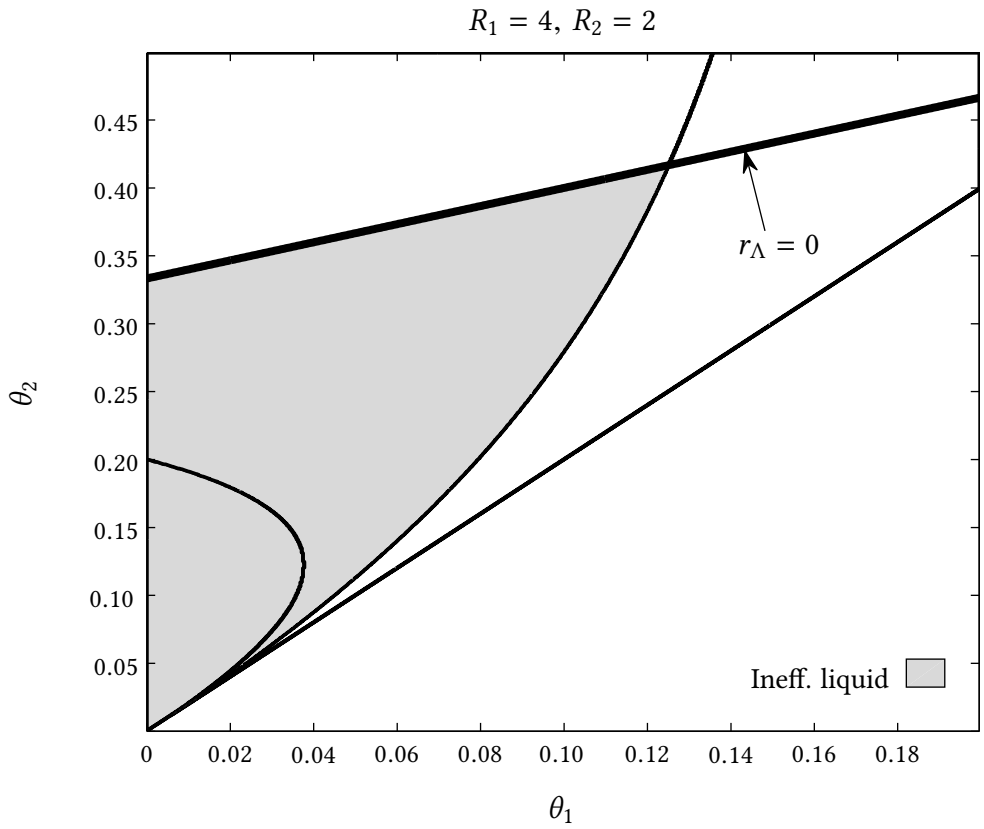

Figure 5: This figure shows the expansion of the inefficiently liquid region when $\boldsymbol{R} = (4, 2)$. In contrast with the case of $\boldsymbol{R} = (4, 3)$, all competitive equilibria in liquid region F_ℓ are constrained Pareto inefficient. For low values of θ_1 and compared to $\boldsymbol{R} = (4, 3)$, higher values of θ_2 can lead to inefficiency.

liquid steady states have a non positive interest rate.

The above lemma suggests that countries with a low level of financial development but high growth opportunities, i.e., a large R_1, may be more prone to this type of constrained inefficiency (Figure 5).[11] These economies suffer from shortages of stores of value due to low liquidity of return to investment. Investment in real estate is a liquid but relatively unproductive investment that has served as an important store of value in these countries. The analysis above suggests that these countries may be investing too much in real estate.

Investment in liquid assets such as real estate can take the form of bubbly equilibria in models with borrowing constraint. Absent uncertainty, these bubbly equilibria are Pareto efficient because they help agents transfer resources across periods.[12] To my knowledge, this is the first paper to show that investment in liquid assets may be inefficient even in an environment without

[11] This model is not quantitative and so it is difficult to rule out the possibility that countries with developed financial sectors may suffer from the same type of inefficiency. Nonetheless, countries with underdeveloped financial sectors are certainly the more likely candidates.

[12] See Caballero and Krishnamurthy (2006) for welfare analysis of such real estate bubbles.

uncertainty.

Countries with underdeveloped financial institutions may also be investing too little in illiquid but highly productive projects such as young firms and small and medium enterprises (SMEs) with high growth potential as opposed to old and large firms. Young firms and SMEs are commonly believed to face severe frictions in financing their operations through credit markets. Therefore, SME finance is a prevalent concern among policy makers in both developed and developing countries.[13] A growing body of research studies the misallocation of credit especially in economies with underdeveloped financial markets. In this line of work, misallocation of credit is the result of the limited net worth of some firms with high marginal product of capital which are facing binding borrowing constraints. Allocation of credit across firms, e.g., young firms or SMEs versus old and large firms, in such environments is not first best.[14] This paper shows that the allocation of credit between SME and large firms or young and old firms may not even be the second best since young firms and SME not only have limited net worth to use as collateral but may also be subject to higher collateral requirements. The higher collateral requirements are due to inadequate records and accounts to document firm performance, as well as, higher levels of credit risk.[15] The latter notion of inefficiency assumes that the planner faces the same contractual, informational, and institutional constraints in reallocating the resources as do private agents. Hence, this paper can provide a stronger case for supporting young firms and the SME sector than what was previously understood.

Investment in tangible assets in financially underdeveloped economies may also be constrained inefficient. Firms with a higher share of tangible assets or inputs such as land and machinery find it easier to pledge collateral and hence enjoy a higher borrowing capacity. As another example, capital-intensive production technologies can pledge a higher fraction of their output than labor-intensive technologies because labor cannot be pledged as collateral.[16]

I close this section by characterizing the set of constrained Pareto efficient allocations in the following proposition:

PROPOSITION 5. *Let* $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$. *If* $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$, *any allocation* $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}$ *that satisfies 7 with equality for all* $t \geq 0$ *is constrained Pareto efficient. Consequently, any competitive equilibrium corresponding to* $(\boldsymbol{\theta}, \boldsymbol{R})$ *is constrained Pareto efficient. If* $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$, *any allocation* $\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}$ *that satisfies 7 with equality for all* $t \geq 0$ and *has* $x_{2t} = 0, t \geq T$ *for some* $T \geq 0$ *is constrained*

[13] See Stevenson (2010) for a discussion on SMEs and why they are important.

[14] For example, see Hsieh and Klenow (2009).

[15] See Global Financial Development Report (2014) by World Bank for SME financing obstacles. Liberti and Mian (2010) document that risk and collateral requirements are correlated, and Dietsch and Petey (2004) show that SMEs are riskier than large firms.

[16] Calvo et al. (2014) show that different liquidities of capital and labor-intensive technologies can explain jobless or wageless recoveries during financial crises in a sample of countries.

Pareto efficient. Hence, any competitive equilibrium in F_i is constrained Pareto efficient.

4.2 Regulated Economy

In this section, I discuss policies that can implement the Pareto improving reallocation proposed in Section 4.1.

Consider any competitive equilibria and suppose that the social planner can dictate the fraction $\alpha_{\ell t}$ of total funds that are invested in the liquid type. In this case, the entrepreneur only chooses the level of new funds raised i_t, and, the maximization problem of the middle aged entrepreneurs takes the following form:

$$\begin{aligned} \max_{i_t \geq 0} \quad & ((1-\alpha_{\ell t})R_1 + \alpha_{\ell t}R_2)\,(i_t + (1+r_{t-1})e) - (1+r_t)i_t \\ s.t. \quad & (1+r_t)i_t \leq (\theta_1(1-\alpha_{\ell t})R_1 + \theta_2\alpha_{\ell t}R_2)\,(i_t + (1+r_{t-1})e)\,. \end{aligned} \tag{IV}$$

The following proposition shows that this type of policy can implement the Pareto improving reallocations in Section 4.1.

Proposition 6. *Any Pareto improving reallocation of the type analyzed in Section 4.1, when δ is small enough in absolute value, can be implemented by regulating the investment portfolios of the entrepreneurs. In an inefficiently liquid equilibrium, a planner chooses a lower liquid investment-to-total investment ratio, and the regulated interest rate is lower than in the unregulated equilibrium. Moreover, given any inefficiently liquid equilibria where $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ or one where $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ in the mixed region, this regulation can implement a Pareto improvement reallocation that results in a constrained Pareto efficient allocation.*

The above regulation is akin to a maximum liquid asset ratio in a perfectly competitive banking sector that lends out the funds deposited by the young to the middle-aged entrepreneurs.[17] The banks should be required to keep the fraction of their assets invested in the liquid type less than or equal to what the social planner chooses in Proposition 6.[18][19]

It is worth noting that the overinvestment in liquid assets is accompanied by too much investment relative to the constrained optimum. To see why, recall that the aggregate investment in any period t is $(2 + r_{t-1})e$. Given that the interest rate is too high in a constrained inefficient equilib-

[17] See Appendix A for more details.

[18] Regulations that require financial institutions to keep a minimum liquid asset ratio have been used in many countries as a monetary or macroprudential instrument. "Liquidity" in the context of these regulations refers to market liquidity: the ability to sell the asset in the market quickly without any significant discount.

[19] For historical records of and motivations behind liquidity requirements, see Gulde et al. (1997). These types of requirements are still in place in many developing countries, and they have been a cornerstone in BASEL III. See http://www.frbsf.org/publications/banking/asiafocus/2011/march.pdf and http://www.bis.org/publ/bcbs188.pdf.

rium, any Pareto improvement would reduce aggregate investment at all periods. Some emerging market economies, e.g., China, are likely examples of such inefficient investment booms.[20]

The fact that the interest rate is too high in constrained inefficient equilibrium is in contrast with the conventional dynamic inefficiency in the overlapping generations models first studied in Samuelson (1958) and Diamond (1965). The conventional dynamic inefficiency implies an interest rate that is too low, which has to be raised by a planner to achieve efficiency.

For this regulation to work, banks should be able to observe and monitor investments by the entrepreneurs in the two types. The following lemma shows that one can reach the Pareto frontier via a simpler and less demanding instrument:

Lemma 5. *Given an inefficiently liquid competitive equilibrium, a social planner can make a Pareto improvement that reaches the Pareto frontier by levying a debt tax (and reimbursing via a lump sum transfer) where the middle-aged entrepreneur has to pay* $(1+\tau)(1+r_t)i_t$ *at* $t+1$ *for all* $t \geq T$ *for some* $T \geq 0$ *and a constant* $\tau > 0$.

The problem in a constrained inefficient equilibrium is that the middle-aged raise too much debt. The excess borrowing bids up the interest rate by a socially inefficient amount. Hence a debt tax is a natural way to penalize the excess borrowing and internalize the pecuniary externality that leads to inefficiency.

4.3 Output and Welfare in the Long Run

In this part of the paper, I show how this model differs from the benchmark economy in terms of the effect of financial development on long-term welfare. As in the previous sections, financial development refers to improvements in contract enforcement, contracting technology, bankruptcy laws, and corporate governance that raise the liquidity of investment. The following proposition shows that in part of the inefficient region, financial development that raises the liquidity of the liquid type lowers long-term output and welfare:

Proposition 7. *Let* $V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ *and* $Y_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ *denote the steady-state utility and aggregate output in the mixed region.* V_m^{ss} *is increasing in* θ_2 *if and only if* $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 2\theta_1 R_1$. *Consequently,* V_m^{ss} *is always increasing in* θ_2 *in the efficient part of the mixed region. Moreover, within the inefficient part of the mixed region, if* θ_2 *is low enough for any given* θ_1, V_m^{ss} *is decreasing in* θ_2. *In the inefficient part of the mixed region, there exists a threshold* θ_1^* *such that given* $\theta_1 < \theta_1^*$, Y_m^{ss} *is decreasing in* θ_2 *if* θ_2 *is low enough. Finally, for any economy* (θ_1, θ_2), *if* Y_m^{ss} *is decreasing in* θ_2, *then* V_m^{ss} *is decreasing in* θ_2 *as well.*

[20] See Lee et al. (2012).

An increase in θ_2 in the inefficient region has two effects. On the one hand, it increases the liquidity of any given portfolio and consequently the investment size. On the other hand, it makes investment in the liquid type more attractive, which encourages the entrepreneurs to substitute the liquid for the productive type. This second effect is detrimental to the output and welfare since investment in the liquid type is inefficient. Hence V_m^{ss} and Y_m^{ss} decrease when the second effect is dominant.

Proposition 7 suggests that certain financial market policies may be harmful for long-term welfare, especially in economies where even most liquid investments are not very liquid. Developing corporate bond markets, loan guarantees, and asset securitization for residential mortgages, have been on the agenda of policy makers in emerging market countries as well as in international organizations such as the International Monetary Fund and the Bank of International Settlements.[21] Proposition 7, however, implies that these policies may have negative welfare consequences, especially in less financially developed economies including many low-income and emerging markets. Low levels of financial development make these economies more likely to lie in the inefficient region. Mortgage loan guarantees, development of asset securitization, and corporate bond markets may raise the liquidity of the relatively more liquid investments such as home mortgages and investments in large and mature firms. Therefore, this type of financial development can reduce long-term output and welfare when there is overinvestment in the relative more liquid sectors of the economy.[22]

Policies that facilitate seizure of collateral by creditors may also end up worsening long-term welfare when an economy is inefficiently liquid. Creating public property registries and enhancing creditor rights in bankruptcy laws are among such policies. These policies result in higher liquidity for investments with a high share of tangible assets or inputs such as land and physical capital. Qian and Strahan (2007), for example, show that higher creditor rights affect collateral requirements more for firms with more tangible assets. Raising the liquidity of investment in tangible assets, however, may lower long-term output and welfare when the economy is overinvesting in tangible assets. In contrast, policies that increase the liquidity of investments in intangibles, e.g., human capital or labor-intensive production, raise long-term output and welfare when there is overinvestment in tangibles. Examples of such policies include improving accounting standards, creating credit records, and establishing information sharing platforms.

The results in this section have an important bearing on the measurement and benchmarking

[21] For the bond market in emerging economies, see Bank for International Settlements Papers, No 11. For discussions of asset securitization in developing countries, see Alles (2001) and Chiquier et al. (2004).

[22] This follows the argument of the loanable funds theory developed in Holmstrom and Tirole (1997). According to this argument, only the well capitalized firms can use the arm's length type of financing, i.e., bond finance, while firms with lower net worth use informed debt, i.e., bank finance.

of financial development.[23] Financial development may or may not correlate with higher output and welfare depending on its composition and the initial level of development: it may lead to constrained inefficient economies or may lower long-term output and welfare in a financially underdeveloped economy. Hence it is essential for any type of measurement or benchmarking to capture the different compositions of credit market developments. Moreover, for low values of liquidities, financial development can lead to lower aggregate output at the steady state. This can speak to some recent evidence on the non-monotonic relationship between financial development and growth, especially at low levels of financial development.[24]

5 Public Liquidity

In this section I study whether and how the introduction of government bonds can improve welfare in a constrained inefficient equilibrium. The effects of introducing government bonds on competitive equilibria can also help to empirically distinguish between this model and models with one type of investment such as Farhi and Tirole (2012), Holmström and Tirole (1998), and Woodford (1990).

5.1 Competitive Equilibrium with Government Bonds

Consider the model in Section 2 with only one difference: the young and middle-aged entrepreneurs at any time $t \geq 0$ can purchase a one-period, risk free government bond sold at par, denoted by b_t^y and b_t^m. A unit of bond purchased at time t is a promise by the government to deliver one unit of consumption good plus the interest in period $t+1$. The final form of the maximization problem of the middle-aged with government bonds (as long as $1 + r_t < R_1$) that can be compared to II is as follows:

$$\begin{aligned} \max_{i_t, b_t^m \geq 0} \quad & \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t)(i_t - b_t^m) + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1})e - \tau_{t+1}^o \\ s.t. \quad & \left(\frac{\theta_1 R_1 (1 + r_{t-1})}{1 + r_t - \theta_1 R_1}\right) e \leq (i_t - b_t^m) \leq \left(\frac{\theta_2 R_2 (1 + r_{t-1})}{1 + r_t - \theta_2 R_2}\right) e \,. \end{aligned} \tag{II$_b$}$$

Φ and Λ are as before, given in II. The important assumption here is that investment in government bonds is perfectly liquid so that bond purchases reduce the total debt payments to $(1 + r_t)(i_t - b_t^m)$. τ_{t+1}^o denotes the lump sum tax that is levied on the old entrepreneurs before

[23] See Beck et al. (2008).
[24] See Levine (2005) page 903.

consumption takes place.[25] I suppose that government balances its budget every period:

$$(1 + r_t)b_t = b_{t+1} + \tau^o_{t+1} . \tag{8}$$

Market clearings dictate that for all $t \geq 0$:

$$i_t + b^y_t = e , \tag{9}$$

$$b^m_t + b^y_t = b_t . \tag{10}$$

In order to ensure the existence of competitive equilibrium in which borrowing constraint is still binding, one needs to restrict the supply of bonds. Let $\sigma_t = \frac{b_t}{e}$ be the normalized supply of bonds for all $t \geq 0$;[26] then one needs the following assumption:

Assumption 2. *$\sigma_t < \min(1-\theta_2 R_2, 1-\frac{\theta_1 R_1}{1-\theta_1})$ for all $t \geq 0$ and $\sigma = \lim_{t\to\infty} \sigma_t < \min(1-\theta_2 R_2, 1-\frac{\theta_1 R_1}{1-\theta_1})$ exists.*

One can redefine the three regions as follows:

Definition 5. *Define $F(\Sigma)$ as the set of $(\boldsymbol{\theta}, \boldsymbol{R})$ that satisfies Assumption 1 and Assumption 2, given the sequence $\Sigma = \{\sigma_t\}_{t=0}^{\infty}$. Then the three regions of $F(\Sigma)$ are defined as:*

The Liquid Region*: $F_\ell(\Sigma) = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F(\Sigma) | \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) < \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right) \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\}$.*

The Mixed Region*: $F_m(\Sigma) = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F(\Sigma) | \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right)\}$.*

The Illiquid Region*: $F_i(\Sigma) = \{(\boldsymbol{\theta}, \boldsymbol{R}) \in F(\Sigma) | 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) < \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right)\}$.*

The existence and uniqueness of competitive equilibria and the steady states with government bonds are established as follows:

Lemma 6. *For any $(\boldsymbol{\theta}, \boldsymbol{R})$ and Σ satisfying Assumption 1 and Assumption 2 and for any given initial condition $1 + r_{-1} < R_1$, there is a unique competitive equilibrium that converges to a unique and stable steady state corresponding to the region in Definition 5 containing $(\boldsymbol{\theta}, \boldsymbol{R})$. The steady-state*

[25] A tax on the young would affect the equilibrium conditions in more subtle ways and would perhaps complicate the problem, e.g., by resulting in multiple steady states. I want to avoid this complication since effects of tax policy are not the focus of my study.

[26] From now on, I use the terms "supply" and "normalized supply" of bonds interchangeably.

interest rates for the three regions are:

$$\begin{cases} 1 + r_{\ell}^{ss}(\Sigma) = \left(\frac{(1-\sigma)\theta_1 R_1}{(1-\sigma)-\theta_1 R_1}\right) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_{\ell}(\Sigma)\,, \\ 1 + r_{m}^{ss}(\Sigma) = 1 + r_{\Lambda}(\boldsymbol{\theta}, \boldsymbol{R}) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_{m}(\Sigma)\,, \\ 1 + r_{i}^{ss}(\Sigma) = \left(\frac{(1-\sigma)\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right) & \text{if } (\boldsymbol{\theta}, \boldsymbol{R}) \in F_{i}(\Sigma)\,. \end{cases}$$

Moreover, at the steady state, the entrepreneurs specialize in the liquid and productive type of investments in regions $F_{\ell}(\Sigma)$ and $F_i(\Sigma)$ respectively but invest strictly positive amounts in both types in $F_m(\Sigma)$.

The following lemma shows the effects of government bonds on the allocation of credit at the steady state:

LEMMA 7. *Let $i_x^{ss} \equiv x_1^{ss} + x_2^{ss}$ denote the total amount of resources invested in the two types by entrepreneurs at the steady state. One has the following:*

$$\frac{\partial i_x^{ss}}{\partial \sigma}|_{(\boldsymbol{\theta},\boldsymbol{R})\in F_{\ell}} = \left(\left(\frac{\theta_2 R_2}{(1-\sigma)-\theta_2 R_2}\right)^2 - 1\right) e\,,$$

$$\frac{\partial i_x^{ss}}{\partial \sigma}|_{(\boldsymbol{\theta},\boldsymbol{R})\in F_{m}} = -e\,.$$

and:

$$\frac{\partial x_1^{ss}}{\partial \sigma}|_{(\boldsymbol{\theta},\boldsymbol{R})\in F_{m}} = \frac{1 + r_{\Lambda}(\boldsymbol{\theta}, \boldsymbol{R}) - \theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1}\, e > 0\,,$$

$$\frac{\partial x_2^{ss}}{\partial \sigma}|_{(\boldsymbol{\theta},\boldsymbol{R})\in F_{m}} = -\frac{1 + r_{\Lambda}(\boldsymbol{\theta}, \boldsymbol{R}) - \theta_1 R_1}{\theta_2 R_2 - \theta_1 R_1}\, e < -e\,.$$

An increase in the long-term supply of public liquidity σ in the liquid region crowds out private investment when public liquidity is scarce and crowds in private investment when public liquidity is abundant.[27] The marginal effect of public liquidity on private investment is strictly increasing in the level of public liquidity, i.e., $\frac{\partial^2 i_x^{ss}}{\partial \sigma^2} > 0$. On the other hand, government bonds always crowd out private investment *one for one* in the mixed region and the marginal effect of public liquidity on private investment is constant, i.e., $\frac{\partial^2 i_x^{ss}}{\partial \sigma^2} = 0$. Public liquidity crowds out the liquid type and crowds in the productive type in the mixed region. The crowding out of the liquid type happens more than proportionally so that the demand for funds and consequently

[27] This is similar to the result in Farhi and Tirole (2012) for the *benchmark economy*.

the interest rate remain unchanged. In contrast to other models which feature the crowding out effect, government bonds crowd out private investment while having no effects on the interest rate.

5.2 Welfare Effects of Government Bond

The effects of government bonds on long-term welfare are characterized as follows:

LEMMA 8. *Let $V_z^{ss}(\Sigma)$ denote the steady-state utility level for region $z \in \{\ell, m, i\}$ given $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_z(\Sigma)$, when the long-run supply of government bonds is σ. Then one has:*

$$\begin{aligned}
\frac{\partial V_\ell^{ss}(\Sigma)}{\partial \sigma}|_{\sigma=0} &= \left(\frac{R_2 - 1}{1 - \theta_2 R_2}\right) r_\ell^{ss} e\,, \\
\frac{\partial V_i^{ss}(\Sigma)}{\partial \sigma}|_{\sigma=0} &= \left(\frac{R_1 - 1}{1 - \theta_1 R_1}\right) r_i^{ss} e\,, \\
\frac{\partial V_m^{ss}(\Sigma)}{\partial \sigma}|_{\sigma=0} &= -r_m^{ss} e\,.
\end{aligned}$$

r_z^{ss} denotes the steady-state interest rate for region $z \in \{\ell, m, i\}$ when there is no government bond in the economy.

Introduction of government bonds in an economy which lies in the inefficient part of the liquid region F_ℓ is harmful to long-term welfare. The reason is that government bonds crowd out the more productive (relative to government bond) investment in the liquid type. This negative long-term effect implies that government bonds cannot Pareto improve constrained inefficient equilibria in the liquid region. In contrast, Lemma 8 implies that the supply of government bonds enhances the steady state utility in the inefficient part of F_m. The following proposition shows that government bonds can Pareto improve the competitive allocation in the mixed region:

PROPOSITION 8. *For any $(\boldsymbol{\theta}, \boldsymbol{R})$ in the inefficient part of F_ℓ, there exists $\epsilon > 0$ such that for any Σ satisfying Assumption 2 with a long-term supply of bonds no more than ϵ, Σ cannot Pareto improve the competitive equilibrium corresponding to $(\boldsymbol{\theta}, \boldsymbol{R})$. Moreover, for any inefficiently liquid equilibria in F_m and also for constrained inefficient equilibria corresponding to the unique point $(\boldsymbol{\theta}^*, \boldsymbol{R}^*) \in F_\ell$ where $r(\boldsymbol{\theta}^*, \boldsymbol{R}^*) = 0$, i.e., where the steady-state interest rate is zero, there exists a small enough sequence of government bonds Σ, which Pareto improves the competitive equilibrium allocation.*

As discussed in the previous subsection, public liquidity crowds out the liquid investment more than proportionally to keep the demand for funds and the interest rate unchanged in the mixed region. This crowding-out effect is also the reason why government bonds can make Pareto

improvement in the mixed region. By substituting one unit of investment in government bonds for one unit in the liquid type, entrepreneurs can pledge more than before, and the borrowing constraint becomes less binding. Entrepreneurs can use that extra amount of liquidity to invest in the productive type while the interest rate does not increase. This raises their consumption and results in a Pareto improvement.

6 Conclusion

This paper introduces a new type of constrained inefficiency in the allocation of credit across investments with different liquidities and returns. Constrained efficiency can be achieved by a regulation akin to a maximum liquid asset ratio in a perfectly competitive banking sector or via a debt tax. The nature of this inefficiency is unconventional in that Pareto improvement reduces the interest rate. Comparative statics reveal non-monotonic effects of technological and financial development, i.e., higher return and liquidity, on the interest rate, credit allocation, and long-term output and welfare. These results have important bearings on the measurement and benchmarking of financial development.

There are many potentially insightful extensions of this stylized model. Bubbles may arise in this model where liquidities are low. Compared to the benchmark economy, bubbles, similar to government debt, may have a different effect on the interest rate, credit, and investment in the mixed region. It will be interesting to see whether bubbly equilibria are efficient or if bubbles can Pareto improve inefficient equilibria in this model. Additionally, the welfare loss of inefficient equilibria can be magnified in an extension with growth externalities. In an extension with endogenous growth where productive type represents the more knowledge-intensive technology that entails knowledge spillovers, inefficient equilibria may feature a lower long-term growth rate relative to the optimum. Finally, while I discussed some of its implications for capital flows, an open-economy version of this model deserves more exploration in future research. Financial openness increases the supply elasticity of funds and makes the interest rate less responsive to the decisions of domestic entrepreneurs. These effects can make inefficiency a less likely outcome. The welfare effects of financial openness both in a Pareto sense and in the long run are other issues which can be studied in a similar vein.

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A Appendix: Implementation of Pareto Improving Reallocation

Suppose that there is free entry into the banking sector that lasts only for one period and is owned by the young. Free entry implies zero profits in equilibrium. Banks are funded by deposits from the young and lend to middle-aged entrepreneurs. In period t, banks are required to keep the share of the liquid-type investment on their balance sheet less than or equal to $\alpha^{sp}_{\ell t}$, i.e., the share chosen by the planner. Under such regulation, middle-aged entrepreneurs choose the maximum possible share $\alpha^{sp}_{\ell t}$, and the resulting allocation coincides with the social planner allocation. To see why, assume that this is the case for all periods up to $t-1$. Then the initial wealth at t is w^{sp}_{t-1}, which is no greater than w^{ce}_{t-1} (initial wealth in the competitive equilibrium) by Proposition 6. But 6 implies that middle-aged would choose a share $\alpha_{\ell t}$ not less than $\alpha^{ce}_{\ell t}$ at a lower level of initial wealth which itself is not less than $\alpha^{sp}_{\ell t}$. Hence the middle-aged choose the maximum possible share of $\alpha^{sp}_{\ell t}$ under regulation which implies that the wealth in the next period is w^{sp}_{t}. Since $w^{sp}_{0} = w^{ce}_{0}$, the same logic applies to the initial period. The claim is thus proven by induction.

B Appendix: Proofs

Proof of Lemma 1. In any equilibrium, the resource constraint binds, and so I can solve for the value of x_{2t} in the above and rewrite the problem as:

$$\begin{aligned} \max_{i_t, x_{1t} \geq 0} \quad & (R_1 - R_2)x_{1t} + (R_2 - (1 + r_t))\, i_t + R_2(1 + r_{t-1})e \\ s.t. \quad & (1 + r_t - \theta_2 R_2)i_t \leq (\theta_1 R_1 - \theta_2 R_2)x_{1t} + \theta_2 R_2(1 + r_{t-1})e\,, \\ & 0 \leq x_{1t} \leq (1 + r_{t-1})e + i_t\,. \end{aligned}$$

One can immediately see from the above that $1 + r_t > \theta_2 R_2$. Otherwise, it must be that $1 + r_t \leq \theta_2 R_2 < R_2$, in which case i_t can be raised unboundedly and there may not be any maximum to the objective function. One must also have $1 + r_t \leq R_1$. Otherwise, the optimal solution to the problem requires that $i_t = 0$. To see why rewrite the above with x_{2t} in the objective function. If $1 + r_t > R_1$, then by Assumption 1 both coefficients of x_{2t} and i_t are strictly negative, and so the best an entrepreneur can do is to set both to zero. This cannot be an equilibrium since market clearing in the capital market cannot be satisfied.

Now suppose that the borrowing constraint does not bind for some $t \geq 0$. Since the coefficient of x_{1t} in the objective function of the above problem, which is $R_1 - R_2$, is strictly positive by Assumption 1, x_{1t} must be at the highest possible value, which is $(1 + r_{t-1})e + i_t$. At this value

the objective function can be written as $(R_1 - (1 + r_t))i_t + D_{t-1}$ where D_{t-1} is determined at $t - 1$. Moreover, the borrowing constraint at this value of x_{1t} is:

$$(1 + r_t - \theta_1 R_1) i_t < \theta_1 R_1 (1 + r_{t-1}) e \,.$$

Since the constraint is not binding, one can raise i_t by an small amount $\epsilon > 0$ so that the constraint is still satisfied and the value of the objective function is increased by $(R_1 - (1 + r_t))\epsilon$. This contradiction shows that the borrowing constraint must always be binding. The rest of the lemma is straightforward by using the borrowing constraint to eliminate x_{1t}. ∎

Proof of Lemma 2. First, I show that these are the only steady-state equilibria for the three regions. Suppose that r_z^{ss} is a steady-state interest rate for $z \in \{\ell, m, i\}$. Consider n steady state of the liquid region. If $\frac{\theta_2 R_2}{1-\theta_2 R_2} < (1 + r_\ell^{ss})$, by 5 both of the upper and lower bounds on the next period interest rate will be strictly smaller than r_ℓ^{ss}. Nor can it be that $(1 + r_\ell^{ss}) < \frac{\theta_2 R_2}{1-\theta_2 R_2}$. In that case using 5, the upper bound for the interest rate in the next period $\theta_2 R_2 (2 + r_\ell^{ss})$, will be strictly bigger than $1 + r_\ell^{ss}$ but strictly less than $\frac{\theta_2 R_2}{1-\theta_2 R_2}$ and so strictly less than $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ (since $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$). Hence given 4 the next period interest rate will be the upper bound itself, which is a contradiction given that it is strictly bigger than $1 + r_\ell^{ss}$. Hence one must have $(1 + r_\ell^{ss}) = \frac{\theta_2 R_2}{1-\theta_2 R_2}$. In a similar fashion, I can show that if there exists an steady state for F_i, it must be $(1 + r_i^{ss}) = \frac{\theta_1 R_1}{1-\theta_1 R_1}$. Finally, suppose that $(1 + r_m^{ss}) < (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$ in the mixed region. Then using 4 and the fact that the economy is at the steady state, one must have $(1 + r_m^{ss}) = \frac{\theta_2 R_2}{1-\theta_2 R_2}$ which gives $\frac{\theta_2 R_2}{1-\theta_2 R_2} < (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$. This is a contradiction given that the economy is in F_m. Similarly one cannot have $(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) < (1 + r_m^{ss})$, and so $(1 + r_m^{ss}) = (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$.

It only remains to check that these steady states exist. As I showed above, the trajectories of the interest rates are consistent with the equilibrium conditions 4 and 5 given an initial interest rate $1 + r_{-1}$ equal to the steady state. The values of i_t are exogenously given and equal to e, and the values of x_{1t} and x_{2t} can be derived from 6. The only condition that remains is that $1 + r_t < R_1$ for all $t \geq 0$. To see this, note that under Assumption 1:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \min(R_1, R_2) \,.$$

Hence, the remaining condition is satisfied for the liquid and mixed regions. The condition is also satisfied in the illiquid region since I assumed that $\frac{\theta_1 R_1}{1-\theta_1 R_1} < R_1$ in Definition 3. Local stability of the steady states in F_ℓ and F_i follows from the fact that $\theta_1 R_1 < \theta_2 R_2 < 1$ by Assumption 1. If $1 + r_z^{ss} \neq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ where $z \in \{\ell, i\}$, suppose without loss of generality that $1 + r_z^{ss} - \epsilon < 1 + r_t < 1 + r_z^{ss}$. For small enough $\epsilon > 0$, the whole interval $[1 + r_z^{ss} - \epsilon, 1 + r_z^{ss}]$ is either strictly below or

above $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. In either case, 4 and 5 imply $1 + r_t < 1 + r_{t+1} = \theta_z R_z(2 + r_t) < 1 + r_z^{ss}$, and hence by Assumption 1, the interest rates starting from a point in the interval $[1 + r_z^{ss} - \epsilon, 1 + r_z^{ss}]$ converge to the steady state value. For the case in which $1 + r_z^{ss} = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ where $z \in \{\ell, i\}$ or the mixed region, F_m, where the steady state interest rate is $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, suppose without loss of generality that $1 + r_{t-1} < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ (the proof for the case $1 + r_{t-1} > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ is very similar). If $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \in [\theta_1 R_1(2 + r_{t-1}), \theta_2 R_2(2 + r_{t-1})]$, then 4 gives $1 + r_t = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Otherwise, suppose that $\theta_2 R_2(2 + r_{t-1}) < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Then 5 implies $1 + r_{t-1} < 1 + r_t = \theta_2 R_2(2 + r_{t-1}) < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Hence, $1 + r_{t+k}, k = 1, 2, 3, \ldots$ converges to $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. The proof is very similar when $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \theta_1 R_1(2 + r_{t-1})$, and so this completes the proof. ∎

Proof of Lemma 3. For the steady-state interest rate observe that:

$$\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_1} = \frac{(1 - \theta_2) R_1 R_2 (R_2 - R_1)}{\Big((1 - \theta_1) R_1 - (1 - \theta_2) R_2\Big)^2} < 0\,, \tag{11}$$

$$\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} = \frac{(1 - \theta_1) R_1 R_2 (R_1 - R_2)}{\Big((1 - \theta_1) R_1 - (1 - \theta_2) R_2\Big)^2} > 0\,. \tag{12}$$

Let $s_j(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{x_j^{ss}}{x_1^{ss} + x_2^{ss}}$ be the share of type $j \in \{1, 2\}$ in total investment at the steady state. By 6:

$$s_1(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{\theta_2 R_2 - (1 - \theta_2 R_2)(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}{(\theta_2 R_2 - \theta_1 R_1)(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}\,,$$

$$s_2(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{(1 - \theta_1 R_1)(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) - \theta_1 R_1}{(\theta_2 R_2 - \theta_1 R_1)(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}\,.$$

Now one can rewrite $s_1(\boldsymbol{\theta}, \boldsymbol{R})$ as:

$$s_1(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{\frac{1}{2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})} - (1 - \theta_2 R_2)}{\theta_2 R_2 - \theta_1 R_1}\,.$$

The numerator of the above is strictly increasing in θ_1 by Proposition 3, and the denominator is strictly decreasing in θ_1. This implies that $s_1(\boldsymbol{\theta}, \boldsymbol{R})$ is strictly increasing in θ_1 when $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ and hence monotone in θ_1 in all three regions. For $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ one has:

$$\frac{\partial s_2(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} = \frac{(\theta_2 R_2 - \theta_1 R_1) \frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} - \Big((1 - \theta_1 R_1)(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) - \theta_1 R_1\Big) R_2 (2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}{\Big((\theta_2 R_2 - \theta_1 R_1)(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))\Big)^2}\,.$$

Arranging terms in the numerator, the above can be written as:

$$\frac{\partial s_2(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} = \frac{\Big(a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)\Big)R_2}{\Big((\theta_2 R_2 - \theta_1 R_1)(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))((1-\theta_1)R_1 - (1-\theta_2)R_2)\Big)^2},$$

where $a(\theta_1)$, $b(\theta_1)$ and $c(\theta_1)$ are:

$$\begin{aligned}
a(\theta_1) =& -R_1 R_2^2(1+R_1)(1-(1+R_1)\theta_1)\,,\\
b(\theta_1) =& R_1 R_2^2(1+R_1) + R_1 R_2\Big((R_1 - R_2)(1+2R_1) - R_1(R_1 R_2 - 1)\Big)\theta_1\\
&- R_1^2 R_2(1+R_1)(2+R_2)\theta_1^2\,,\\
c(\theta_1) =& R_1 R_2(R_1 - R_2) - R_1 R_2(R_1(2+R_1) - R_2)\theta_1 + R_1^2(1+R_2)(R_1 R_2 - R_1 + R_2)\theta_1^2\\
&+ R_1^3(1+R_1)\theta_1^3\,.
\end{aligned}$$

To show that $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ has at most one (interior) maximum, it is enough to show that given any θ_1, $a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)$ has at most one root as a quadratic polynomial of θ_2 inside F. By Proposition 2, $\theta_1 \leq \frac{1}{1+R_1}$, and therefore $a(\theta_1) \leq 0$. In the next step, I show that $c(\theta_1) > 0$ for all θ_1 by proving that $\widetilde{c}(\theta_1) = R_1^{-}1\Big(c(\theta_1) - R_1^3(1+R_1)\theta_1^3\Big) > 0$ inside F. If $\widetilde{c}(\theta_1)$ has no roots, then $\widetilde{c}(\theta_1) > 0$ since $\widetilde{c}(0) > 0$. Therefore, suppose θ_1^* is the smallest root of $\widetilde{c}(\theta_1) = 0$:

$$\begin{aligned}
\theta_1^* =& \frac{R_2(R_1(2+R_1) - R_2) - \sqrt{\Delta}}{2R_2(R_1 - R_2)}\,,\\
\Delta =& (R_2(R_1(2+R_1) - R_2))^2 - 4R_1 R_2(R_1 - R_2)(1+R_2)(R_1 R_2 - R_1 + R_2)\,.
\end{aligned}$$

Now one has:

$$\begin{aligned}
&\theta_1^* = \frac{R_2(R_1(2+R_1) - R_2) - \sqrt{\Delta}}{2R_2(R_1 - R_2)} > \frac{1}{1+R_1} \quad \Leftrightarrow\\
&(1+R_1)^2\Big(R_2(R_1(2+R_1) - R_2))^2 - 4R_1 R_2(R_1 - R_2)(1+R_2)(R_1 R_2 - R_1 + R_2)\Big) <\\
&\Big(R_2(1+R_1)(R_1(2+R_1) - R_2)) - 2R_2(R_1 - R_2)\Big)^2 \quad \Leftrightarrow\\
&(1+R_1)\Big(R_1(1+R_1)(1+R_2)(R_1(R_2 - 1) + R_2) - R_2(R_1(2+R_1) - R_2)\Big) > -R_2(R_1 - R_2)\,.
\end{aligned}$$

The last inequality holds since:

$$\begin{aligned}
&R_1(1+R_1)(1+R_2)(R_1(R_2 - 1) + R_2) - R_2(R_1(2+R_1) - R_2) >\\
&R_1 R_2(1+R_1)(1+R_2) - R_1 R_2(2+R_1) > R_1^2 R_2 > 0 > -R_2(R_1 - R_2)\,.
\end{aligned}$$

Hence $\theta_1^* > \frac{1}{1+R_1}$, and since $\theta_1 \leq \frac{1}{1+R_1}$ in F, one must have $c(\theta_1) > 0$ in F. Now since $a(\theta_1) \leq 0$ and $c(\theta_1) > 0$ in F, at least one root of $a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)$ for any given θ_1 has to be non-positive. Therefore, $a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)$ has at most one root in F for any θ_1, and consequently $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ has at most one (interior) maximum. Note that when $\boldsymbol{\theta}$ is on the boundary of F_m and F_i, $s_2(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ and hence $\frac{\partial s_2(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_2} > 0$ given any θ_1. Now suppose $\widetilde{\theta_1}$ is the value for which the vertical line $\theta_1 = \widetilde{\theta_1}$ is tangent to the boundary of F_ℓ. Observe that when θ_2 increases along $\theta_1 = \widetilde{\theta_1}$ line, $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ reaches the maximum of one at the point of tangency. Therefore, beyond the point of tangency $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ must be strictly decreasing in θ_2. This implies that for the particular value of $\theta_1 = \widetilde{\theta_1}$, there is a unique maximum for $s_2(\boldsymbol{\theta}, \boldsymbol{R})$. Hence, by continuity there must be a unique maximum for $s_2(\boldsymbol{\theta}, \boldsymbol{R})$ over the range of θ_2 given any θ_1 in a neighborhood of $\widetilde{\theta_1}$, which completes the proof. ∎

Proof of Proposition 1. First I show that for all $t \geq 0$, $1 + r_t < R_1$. Suppose $1 + r_{t-1} < R_1$ for some $t \geq 0$. Consider the window defined by 5, where $1 + r_t \in [\theta_1 R_1(2 + r_{t-1}), \theta_2 R_2(2 + r_{t-1})]$. If $\theta_2 R_2(2 + r_{t-1}) \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, 4 implies $1 + r_t = \theta_2 R_2(2 + r_{t-1}) \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < R_1$. The last inequality holds by Assumption 1. If $\theta_1 R_1(2 + r_{t-1}) < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \theta_2 R_2(2 + r_{t-1})$, by 4 I get $1 + r_t = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < R_1$. Finally, consider the case $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \theta_1 R_1(2 + r_{t-1})$. The equilibrium path of interest rate, given by 4, implies that $1 + r_t = \theta_1 R_1(2 + r_{t-1}) \leq \max(1 + r_{t-1}, \frac{\theta_1 R_1}{1-\theta_1 R_1}) < R_1$. The first inequality holds because the value of $\theta_1 R_1(2 + r_{t-1})$ is always between $1 + r_{t-1}$ and $\frac{\theta_1 R_1}{1-\theta_1 R_1}$. The second inequality is obtained by the assumption that $1 + r_{t-1} < R_1$ and definition of F. By induction, $1 + r_{-1} < R_1$ implies $1 + r_t < R_1$ for all $t \geq 0$. This proves the necessary condition for the interest rates in the competitive equilibrium.

In the second step, I prove the existence and uniqueness. I show that given $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and the initial condition $1 + r_{-1}$, a unique path of interest rates is defined by 4 and 5. Note that given the path of interest rates, I can simply solve for (x_{1t}, x_{2t}, i_t) for all $t \geq 0$ using 3 and 6 in each period. Suppose I have determined the unique interest rate $1 + r_{t-1}$ for $t - 1$. Consider the window, defined by 5, where $1 + r_t \in [\theta_1 R_1(2 + r_{t-1}), \theta_2 R_2(2 + r_{t-1})]$. If $\theta_2 R_2(2 + r_{t-1}) \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ or $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq \theta_1 R_1(2 + r_{t-1})$, using 4 gives $1 + r_t = \theta_2 R_2(2 + r_{t-1})$ and $1 + r_t = \theta_1 R_1(2 + r_{t-1})$ respectively. Finally, suppose $\theta_1 R_1(2 + r_{t-1}) < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \theta_2 R_2(2 + r_{t-1})$. Then, if $1 + r_t > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, by 4 one must have $1 + r_t = \theta_1 R_1(2 + r_{t-1}) < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Similarly, if $1 + r_t < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, by 4 one must have $1 + r_t = \theta_2 R_2(2 + r_{t-1}) > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. The two contradictions show that one must have $1 + r_t = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Hence, I have shown that given $1 + r_{t-1}$, there is a uniquely determined interest rate at time t, that is, $1 + r_t$. Therefore, by induction, I have shown that given an initial condition $1 + r_{-1}$, there is a unique path of interest rates for all $t \geq 0$.

In the third and final step, I show that the unique equilibrium path of the interest rates defined in step two converges to the unique steady state characterized in Lemma 2, for given $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and an initial condition $1 + r_{-1}$. Consider the case $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$ first. Note that if $1 + r_{t-1} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$,

using 4 and 5 implies $1 + r_t = \theta_2 R_2(2 + r_{t-1}) \leq \max(1 + r_{t-1}, \frac{\theta_2 R_2}{1-\theta_2 R_2}) \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Hence, if $1 + r_{-1} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, the path of interest rates is defined as $1 + r_t = \theta_2 R_2(2 + r_{t-1})$ for all $t \geq 0$. This path is clearly convergent to $1 + r_\ell^{ss} = \frac{\theta_2 R_2}{1-\theta_2 R_2}$. Now suppose $1 + r_{-1} > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, which implies $1 + r_{-1} > \frac{\theta_2 R_2}{1-\theta_2 R_2}$. Define the series $\{1 + \bar{r}_t\}_{t=-1}^{\infty}$ as $1 + \bar{r}_t = \theta_2 R_2(2 + \bar{r}_{t-1})$ for all $t \geq 0$ and $1 + \bar{r}_{-1} = 1 + r_{-1}$. If $1 + r_{t-1} \leq 1 + \bar{r}_{t-1}$, 5 implies $1 + r_t \leq \theta_2 R_2(2 + r_{t-1}) \leq \theta_2 R_2(2 + \bar{r}_{t-1}) = 1 + \bar{r}_t$. Hence by induction one must have $1 + r_t \leq 1 + \bar{r}_t$ for all $t \geq 0$. Since by Assumption 1 $\theta_2 R_2 < 1$, this immediately implies that there is a finite t_0 for which $1 + r_{t_0} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Therefore, this case is similar to the previous part of the proof and so convergence is established.

Now consider the case $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ where Definition 3 implies that $\frac{\theta_1 R_1}{1-\theta_1 R_1} < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \frac{\theta_2 R_2}{1-\theta_2 R_2}$. Without loss of generality, suppose $1 + r_{-1} > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Define the series $\{1 + \underline{r}_t\}_{t=-1}^{\infty}$ as $1 + \underline{r}_t = \theta_1 R_1(2 + \underline{r}_{t-1})$ for all $t \geq 0$ and $1 + \underline{r}_{-1} = 1 + r_{-1}$. It is easy to see that there is a finite and unique $t_0 \geq 0$ such that $1 + \underline{r}_{t_0} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 1 + \underline{r}_{t_0-1}$. Now note that if $1 + r_{t-1} > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ for some $t \geq 0$, one must have $\theta_2 R_2(2 + r_{t-1}) \geq \min(1 + r_{t-1}, \frac{\theta_2 R_2}{1-\theta_2 R_2}) > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, and therefore 4 and 5 give $1 + r_t = \max(\theta_1 R_1(2 + r_{t-1}), 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))$. Using this observation and by induction, for $-1 \leq t \leq t_0 - 1$ one must have $1 + r_t = 1 + \underline{r}_t > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, and so $\theta_2 R_2(2 + r_t) \geq \min(1 + r_{t-1}, \frac{\theta_2 R_2}{1-\theta_2 R_2}) > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Using 4 and the definition of t_0, this implies that $1 + r_{t_0} = \max(\theta_1 R_1(2 + r_{t_0-1}), 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Therefore, the path of interest rates converges to the steady state-interest rate, $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, in finite periods. The proof for the case $1 + r_{-1} < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ is very similar. Finally, if $1 + r_{-1} = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$, the economy is already in the steady state, and all future interest rates will be the same.

The proof for the illiquid region is very similar to the case of liquid region, and so I do not provide it here. ■

Proof of Proposition 2. First, I compute the boundaries of the illiquid and liquid regions as functions of θ_1. For the illiquid region the defining boundary is characterized by:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{\theta_1 R_1}{1 - \theta_1 R_1}\right).$$

Using 1 and solving the above as a function of θ_1, I get:

$$\theta_2^i(\theta_1) = \left(\frac{\theta_1 R_1(1 - \theta_1(1 + R_2))}{R_2(1 - \theta_1(1 + R_1))}\right).$$

This function is strictly increasing and convex in θ_1 since $\theta_1 R_1$ is increasing and:

$$\frac{d}{d\theta_1}\left(\frac{(1 - \theta_1(1 + R_2))}{(1 - \theta_1(1 + R_1))}\right) = \frac{R_1 - R_2}{(1 - \theta_1(1 + R_1))^2} > 0.$$

Also observe that $\theta_2^i(0) = 0$, and so no matter how close to the origin, there are illiquid equilibria in any neighborhood of the $\boldsymbol{\theta} = \mathbf{0}$. Now the characterizing equation for the liquid region is:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{\theta_2 R_2}{1 - \theta_2 R_2} \right) .$$

Collecting terms involving θ_1 or θ_2 on different sides, I obtain two distinct curves:

$$\overline{\theta}_2^\ell(\theta_1) = \left(\frac{(\theta_1 R_1(1+R_2) + R_2) + \sqrt{(\theta_1 R_1(1+R_2) + R_2)^2 - 4\theta_1 R_1 R_2(1+R_1)}}{2R_2(1+R_1)} \right) ,$$
$$\underline{\theta}_2^\ell(\theta_1) = \left(\frac{(\theta_1 R_1(1+R_2) + R_2) - \sqrt{(\theta_1 R_1(1+R_2) + R_2)^2 - 4\theta_1 R_1 R_2(1+R_1)}}{2R_2(1+R_1)} \right) .$$

Note that obviously $\underline{\theta}_2^\ell(\theta_1) \leq \overline{\theta}_2^\ell(\theta_1)$ and $\underline{\theta}_2^\ell(0) = 0$, and so the lower boundary characterizing the liquid region passes through the origin. This means that there are liquid steady-state equilibria at any neighborhood of the origin.

Now let $\Delta(\theta_1) \equiv (\theta_1 R_1(1+R_2) + R_2)^2 - 4\theta_1 R_1 R_2(1+R_1)$. Then the two curves $\underline{\theta}_2^\ell(\theta_1)$ and $\overline{\theta}_2^\ell(\theta_1)$ touch each other when $\Delta(\theta_1) = 0$. This equation has two roots:

$$\overline{\theta}_1 = \left(\frac{R_2 \left((2(1+R_1) - (1+R_2)) + \sqrt{4(1+R_1)(R_1 - R_2)} \right)}{R_1(1+R_2)^2} \right) ,$$
$$\underline{\theta}_1 = \left(\frac{R_2 \left((2(1+R_1) - (1+R_2)) - \sqrt{4(1+R_1)(R_1 - R_2)} \right)}{R_1(1+R_2)^2} \right) .$$

The smaller root is less than $\frac{1}{1+R_1}$ since:

$$\begin{aligned} \underline{\theta}_1 < \frac{1}{1+R_1} &\Leftrightarrow (1+R_1)R_2 \left((2(1+R_1) - (1+R_2)) - \sqrt{4(1+R_1)(R_1 - R_2)} \right) \\ &< R_1(1+R_2)^2 \Leftrightarrow (1+R_1)R_2(2(1+R_1) - (1+R_2)) - R_1(1+R_2)^2 \\ &< (1+R_1)R_2\sqrt{4(1+R_1)(R_1 - R_2)} \Leftrightarrow (R_1 - R_2)(2R_1R_2 + R_2 - 1) < \\ &\qquad (1+R_1)R_2\sqrt{4(1+R_1)(R_1 - R_2)} . \end{aligned}$$

If I square both sides, cancel $R_1 - R_2$, and collect the terms, I get:

$$\Leftrightarrow R_1 < 4R_1^2R_2^3 + 8R_1^2R_2^2 + 4R_1R_2^3 + 7R_1R_2^2 + R_2^3 + 4R_1^2R_2 + 2R_2^2 + 2R_1R_2 + R_2 .$$

This is obviously the case given Assumption 1. The bigger root is greater than $\frac{1}{1+R_1}$ since:

$$\begin{aligned}\overline{\theta}_1 > \frac{1}{1+R_1} &\Leftrightarrow (1+R_1)R_2\left((2(1+R_1)-(1+R_2))+\sqrt{4(1+R_1)(R_1-R_2)}\right)\\ &> R_1(1+R_2)^2 \Leftrightarrow (1+R_1)R_2(2(1+R_1)-(1+R_2))-R_1(1+R_2)^2\\ &> -(1+R_1)R_2\sqrt{4(1+R_1)(R_1-R_2)} \Leftrightarrow (R_1-R_2)(2R_1R_2+R_2-1) >\\ &\qquad -(1+R_1)R_2\sqrt{4(1+R_1)(R_1-R_2)}\,.\end{aligned}$$

The last inequality is obvious given that one term is positive and the other is negative. Therefore the point at which the two curves $\overline{\theta}_2^{\ell}(\theta_1)$ and $\underline{\theta}_2^{\ell}(\theta_1)$ touch each other inside F is $\underline{\theta}_1$. The fact that $\underline{\theta}_1 < \frac{1}{1+R_1}$ proves that for high θ_1 there is no liquid steady state.

Next, I prove that $\overline{\theta}_2^{\ell}(\theta_1)$ is strictly decreasing and $\underline{\theta}_2^{\ell}(\theta_1)$ is strictly increasing. The derivatives are:

$$\begin{aligned}\frac{d\overline{\theta}_2^{\ell}(\theta_1)}{d\theta_1} &= C_0\left(R_1(1+R_2)+(R_1(1+R_2)(\theta_1R_1(1+R_2)+R_2)-2R_1R_2(1+R_1))\,\Delta(\theta_1)^{-\frac{1}{2}}\right),\\ \frac{d\underline{\theta}_2^{\ell}(\theta_1)}{d\theta_1} &= C_0\left(R_1(1+R_2)-(R_1(1+R_2)(\theta_1R_1(1+R_2)+R_2)-2R_1R_2(1+R_1))\,\Delta(\theta_1)^{-\frac{1}{2}}\right).\end{aligned}$$

C_0 is just a constant. It is easy to see that the term in parentheses just before $\Delta(\theta_1)^{-\frac{1}{2}}$ is always negative for $\theta_1 \le \underline{\theta}_1$. Hence, $\frac{d\underline{\theta}_2^{\ell}(\theta_1)}{d\theta_1}$ should be strictly positive. Now for the other case:

$$\begin{aligned}&\frac{d\overline{\theta}_2^{\ell}(\theta_1)}{d\theta_1} < 0\\ \Leftrightarrow\ & R_1^2(1+R_2)^2\Delta(\theta_1) < \left(R_1^2(1+R_2)^2\theta_1 - R_1R_2(2(1+R_1)-(1+R_2))\right)^2\\ \Leftrightarrow\ & (1+R_2)^2\Delta(\theta_1) < \left(R_1(1+R_2)^2\theta_1 - R_2(2(1+R_1)-(1+R_2))\right)^2\\ \Leftrightarrow\ & (1+R_2) < 2(1+R_1)-(1+R_2)\,.\end{aligned}$$

The last statement is correct given Assumption 1. In the last step I have used the definition of $\Delta(\theta_1)$ to cancel out all terms. What I proved shows that for any $\boldsymbol{\theta} \in F_\ell$ one must have $\boldsymbol{\theta} \le (\frac{1}{1+R_1}, \frac{1}{1+R_1})$. This is because I showed that $\underline{\theta}_1 < \frac{1}{1+R_1}$ and that $\frac{d\overline{\theta}_2^{\ell}(\theta_1)}{d\theta_1}$ is strictly decreasing while $\overline{\theta}_2^{\ell}(\theta_1)$ stays above $\underline{\theta}_2^{\ell}(\theta_1)$ and intersects with $\theta_1 = 0$ at $\frac{1}{1+R_1}$.

In the next step I want to prove that the liquid region lies above the illiquid region. First, I observe the following:

$$\frac{\partial r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})}{\partial\theta_2} = \frac{(1-\theta_1)(R_1-R_2)R_1R_2}{((1-\theta_1)R_1-(1-\theta_2)R_2)^2} > 0\,.$$

Now suppose that $r_\Lambda(\theta_1, \theta_2, \boldsymbol{R}) \geq \frac{\theta_2 R_2}{1-\theta_2 R_2}$ and $r_\Lambda(\theta_1, \theta_2', \boldsymbol{R}) \leq \frac{\theta_1 R_1}{1-\theta_1 R_1}$ where $(\theta_1, \theta_2, \boldsymbol{R})$ and $(\theta_1, \theta_2', \boldsymbol{R})$ are in F. Then, if $\theta_2 \leq \theta_2'$, by the derivation above $r_\Lambda(\theta_1, \theta_2, \boldsymbol{R}) \leq r_\Lambda(\theta_1, \theta_2', \boldsymbol{R})$ and hence:

$$\frac{\theta_2 R_2}{1-\theta_2 R_2} \leq r_\Lambda(\theta_1, \theta_2, \boldsymbol{R}) \leq r_\Lambda(\theta_1, \theta_2', \boldsymbol{R}) \leq \frac{\theta_1 R_1}{1-\theta_1 R_1}\,.$$

This is not possible since it implies that $\theta_2 R_2 \leq \theta_1 R_1$ and hence $(\theta_1, \theta_2, \boldsymbol{R})$ cannot be in F. In the last step of the proof, I show that $(\frac{1}{R_2}, \frac{1}{1+R_1}) \in F_i$. First, note that:

$$\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_1} = \frac{(1-\theta_2)(R_2-R_1)R_1R_2}{((1-\theta_1)R_1-(1-\theta_2)R_2)^2} < 0\,.$$

Second, observe that $\theta_2^i(\theta_1)$ is strictly increasing, passes through the origin, and also converges to infinity when θ_1 gets close to $\frac{1}{1+R_1}$. This means that $\theta_2^i(\theta_1)$ cuts the horizontal border of F that is $\theta_2 = \frac{1}{R_2}$ at an interior point, say, $(\bar{\theta}_1, \frac{1}{R_2})$ where $\bar{\theta}_1 < \frac{1}{1+R_1}$. At this point $\theta_2^i(\bar{\theta}_1) = \frac{\bar{\theta}_1 R_1}{1-\bar{\theta}_1 R_1}$. But since I have proven above that $\frac{\partial r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})}{\partial \theta_1} < 0$, for any $\theta_1 \in (\bar{\theta}_1, \frac{1}{1+R_1})$ I obtain:

$$\theta_2^i(\theta_1) < \theta_2^i(\bar{\theta}_1) = \frac{\bar{\theta}_1 R_1}{1-\bar{\theta}_1 R_1} < \frac{\theta_1 R_1}{1-\theta_1 R_1}\,.$$

This means that $\theta_1 \in F_i$ for $\theta_1 \in (\bar{\theta}_1, \frac{1}{1+R_1})$.

■

Proof of Proposition 3. When $R_1 > R_2 > 1$ and $1 > \theta_1 R_1 > \theta_2 R_2$, entrepreneurs only invest in type 1 since type 2 is dominated in terms of both liquidity and return. Hence, this economy collapses to the economy in Farhi and Tirole (2012) with only one investment type, (θ_1, R_1), and no bubbles or outside liquidity. Farhi and Tirole (2012) show in their Proposition 5 that under the assumption that $R_1 > 1$, all competitive equilibria are Pareto efficient and hence constrained Pareto efficient as well. ■

Proof of Proposition 4. I proceed in two steps. First I prove some comparative statics regarding steady-state utility levels, and then I complete the proof by considering the transition dynamics.

Let V_ℓ^{ss} and V_i^{ss} be the steady-state utility levels in the liquid and illiquid regions. For any values of $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$, the following statements are correct. $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ and $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ have the same sign. $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_\ell$. $V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is negative if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_i$.

The proofs of the above statements are as follows. First I show that the steady-state level of

utility for any values of α_ℓ for which $\left(\frac{\gamma_\alpha}{1-\gamma_\alpha}\right) < R_\alpha$ is given by:

$$V_\alpha^{ss} = \left(\frac{R_\alpha - \gamma_\alpha}{1-\gamma_\alpha}\right) e = \left(\frac{(1-\alpha_\ell)(1-\theta_1)R_1 + \alpha_\ell(1-\theta_2)R_2}{(1-\alpha_\ell)(1-\theta_1 R_1) + \alpha_\ell(1-\theta_2 R_2)}\right) e\,.$$

Additionally, the steady state utility levels for the three regions in Definition 3 are $V_\ell^{ss} = \left(\frac{(1-\theta_2)R_2}{1-\theta_2 R_2}\right) e$, $V_i^{ss} = \left(\frac{(1-\theta_1)R_1}{1-\theta_1 R_1}\right) e$, and:

$$V_m^{ss} = \left(\frac{(\theta_2-\theta_1)^2 R_1^2 R_2^2}{(\theta_2 R_2 - \theta_1 R_1)((1-\theta_1)R_1 - (1-\theta_2)R_2)}\right) e\,.$$

Moreover suppose $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$ and that $\widetilde{\alpha}_\ell = \left(\frac{x_2^{ss}(\boldsymbol{\theta},\boldsymbol{R})}{x_1^{ss}(\boldsymbol{\theta},\boldsymbol{R}) + x_2^{ss}(\boldsymbol{\theta},\boldsymbol{R})}\right)$. Then in the regulated economy corresponding to $\widetilde{\alpha}_\ell$ one has:

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 1 + r_{\widetilde{\alpha}_\ell}^{ss} = \left(\frac{(1-\widetilde{\alpha}_\ell)\theta_1 R_1 + \widetilde{\alpha}_\ell \theta_2 R_2}{(1-\widetilde{\alpha}_\ell)(1-\theta_1 R_1) + \widetilde{\alpha}_\ell(1-\theta_2 R_2)}\right),$$

$$V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) = V_{\widetilde{\alpha}_\ell}^{ss} = \left(\frac{(1-\widetilde{\alpha}_\ell)(1-\theta_1)R_1 + \widetilde{\alpha}_\ell(1-\theta_2)R_2}{(1-\widetilde{\alpha}_\ell)(1-\theta_1 R_1) + \widetilde{\alpha}_\ell(1-\theta_2 R_2)}\right) e\,.$$

To see the above, let $R_{\alpha t} = (1-\alpha_{\ell t})R_1 + \alpha_{\ell t} R_2$ and $\gamma_{\alpha t} = \theta_1(1-\alpha_{\ell t})R_1 + \theta_2 \alpha_{\ell t} R_2$ be the return and liquidity of the regulated portfolio at time t. Problem IV is the maximization problem of an entrepreneur that has access only to one type of investment project with a return of $R_{\alpha t}$ and liquidity of $\gamma_{\alpha t}$. The optimal solution to IV is:

$$\begin{cases} i_t = \left(\frac{(1+r_{t-1})\gamma_{\alpha t}}{(1+r_t)-\gamma_{\alpha t}}\right) e & \text{if } R_{\alpha t} \geq 1 + r_t\,, \\ i_t = 0 & \text{if } R_{\alpha t} < 1 + r_t\,. \end{cases}$$

Note that $\gamma_{\alpha t} < 1$ by Assumption 1. Also note that for any $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$, there exists an $\epsilon > 0$ such that $\left(\frac{\gamma_{\alpha t}}{1-\gamma_{\alpha t}}\right) < R_{\alpha t}$ for all $\alpha_{\ell t} \in [0, \epsilon)$. This is because the inequality holds for $\alpha_{\ell t} = 0$ according to the definition of F, and so by continuity it holds in a neighborhood of zero. If $\{\alpha_{\ell t}\}_{t=0}^{\infty}$ are all set to α_ℓ and this value is in the neighborhood above, the steady-state equilibrium of the regulated economy is

$$1 + r_\alpha^{ss} = \left(\frac{\gamma_\alpha}{1-\gamma_\alpha}\right),$$

where variables without time subscript correspond to α_ℓ. Using the values of steady-state interest rates in Lemma 2, market clearings, and the objective function in II, deriving V_ℓ^{ss}, V_i^{ss} and V_m^{ss} is straightforward. For the regulated economy, recall that by IV and the above equation, the

objective function when the social planner sets $\alpha_\ell = \alpha$ will be:

$$\begin{aligned} V_\alpha^{ss} &= (((1-\alpha)R_1 + \alpha R_2)(2 + r_\alpha^{ss}) - (1 + r_\alpha^{ss}))e \\ &= (R_\alpha(1 + \frac{\gamma_\alpha}{1-\gamma_\alpha}) - \frac{\gamma_\alpha}{1-\gamma_\alpha})e \\ &= \left(\frac{R_\alpha - \gamma_\alpha}{1-\gamma_\alpha}\right) e\,. \end{aligned}$$

Note that the numerator and denominator of V_α^{ss} are weighted averages of those of V_ℓ^{ss} and V_i^{ss}. This implies that V_α^{ss} always lies between the two values of V_ℓ^{ss} and V_i^{ss}. For last part of the proposition, observe that 6 gives:

$$\widetilde{\alpha}_\ell = \frac{(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})) - \theta_1 R_1(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}{(\theta_2 R_2 - \theta_1 R_1)(2 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}\,.$$

The interest rate is:

$$\begin{aligned} &1 + r_{\widetilde{\alpha}_\ell}^{ss} = \frac{\gamma_{\widetilde{\alpha}_\ell}}{1 - \gamma_{\widetilde{\alpha}_\ell}} = \\ &\frac{((1 + r_\Lambda) - \theta_1 R_1(2 + r_\Lambda))\theta_2 R_2 + (\theta_2 R_2(2 + r_\Lambda) - (1 + r_\Lambda))\theta_1 R_1}{((1 + r_\Lambda) - \theta_1 R_1(2 + r_\Lambda))(1 - \theta_2 R_2) + (\theta_2 R_2(2 + r_\Lambda) - (1 + r_\Lambda))(1 - \theta_1 R_1)} = \\ &\frac{(\theta_2 R_2 - \theta_1 R_1)(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))}{(\theta_2 R_2 - \theta_1 R_1)} = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\,. \end{aligned}$$

Therefore the utility levels at the steady state should be the same. Note that $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \min(R_1, R_2) \leq R_{\widetilde{\alpha}_\ell}$ by Assumption 1, and so the proof is complete.

In the second step, I prove the lemma. Consider $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$. Note that:

$$\begin{aligned} &V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) > V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) \Leftrightarrow \frac{(1-\theta_2)R_2}{1-\theta_2 R_2} > \frac{(1-\theta_1)R_1}{1-\theta_1 R_1} \Leftrightarrow \\ &\frac{R_2 - 1}{1-\theta_2 R_2} > \frac{R_1 - 1}{1-\theta_1 R_1} \Leftrightarrow (R_2 - 1)(1-\theta_1 R_1) > (R_1 - 1)(1-\theta_2 R_2) \Leftrightarrow \\ &(R_2 - \theta_1 R_1 R_2 + \theta_1 R_1) > (R_1 - \theta_2 R_1 R_2 + \theta_2 R_2) \Leftrightarrow \\ &(\theta_2 - \theta_1) R_1 R_2 > (1-\theta_1)R1 - (1-\theta_2)R_2 \Leftrightarrow \\ &1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = \left(\frac{(\theta_2 - \theta_1)R_1 R_2}{(1-\theta_1)R1 - (1-\theta_2)R_2}\right) > 1\,. \end{aligned}$$

Now define the following terms:

$$\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \left(\frac{(\theta_2 - \theta_1)R_1R_2 - ((1-\theta_1)R1 - (1-\theta_2)R_2)}{(\theta_2 R_2 - \theta_1 R_1)} \right) e \,,$$

$$\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \frac{(\theta_2 R_2((1-\theta_1)R1 - (1-\theta_2)R_2) - (1-\theta_2 R_2)(\theta_2 - \theta_1)R_1R_2}{(1-\theta_2 R_2)((1-\theta_1)R1 - (1-\theta_2)R_2)} \,.$$

Note that the denominators of $\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ and $\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ are strictly positive. Moreover, one can easily see that the numerator of $\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 1$ and that the numerator of $\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \frac{\theta_2 R_2}{1-\theta_2 R_2} = 1 + r_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ or equivalently $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_\ell$. $\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is the welfare gains per unit of reduction in x_1 of investing the freed resources in x_2, and $\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R})$ is the maximum amount of reduction in x_1 that can possibly occur (see Section 4.1). Given that

$$V_m^{ss} = \left(\frac{(\theta_2 - \theta_1)^2 R_1^2 R_2^2}{(\theta_2 R_2 - \theta_1 R_1)((1-\theta_1)R_1 - (1-\theta_2)R_2)} \right) e \,.$$

Now I want to compute and simplify $V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) + \Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{DEN}{NUM}$. The common denominator and the numerator are:

$$\begin{aligned}
DEN &= (1-\theta_2 R_2)((1-\theta_1)R1 - (1-\theta_2)R_2)(\theta_2 R_2 - \theta_1 R_1) \,, \\
NUM &= (1-\theta_2 R_2)(\theta_2 - \theta_1)R_1R_2 \Big((\theta_2 - \theta_1)R_1R_2 \\
&\quad - \Big((\theta_2 - \theta_1)R_1R_2 - ((1-\theta_1)R1 - (1-\theta_2)R_2) \Big) \Big) \\
&\quad + \theta_2 R_2 \left((1-\theta_1)R1 - (1-\theta_2)R_2\right) \Big((\theta_2 - \theta_1)R_1R_2 - ((1-\theta_1)R1 - (1-\theta_2)R_2) \Big) \,, \\
&= ((1-\theta_1)R1 - (1-\theta_2)R_2)) \Big((\theta_2 - \theta_1)R_1R_2 + \theta_2 R_2^2 - \theta_2 R_1 R_2 - \theta_2 R_2(\theta_2 R_2 - \theta_1 R_1) \Big) \,, \\
&= ((1-\theta_1)R1 - (1-\theta_2)R_2))(1-\theta_2)R_2(\theta_2 R_2 - \theta_1 R_1) \,.
\end{aligned}$$

Therefore:

$$\begin{aligned}
V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) + \Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R}) &= \frac{\theta_2 R_2}{1-\theta_2 R_2} = V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) \,, \Rightarrow \\
V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) &= \Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_\ell(\boldsymbol{\theta}, \boldsymbol{R}) \,.
\end{aligned}$$

By the last equation, it is obvious that the sign of $V_\ell^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if

$r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_\ell$. For the last case, define:

$$\Omega_i(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \left(\frac{((1-\theta_1)R1 - (1-\theta_2)R_2) - (\theta_2 - \theta_1)R_1R_2}{(\theta_2 R_2 - \theta_1 R_1)} \right) e \,,$$

$$\Gamma_i(\boldsymbol{\theta}, \boldsymbol{R}) \equiv \frac{(1-\theta_1 R_1)(\theta_2 - \theta_1)R_1R_2 - \theta_1 R_1((1-\theta_1)R1 - (1-\theta_2)R_2)}{(1-\theta_1 R_1)((1-\theta_1)R1 - (1-\theta_2)R_2)} \,.$$

Note that $\Omega_i(\boldsymbol{\theta}, \boldsymbol{R}) = -\Omega_\ell(\boldsymbol{\theta}, \boldsymbol{R})$. Similar simplifications lead to:

$$V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) = \Omega_i(\boldsymbol{\theta}, \boldsymbol{R})\Gamma_i(\boldsymbol{\theta}, \boldsymbol{R}) \,.$$

Hence $V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) - V_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is positive if and only if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \notin F_i$.

Now I complete the proof of the main theorem. By Proposition 2, the competitive equilibrium converges to a unique steady state corresponding to $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell \cup F_m$. This implies that there exist $T \geq 0$ and $\varepsilon > 0$ such that $x_{2t} \geq \varepsilon$ for $t \geq T$. Suppose one reduces x_{2t} for $t \geq T+1$ by $\delta + \epsilon$, increases x_{1t} for $t \geq T+1$ by ϵ, and reduces x_{2T} and increases x_{1T} both by $\frac{1}{\theta_2 R_2 - \theta_1 R_1}\delta$. Moreover, $\delta > 0$, $\epsilon > 0$ are such that $\epsilon + \delta < \varepsilon$ and:

$$\begin{aligned} \delta &= (\theta_2 R_2 - \theta_1 R_1)\epsilon + \theta_2 R_2 \delta \,, \\ \epsilon &= \frac{1 - \theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1}\delta \,. \end{aligned}$$

Similar to what is shown in the text, this reallocation reduces the debt payments of each generation from T onward by δ and leaves all middle-aged entrepreneurs at or after T strictly better off when $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$. If $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$, the reallocation does not affect the utility of the middle-aged after T but increases the utility of the middle-aged at T. This proves that the competitive equilibrium is constrained Pareto inefficient whenever $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$.

If $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_m$, then by definition $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ implies a strictly negative interest rate at the steady state. If $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$, by the first part of the proof on steady-state utility levels, $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ implies:

$$\frac{\theta_2 R_2}{1 - \theta_2 R_2} < \frac{\theta_1 R_1}{1 - \theta_1 R_1} \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 1 \,.$$

Hence by Lemma 2, the steady-state interest rate is strictly negative, which completes the proof. ■

Proof of Lemma 4. The straight line corresponding to $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ is $\theta_2^\Lambda(\theta_1) = \left(\frac{R_1(R_2-1)}{R_2(R_1-1)}\right)\theta_1 + \left(\frac{R_1-R_2}{R_2(R_1-1)}\right)$. This line intersects horizontal line $\theta_1 = 0$ at $\theta_2^\Lambda(0) = \frac{R_1-R_2}{R_2(R_1-1)}$, which implies $\theta_2^\Lambda(0) > 0$. Hence, Proposition 2 and Proposition 4 imply that a strictly positive neighborhood of the origin, i.e., $\boldsymbol{\theta} = 0$, corresponds to inefficiently liquid equilibria. Since, by Proposition 2, any neighborhood of the origin contains liquid equilibria, it follows by Proposition 4 that there are inefficiently liquid equilibria in any small enough neighborhood of the origin. Note that by Proposition 2, the boundary of F_ℓ cuts the vertical axis $\theta_1 = 0$ at the origin and $\boldsymbol{\theta} = (0, \frac{1}{1+R_1})$ and also that the upper part of the boundary is negatively sloped in the (θ_1, θ_2) plane. Therefore, it follows that the $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ line passes through F_ℓ if and only if its intersection with $\theta_1 = 0$, that is $(0, \theta_2^\Lambda(0)$, lies below or at $\boldsymbol{\theta} = (0, \frac{1}{1+R_1})$. This is the case whenever $\frac{R_1-R_2}{R_2-1} \leq 1$.

For the last part, let S_i denote the unique intersection of $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ with the boundary of F_i. Observe that by Proposition 2, the inefficiently liquid region lies above the convex inner boundary of F_i and below $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$. This completes the proof. ∎

Proof of Proposition 5. For part of this proof, I use some of the results in Ghate and Smith (2005), specially their Theorem 2.6. This theorem shows that complementary slackness conditions are sufficient for optimality in a linear programming with infinite variables and infinite number of constraints when feasible points, constraints, and objective functions of both primal and dual problems are elements of appropriate spaces. A necessary condition for this result is that the feasible points of the primal problem, i.e., feasible allocations $\{c_t, x_{1t}, x_{2t}\}_{t=0}^\infty$, lie in ℓ_∞. To see this, note that by 7 and Assumption 1:

$$x_{1t} + x_{2t} \leq \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} + e \leq \theta_1 R_1 (x_{1t-1} + x_{2t-1}) + e\,.$$

This together with 7 gives:

$$\begin{cases} x_{1t} + x_{2t} \leq i_{-1} + \frac{e}{1-\theta_1 R_1}\,, \\ \\ c_t \leq R_1 (i_{-1} + \frac{e}{1-\theta_1 R_1}) + e\,. \end{cases}$$

i_{-1} is the total investment at $t = -1$, which is an initial condition to the problem. The above proves that $\{c_t, x_{1t}, x_{2t}\}_{t=0}^\infty \in \ell_\infty$ for any feasible allocation.

Now let $(\boldsymbol{\theta}, \boldsymbol{R}) \in F$ and consider an allocation $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^\infty$ that satisfies 7 with equality for all $t \geq 0$. If there exists a series of strictly positive weights $\{\lambda_t\}_{t=0}^\infty \in \ell_1$ such that $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^\infty$ solves:

$$
\begin{aligned}
\max_{\{c_t, x_{1t}, x_{2t}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \lambda_t c_t \\
s.t. \quad & c_t + x_{1t} + x_{2t} \leq R_1 x_{1t-1} + R_2 x_{2t-1} + e \\
& x_{1t} + x_{2t} \leq \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} + e \\
& c_t \geq 0\,,\ x_{1t} \geq 0\,,\ x_{2t} \geq 0\,,
\end{aligned}
$$

then $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient. Let $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ be the Lagrange multipliers for resource constraint, borrowing constraint, and non-negativity constraints on x_{1t}, x_{2t}, and c_t respectively. As discussed above, any feasible allocation is bounded. Hence the sufficient conditions for $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ to be a maximum are:

$$
\begin{cases}
\lambda_t - \eta_t + \delta_{ct} = 0\,, \\
(R_1\eta_{t+1} - \eta_t) + (\theta_1 R_1 \gamma_{t+1} - \gamma_t) + \delta_{1t} = 0\,, \\
(R_2\eta_{t+1} - \eta_t) + (\theta_2 R_2 \gamma_{t+1} - \gamma_t) + \delta_{2t} = 0\,, \\
\eta_t \geq 0\,,\ \gamma_t \geq 0\,,\ \delta_{1t} \geq 0\,,\ \delta_{2t} \geq 0\,,\ \delta_{ct} \geq 0\,, \\
\delta_{1t} x_{1t} = 0\,,\ \delta_{2t} x_{2t} = 0\,,\ \delta_{ct} c_t = 0\,,
\end{cases}
\tag{SC}
$$

for $t \geq 0$, provided that $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty} \in \ell_1$. First consider the case where $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$. In this case, if I set $\{\delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ to zero, solving the first three series of equations in SC, I obtain the following for $t \geq 0$:

$$
\begin{aligned}
\eta_t &= \lambda_t\,, \quad \gamma_{t+1} = \frac{R_1 - R_2}{\theta_2 R_2 - \theta_1 R_1}\,\lambda_{t+1}\,, \\
\lambda_{t+2} &= \frac{(1-\theta_1)R_1 - (1-\theta_2)R_2}{(\theta_2 - \theta_1)R_1 R_2}\,\lambda_{t+1}\,, \\
\lambda_1 &= \frac{\theta_2 R_2 - \theta_1 R_1}{(\theta_2 - \theta_1)R_1 R_2}\,(\lambda_0 + \gamma_0)\,.
\end{aligned}
$$

The coefficient in the second difference equation above is $(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))^{-1}$. Therefore, for any positive λ_0 and γ_0, λ_1 is given by the above and

$$
\lambda_t = \left(1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})\right)^{-(t-1)}\,\lambda_1\,.
$$

Since $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) > 0$, the resulting $\{\lambda_t\}_{t=0}^{\infty}$ and consequently all $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ lie in ℓ_1. Therefore, all the conditions above which are sufficient for optimality are satisfied, and $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$

is constrained Pareto efficient.

Now let $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ and consider a feasible allocation $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^\infty$ for which there exists a $T \geq 0$ such that $x_{2t}^* = 0$ for $t \geq T$. If one sets $\{\delta_{1t}, \delta_{ct}\}_{t=0}^\infty$ to zero, the first three sets of sufficient conditions in SC give the following for $t \geq 0$:

$$\eta_t = \lambda_t\,, \quad \gamma_{t+1} = \frac{R_1 - R_2}{\theta_2 R_2 - \theta_1 R_1}\,\lambda_{t+1} - \frac{1}{\theta_2 R_2 - \theta_1 R_1}\,\delta_{2t}\,,$$

$$\lambda_{t+2} = \frac{(1-\theta_1)R_1 - (1-\theta_2)R_2}{(\theta_2 - \theta_1)R_1 R_2}\,\lambda_{t+1} + \frac{\theta_2 R_2}{(\theta_2 - \theta_1)R_1 R_2}\,\delta_{2t+1} - \frac{1}{(\theta_2 - \theta_1)R_1 R_2}\,\delta_{2t}\,,$$

$$\lambda_1 = \frac{\theta_2 R_2 - \theta_1 R_1}{(\theta_2 - \theta_1)R_1 R_2}\,(\lambda_0 + \gamma_0 - \delta_{20})\,.$$

Given any positive λ_0 and γ_0, suppose one sets $\delta_{2t} = 0$ for $0 \leq t \leq T-1$. This implies $\lambda_t = \rho^{t-1}\lambda_1$ for $1 \leq t \leq T$, where $\rho = (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))^{-1} > 1$. Moreover, let $\delta_{2T} = \alpha'\lambda_T$ and $\delta_{2t} = \alpha\lambda_t$ for $t \geq T+1$, where α and α' are positive constants to be determined. For $t \geq T+2$, the above equations lead to the following difference equation:

$$\lambda_{t+1} = \Big(\rho + \frac{\theta_2 R_2}{(\theta_2 - \theta_1)R_1 R_2}\,\alpha\Big)\,\lambda_t - \frac{1}{(\theta_2 - \theta_1)R_1 R_2}\,\alpha\lambda_{t-1}\,.$$

This difference equation has a solution of the form $\lambda_{t+1} = m\lambda_t$ where m is the smallest root of the characteristic equation:

$$m = \frac{1}{2}\Big(\rho + \frac{\theta_2 R_2}{(\theta_2 - \theta_1)R_1 R_2} - \sqrt{\Big(\rho + \frac{\theta_2 R_2}{(\theta_2 - \theta_1)R_1 R_2}\Big)^2 - \frac{4\alpha}{(\theta_2 - \theta_1)R_1 R_2}}\;\Big)\,.$$

It is easy to see that

$$m < 1 \quad \Leftrightarrow \quad \alpha < \frac{(\theta_2 - \theta_1)R_1 R_2(\rho - 1)}{1 - \theta_2 R_2}\,.$$

Hence, if α is small enough, and given the appropriate initial condition, i.e., $\lambda_{T+2} = m\lambda_{T+1}$, one can generate $\{\lambda_t\}_{t=0}^\infty \in \ell_1$. For time $T+1$ and $T+2$, the difference equation becomes:

$$\lambda_{T+1} = \Big(\rho + \frac{\theta_2 R_2}{(\theta_2 - \theta_1)R_1 R_2}\,\alpha'\Big)\,\lambda_T\,,$$

$$\lambda_{T+2} = \Big(\rho + \frac{\theta_2 R_2}{(\theta_2 - \theta_1)R_1 R_2}\,\alpha\Big)\,\lambda_{T+1} - \frac{1}{(\theta_2 - \theta_1)R_1 R_2}\,\alpha'\lambda_T\,.$$

Therefore $\lambda_{T+2} = m\lambda_{T+1}$ if and only if:

$$\Big(\rho + \frac{\theta_2 R_2}{(\theta_2 - \theta_1)R_1 R_2}\,\alpha - m\Big)\Big(\rho + \frac{\theta_2 R_2}{(\theta_2 - \theta_1)R_1 R_2}\,\alpha'\Big) = \frac{1}{(\theta_2 - \theta_1)R_1 R_2}\,\alpha' .$$

The above equation is linear in α'. Note that one always has $\theta_2 R_2 \rho < 1$ and hence for small enough α there is a strictly positive solution for α'. Therefore a small enough $\alpha > 0$ defines unique values of $0 < m < 1$ and $\alpha' > 0$ such that $\{\lambda_t\}_{t=0}^{\infty}$ and $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ are in ℓ_1 and satisfy SC. This proves that $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient.

Finally, let $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ and consider a feasible allocation $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ for which there exists a $T \geq 0$ such that $x_{2t}^* = 0$ for $t \geq T$. Setting $\{\delta_{1t}, \delta_{ct}\}_{t=0}^{\infty}$ and $\{\delta_{2t}\}_{t=0}^{T-1}$ to zero implies $\lambda_t = \lambda_1$ for $1 \leq t \leq T$. Using SC for $k \geq 1$, one can obtain

$$\lambda_{T+k} = \lambda_T - \zeta\Big(\sum_{j=0}^{k-2} \delta_{2T+j}\Big) + \nu\delta_{2T+k-1} ,$$

where $\zeta = \frac{1-\theta_2 R_2}{(\theta_2-\theta_1)R_1 R_2}$ and $\nu = \frac{1}{(\theta_2-\theta_1)R_1 R_2}$. To satisfy the above condition, for any $j \geq 0$ define

$$\delta_{2T+j} = \Big(\frac{\lambda_T}{\lambda_T + \zeta}\Big)^{j+1} ,$$

$$\lambda_{T+j} = (\lambda_T + \zeta + \nu)\Big(\frac{\lambda_T}{\lambda_T + \zeta}\Big)^{j} .$$

It is easy to see that $\{\lambda_t\}_{t=0}^{\infty}$ and $\{\eta_t, \gamma_t, \delta_{1t}, \delta_{2t}, \delta_{ct}\}_{t=0}^{\infty}$ lie in ℓ_1 and satisfy SC. This proves that $\{c_t^*, x_{1t}^*, x_{2t}^*\}_{t=0}^{\infty}$ is constrained Pareto efficient and completes the proof. ■

Proof of Proposition 6. First, I prove that the proposed regulation can implement Pareto improving reallocations used in Proposition 4. Consider an inefficiently liquid equilibrium corresponding to $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$. Similar to the proof of Proposition 4, there exists $T \geq 0$ and $\varepsilon > 0$ such that $x_{2t} \geq \varepsilon$ for $t \geq T$. Suppose one reduces x_{2t} for $t \geq T + 1$ by $\delta + \epsilon$, increases x_{1t} for $t \geq T + 1$ by ϵ, and reduces x_{2T} and increases x_{1T} both by $\frac{1}{\theta_2 R_2 - \theta_1 R_1}\delta$. Moreover, $\delta > 0$ and $\epsilon > 0$ are such that $\epsilon + \delta < \varepsilon$ and

$$\delta = (\theta_2 R_2 - \theta_1 R_1)\epsilon + \theta_2 R_2 \delta ,$$

$$\epsilon = \frac{1 - \theta_2 R_2}{\theta_2 R_2 - \theta_1 R_1}\,\delta .$$

This reduces the debt payments of generations on or after T exactly by δ. It has already been shown in the text that the above reallocation is a Pareto improvement. Let $\{\delta_t, \kappa_t, \nu_t\}_{t=0}^{\infty}$ be defined

as the decrease or increase in $(1 + r_t)e$, x_{1t} and x_{2t} respectively, as above. Then one has

$$\delta_t = \theta_2 R_2 \nu_t - \theta_1 R_1 \kappa_t \,,$$
$$\delta_{t-1} = \nu_t - \kappa_t \,.$$

Let $\{\alpha_{\ell t}\}_{t=0}^{\infty}$ be the fraction of the liquid type in total investment for the original competitive equilibrium. Now define $\{\widetilde{\alpha}_{\ell t}\}_{t=0}^{\infty}$ as follows:

$$\widetilde{\alpha}_{\ell t} = \frac{x_{2t} - \nu_t}{x_{1t} + x_{2t} - \delta_{t-1}} \,.$$

Suppose the planner regulates the portfolios according to $\{\widetilde{\alpha}_{\ell t}\}_{t=0}^{\infty}$. Let $\{\widetilde{x}_{1t}, \widetilde{x}_{2t}, \widetilde{r}_t\}_{t=0}^{\infty}$ be the prices and quantities in the regulated equilibrium. Define $r^*_{-1} = r_{-1}$ and $\{r^*_t\}_{t=1}^{\infty}$ recursively:

$$1 + r^*_t = (\widetilde{\alpha}_{\ell t}\theta_2 R_2 + (1 - \widetilde{\alpha}_{\ell t})\theta_1 R_1)(2 + r^*_{t-1}) \,.$$

By IV and market clearing, r^*_t is an upper bound for $\widetilde{r}_t$ for all t. Now suppose $(1 + r^*_{t-1})e = \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} - \delta_{t-1}$, which is true for $t = 0$ by assumption. Then one has

$$\widetilde{\alpha}_{\ell t}\theta_2 R_2 + (1 - \widetilde{\alpha}_{\ell t})\theta_1 R_1 = \frac{\theta_2 R_2(x_{2t} - \nu_t) + \theta_1 R_1(x_{1t} + \kappa_t)}{x_{1t} + x_{2t} - \delta_{t-1}} \,.$$

However, by resource constraint of the original competitive equilibrium, and using recursive equations above defining $\{\delta_t, \kappa_t, \nu_t\}$:

$$\theta_2 R_2(x_{2t} - \nu_t) + \theta_1 R_1(x_{1t} + \kappa_t) = \theta_1 R_1 x_{1t} + \theta_2 R_2 x_{2t} - \delta_t \,,$$
$$x_{1t} + x_{2t} - \delta_{t-1} = \theta_1 R_1 x_{1t-1} + \theta_2 R_2 x_{2t-1} - \delta_{t-1} + e \,.$$

Hence, it must be that $(1 + r^*_t)e = \theta_1 R_1 x_{1t} + \theta_2 R_2 x_{2t} - \delta_t$, and so by induction this holds for all $t \geq 0$. Now note that in the original competitive equilibrium one must have $1 + r_t < R_{\alpha,t} \equiv (1-\alpha_{\ell t})R_1 + \alpha_{\ell t} R_2$ for all t. Whether entrepreneurs specialize in the liquid type or mix at time t, the interest rate has to be no bigger than $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Using Assumption 1, one has $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < R_2$ and so

$$1 + r_t \leq 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < R_2 \leq R_{\alpha\ell,t} \,.$$

Observe that $1 + r^*_t < 1 + r_t$ and $R_{\widetilde{\alpha},t} \geq R_{\alpha\ell,t}$. The latter is true because $\widetilde{\alpha}_{\ell t} \geq \alpha_{\ell t}$ by construction. Hence $1 + r^*_t < R_{\widetilde{\alpha},t}$ for all t, which immediately implies that the borrowing constraints are binding in the regulated equilibrium and that $\widetilde{r}_t = r^*_t$ for all t. Thus, the allocation induced by the

regulation coincides with the Pareto superior allocation given at the beginning.

In the second part, I show that this type of regulation can make a Praeto improvement that reaches the Pareto frontier given by Proposition 5. Consider an inefficiently liquid equilibrium corresponding to $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \leq 0$. Since it converges to the steady state by Proposition 1, for an arbitrarily small $\epsilon > 0$ one can choose T such that the differences between equilibrium values of the interest rate, investments in the two types, and utility of the old generations and their steady-state values are all less than ϵ for $t \geq T$. Similar to the first part, suppose $\{\widetilde{\alpha}_{\ell t}\}_{t=0}^{\infty}$, $\{\widetilde{x}_{1t}, \widetilde{x}_{2t}, \widetilde{r}_t\}_{t=0}^{\infty}$ be the liquid fraction of investment, the prices and quantities in the regulated equilibrium. Now define $\widetilde{\alpha}_{\ell t} = \alpha_{\ell t}$ if $t \leq T-1$ and $\widetilde{\alpha}_{\ell t} = 0$ if $t \geq T$. In other words, let's replicate the original competitive equilibrium allocation up to time $T-1$ and then completely shut down investment in the liquid type on or after T.

Similar to the first part, it is easy to show that borrowing constraints are binding in the regulated equilibrium and that the regulated equilibrium converges to a new steady state with $\widetilde{x}_1^{ss}$, $\widetilde{x}_2^{ss}$ and $\widetilde{r}^{ss}$. The new steady-state interest rate $1 + \widetilde{r}^{ss} = \frac{\theta_1 R_1}{1-\theta_1 R_1}$ is below $\{1 + \widetilde{r}_t\}_{t=T}^{\infty}$ and strictly so, at least for $1 + \widetilde{r}_T$ for small enough ϵ. The reason is that the original steady-state interest rate is either $1 + r^{ss} = 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ or $1 + r^{ss} = \frac{\theta_2 R_2}{1-\theta_2 R_2}$ and in either case strictly bigger than $1 + \widetilde{r}^{ss} = \frac{\theta_1 R_1}{1-\theta_1 R_1}$. Since ϵ is small, the whole sequence of $\{1 + \widetilde{r}_t\}_{t=T}^{\infty}$ has to lie above $1 + r^{ss}$ and strictly so, at least for $t = T$.

This implies that utilities of the middle-aged for $t \geq T$ have to be above their new steady-state value, i.e., $V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$, and strictly above for $t = T$, in the regulated economy because initial wealth of the middle-aged is bigger than the steady state level. By the first part of Proposition 4, if $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 0$ one has $V_i^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) > V_z^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ for $z \in \{m, \ell\}$, where $V_z^{ss}(\boldsymbol{\theta}, \boldsymbol{R})$ is the steady-state utility for the original steady state. Hence if ϵ is small enough, all of the middle-aged at or after T are better off, while the middle-aged before T are left as well off. If $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = 0$ and $(\boldsymbol{\theta}, \boldsymbol{R}) \in F_\ell$, then after some T, the original equilibrium reaches the steady state level, and so it is still true that at least the middle-aged at T are strictly better off, while all others are at least as well off. ■

Proof of Lemma 5. Suppose that the social planner levies taxes $\{\tau_t\}_{t=0}^{\infty}$ on the old at $t+1$. The middle-aged problem changes to:

$$\begin{aligned} c_{t+1}^o \equiv \max_{i_t, x_{1t}, x_{2t} \geq 0} \quad & R_1 x_{1t} + R_2 x_{2t} - (1 + \tau_t)(1 + r_t) i_t + T_t \\ s.t. \quad & x_{1t} + x_{2t} \leq (1 + r_{t-1}) e + i_t \,, \\ & (1 + r_t) i_t \leq \theta_1 R_1 x_{1t} + \theta_2 R_2 x_{2t} \,. \end{aligned}$$

Note that the tax is rebated back to the agent, i.e., $T_t = \tau_t(1+r_t)i_t$. This ensures that any allocation that solves the above problem for any sequence of $\{\tau_t\}_{t=0}^{\infty}$ satisfies 7 and hence is feasible for a constrained social planner. After simplifying the objective function using the budget and the binding borrowing constraints, as in the problem without taxes, we end up with

$$\begin{aligned} \max_{i_t} \quad & \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t, \tau_t) i_t + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1}) e + T_t \\ s.t. \quad & \left(\frac{\theta_1 R_1 (1 + r_{t-1})}{1 + r_t - \theta_1 R_1}\right) e \leq i_t \leq \left(\frac{\theta_2 R_2 (1 + r_{t-1})}{1 + r_t - \theta_2 R_2}\right) e \,, \end{aligned} \tag{II}$$

where

$$\begin{aligned} \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t, \tau_t) &\equiv \left(\frac{(\theta_2 - \theta_1) R_1 R_2}{\theta_2 R_2 - \theta_1 R_1}\right) - \left(\frac{(1 - (1 + \tau_t)\theta_1) R_1 - (1 - (1 + \tau_t)\theta_2) R_2}{\theta_2 R_2 - \theta_1 R_1}\right)(1 + r_t) \,, \\ \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1}) &\equiv \left(\frac{(\theta_2 - \theta_1) R_1 R_2}{\theta_2 R_2 - \theta_1 R_1}\right)(1 + r_{t-1}) \,. \end{aligned}$$

After solving the above linear maximization, we get

$$\begin{cases} i_t = \left(\frac{\theta_2 R_2 (1+r_{t-1})}{1+r_t-\theta_2 R_2}\right) e \,, & \text{if } r_t < r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; \tau_t) \,, \\ i_t \in \left[\left(\frac{\theta_1 R_1 (1+r_{t-1})}{1+r_t-\theta_1 R_1}\right) e, \left(\frac{\theta_2 R_2 (1+r_{t-1})}{1+r_t-\theta_2 R_2}\right) e\right] \,, & \text{if } r_t = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; \tau_t) \,, \\ i_t = \left(\frac{\theta_1 R_1 (1+r_{t-1})}{1+r_t-\theta_1 R_1}\right) e \,, & \text{if } r_t > r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; \tau_t) \,. \end{cases} \tag{13}$$

We also get

$$r_t = \begin{cases} \theta_2 R_2 (2 + r_{t-1}) - 1 & \text{if } \theta_2 R_2 (2 + r_{t-1}) < 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; \tau_t) \,, \\ \theta_1 R_1 (2 + r_{t-1}) - 1 & \text{if } \theta_1 R_1 (2 + r_{t-1}) > 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; \tau_t) \,, \\ r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; \tau_t) & \text{otherwise} \end{cases}$$

where

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; \tau_t) = \frac{(\theta_2 - \theta_1) R_1 R_2}{(1 - (1 + \tau_t)\theta_1) R_1 - (1 - (1 + \tau_t)\theta_2) R_2} \,.$$

Note that we have $\frac{\partial r_\Lambda(\boldsymbol{\theta},\boldsymbol{R};\tau)}{\partial \tau} < 0$ and that $r_\Lambda(\boldsymbol{\theta},\boldsymbol{R};\tau)$ converges to zero as τ gets large. Now consider an inefficiently liquid equilibrium corresponding to $r_\Lambda(\boldsymbol{\theta},\boldsymbol{R};\tau) \leq 0$. Since it converges to the steady state by Proposition 1, for an arbitrarily small $\epsilon > 0$, one can choose T such that the differences between equilibrium values of the interest rate, investments in the two types, and utility of the old generations and their steady state values are all less than ϵ for $t \geq T$. Similar to the second part of Proposition 6, the social planner can shut down any investment in the liquid type for all $t \geq T$, this time using a debt tax. To do that, the social planner can set $\tau_t = 0$ for $t < T$ and $\tau_t = \tau$ for $t \geq T$, such that $1 + r_\Lambda(\boldsymbol{\theta},\boldsymbol{R};\tau) < \theta_1 R_1(2 + r_{t-1})$ for all $t \geq T$. Such a value for τ exists because T is large so that the sequence of interest rates $\{r_{t-1}\}_{t=T}^{\infty}$ stays close enough to their steady-state value and also so that $r_\Lambda(\boldsymbol{\theta},\boldsymbol{R};\tau)$ can be made small enough by choosing a large enough τ. Under this debt tax, the economy will specialize in the productive type for $t \geq T$, and similar to Proposition 6, we can show that at least one agent is strictly better off while all agents are at least as well off. And since there is no investment in the liquid type in the new allocation for $t \geq T$, by Proposition 5, the new allocation is constrained Pareto efficient. ■

Proof of Proposition 7. By Proposition 4 we have:

$$V_m^{ss} = \left(\frac{(\theta_2-\theta_1)^2 R_1^2 R_2^2}{(\theta_2 R_2 - \theta_1 R_1)((1-\theta_1)R_1 - (1-\theta_2)R_2)}\right) e\,.$$

Normalizing $e = 1$ to simplify the exposition and taking the derivative, we get:

$$\frac{\partial V_m^{ss}(\boldsymbol{\theta},\boldsymbol{R})}{\partial \theta_2} = \Omega\Big\{2(\theta_2-\theta_1)(\theta_2 R_2 - \theta_1 R_1)((1-\theta_1)R_1 - (1-\theta_2)R_2) - (\theta_2-\theta_1)^2\Big(R_2((1-\theta_1)R_1 - (1-\theta_2)R_2) + R_2(\theta_2 R_2 - \theta_1 R_1)\Big)\Big\}$$

where

$$\Omega = \left\{\frac{R_1 R_2}{(\theta_2 R_2 - \theta_1 R_1)((1-\theta_1)R_1 - (1-\theta_2)R_2)}\right\}^2,$$

hence the sign of $\frac{\partial V_m^{ss}(\boldsymbol{\theta},\boldsymbol{R})}{\partial \theta_2}$ is the sign of its numerator. Therefore, we have:

$$
\begin{aligned}
&\frac{\partial V_m^{ss}(\boldsymbol{\theta},\boldsymbol{R})}{\partial \theta_2} > 0 \\
\iff & 2(\theta_2-\theta_1)(\theta_2 R_2-\theta_1 R_1)((1-\theta_1)R_1-(1-\theta_2)R_2)- \\
& (\theta_2-\theta_1)^2\Big(R_2((1-\theta_1)R_1-(1-\theta_2)R_2)+R_2(\theta_2 R_2-\theta_1 R_1)\Big) > 0 \\
\iff & 2(\theta_2 R_2-\theta_1 R_1)((1-\theta_1)R_1-(1-\theta_2)R_2)- \\
& (\theta_2-\theta_1)R_2\Big(((1-\theta_1)R_1-(1-\theta_2)R_2)+(\theta_2 R_2-\theta_1 R_1)\Big) > 0 \\
\iff & 1-\frac{(\theta_2-\theta_1)R_2}{(1-\theta_1)R_1-(1-\theta_2)R_2} > \frac{\theta_1(R_1-R_2)}{\theta_2 R_2-\theta_1 R_1} \\
\iff & \frac{(\theta_1+\theta_2)((1-\theta_1)R_1-(1-\theta_2)R_2)-(\theta_2-\theta_1)R_2(\theta_2 R_2-\theta_1 R_1)}{(1-\theta_1)R_1-(1-\theta_2)R_2} > 2\theta_1 R_1 \\
\iff & \frac{(\theta_2-\theta_1)R_2(R_1-R_2)}{(1-\theta_1)R_1-(1-\theta_2)R_2}+2\theta_1 R_2 > 2\theta_1 R_1 \\
\iff & \frac{(\theta_2-\theta_1)R_1 R_2}{(1-\theta_1)R_1-(1-\theta_2)R_2} = 1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}) > 2\theta_1 R_1\,.
\end{aligned}
$$

As for the second part, note that the boundary of F_i and F_m is defined by $1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}) = \frac{\theta_1 R_1}{1-\theta_1 R_1}$. If the statement is true on the boundary, it will be true for the interior of the efficient region of F_m. The reason is that for any given value of $1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})$, θ_1 reaches its maximum value on the boundary of F_i and F_m. We know that an equilibrium is efficient in F_m if and only if $r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}) > 0$. Hence an equilibrium on the boundary of F_i and F_m is efficient if and only if

$$
\frac{\theta_1 R_1}{1-\theta_1 R_1} > 1 \iff \theta_1 R_1 > \frac{1}{2} \iff \frac{\theta_1 R_1}{1-\theta_1 R_1} > 2\theta_1 R_1 \iff 1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}) > 2\theta_1 R_1\,.
$$

For the third part, note that on the boundary of F_i and F_m we have $1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}) = \frac{\theta_1 R_1}{1-\theta_1 R_1} < 1$, then $\frac{\theta_1 R_1}{1-\theta_1 R_1} < 1$, and hence $1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}) < 2\theta_1 R_1$. Therefore, for any given θ_1, if θ_2 is close enough to the boundary of F_i and F_m, one has $1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}) < 2\theta_1 R_1$, because $1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})$ is strictly increasing in θ_2. We now prove the last two parts of the proposition about $Y_m^{ss}(\boldsymbol{\theta},\boldsymbol{R})$. To compute $Y_m^{ss}(\boldsymbol{\theta},\boldsymbol{R})$, we first use 6 and the value of $r_\Lambda(\boldsymbol{\theta},\boldsymbol{R})$ to get

$$
\begin{cases}
x_1^{ss} = \frac{R_2\Big(\theta_2(\theta_2-\theta_1)R_1R_2-(1-\theta_2)(\theta_2R_2-\theta_1R_1)\Big)}{(\theta_2R_2-\theta_1R_1)((1-\theta_1)R_1-(1-\theta_2)R_2)}\,, \\[2ex]
x_2^{ss} = \frac{R_1\Big((1-\theta_1)(\theta_2R_2-\theta_1R_1)-\theta_1(\theta_2-\theta_1)R_1R_2\Big)}{(\theta_2R_2-\theta_1R_1)((1-\theta_1)R_1-(1-\theta_2)R_2)}\,.
\end{cases}
$$

Hence, we have

$$Y_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) = R_1 x_1^{ss} + R_2 x_2^{ss} = \frac{R_1 R_2 (\theta_2 - \theta_1)\Big((\theta_2 R_2 - \theta_1 R_1) - (\theta_2 - \theta_1) R_1 R_2\Big)}{(\theta_2 R_2 - \theta_1 R_1)((1-\theta_1)R_1 - (1-\theta_2)R_2)} ,$$

which we can rewrite as

$$\begin{cases} Y_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{(\theta_2-\theta_1)R_1R_2}{(1-\theta_1)R_1-(1-\theta_2)R_2}\left(1 + \frac{(\theta_2-\theta_1)R_1R_2}{\theta_2R_2-\theta_1R_1}\right) \Rightarrow \\ \\ \\ Y_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) = (1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}))(1 + \Delta) , \end{cases}$$

where $\Delta = \frac{(\theta_2-\theta_1)R_1R_2}{\theta_2R_2-\theta_1R_1}$. Now we note that $\Delta = \frac{(1-\theta_1)R_1(1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}))}{(1+r_\Lambda(\boldsymbol{\theta},\boldsymbol{R}))-\theta_1R_1}$. We have

$$Y_m^{ss}(\boldsymbol{\theta}, \boldsymbol{R}) = (1 + r_\Lambda)\left(1 + \frac{(1-\theta_1)R_1(1+r_\Lambda)}{(1+r_\Lambda) - \theta_1 R_1}\right) \Rightarrow$$

$$\frac{\partial Y_m^{ss}}{\partial r_\Lambda} = \frac{(1 + (1-\theta_1)R_1)\Big\{((1+r_\Lambda) - \theta_1R_1)^2 - (\theta_1R_1)^2\frac{(1-\theta_1)R_1}{1+(1-\theta_1)R_1}\Big\}}{((1+r_\Lambda) - \theta_1R_1)^2} .$$

We also note that $\theta_1 R_1 < \frac{\theta_1 R_1}{1-\theta_1 R_1} \leq 1 + r_\Lambda$ in F_m and that r_Λ is a strictly increasing function of θ_2, and therefore we have

$$\frac{\partial Y_m^{ss}}{\partial \theta_2} < 0 \iff \frac{\partial Y_m^{ss}}{\partial r_\Lambda} < 0 \iff 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \left\{1 + \sqrt{\frac{(1-\theta_1)R_1}{1 + (1-\theta_1)R_1}}\right\}\theta_1 R_1 .$$

Now, on the boundary of F_i and F_m we have $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{\theta_1 R_1}{1-\theta_1 R_1}$. Hence, on this boundary we have

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \left\{1 + \sqrt{\frac{(1-\theta_1)R_1}{1 + (1-\theta_1)R_1}}\right\}\theta_1 R_1 \iff \frac{\frac{1}{1-\theta_1 R_1}}{1 + \sqrt{\frac{(1-\theta_1)R_1}{1+(1-\theta_1)R_1}}} < 1 .$$

But as $\theta_1 \to 0$, the right-hand side of the inequality above converges to $\left(1 + \sqrt{\frac{R_1}{1+R_1}}\right)^{-1} < 1$. Therefore, for a small enough value of θ_1, the condition above holds at the boundary of F_i and F_m. Then, for small enough θ_1, the equilibrium on the boundary of F_i and F_m will be inefficient (since $1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) = \frac{\theta_1 R_1}{1-\theta_1 R_1} \leq 1$) and satisfy the above condition, implying $\frac{\partial Y_m^{ss}(\boldsymbol{\theta},\boldsymbol{R})}{\partial \theta_2} < 0$. For these small enough values of θ_1 and by continuity, if θ_2 is close enough to the value on the boundary,

the condition still holds and we still have $\frac{\partial Y_m^{ss}(\boldsymbol{\theta},\boldsymbol{R})}{\partial \theta_2} < 0$. Finally, we note that

$$1 + \sqrt{\frac{(1-\theta_1)R_1}{1+(1-\theta_1)R_1}} < 2$$

Therefore, we have

$$1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < \left\{1 + \sqrt{\frac{(1-\theta_1)R_1}{1+(1-\theta_1)R_1}}\right\}\theta_1 R_1 \Rightarrow 1 + r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) < 2\theta_1 R_1 \,.$$

This finalizes the proof. ∎

Proof of Lemma 6. I only show that, given the assumptions on parameters and the initial value, competitive equilibrium is unique, borrowing constraint is always binding, and the interest rate evolves according to a law of motion similar to 4. The rest of the proof is very similar to Proposition 1 and Lemma 2, and so is not provided here.

First, note that $1 + r_t \leq R_1$ for any t. Suppose $1 + r_t > R_1$, in which case the middle-aged do not invest in any of the two investment types since their returns are strictly less than the interest rate. The resource constraint dictates that $i_t - b_t^m = -(1 + r_{t-1})e < 0$. Using 9, in equilibrium one must have $i_t - b_t^m = e - b_t$, which is strictly positive given Assumption 2.

Now I want to show $1 + r_t < R_1$ using induction. This is true for $t = -1$ by assumption. Suppose $1 + r_t = R_1$, while $1 + r_{t-1} < R_1$. Then the entrepreneurs at t do not invest in the liquid type since they can always raise their consumption by reducing their investment in the liquid type. The resource constraint then gives $x_{1t} = (1 + r_{t-1})e + (i_t - b_t^m)$. Substituting this into the borrowing constraint yields

$$(1 + r_t - \theta_1 R_1)(i_t - b_t^m) \leq \theta_1 R_1 (1 + r_{t-1})e \,.$$

In equilibrium, by 9 and the fact that $1 + r_t = R_1$ and $1 + r_{t-1} < R_1$, one has

$$(1-\theta_1)(e - b_t) \leq \theta_1(1 + r_{t-1})e < \theta_1 R_1 e \Rightarrow \sigma_t > 1 - \frac{\theta_1 R_1}{1-\theta_1} \,,$$

which contradicts Assumption 2. Hence it must be that $1 + r_t < R_1$. This implies that borrowing constraint has to be binding at any t; otherwise, the entrepreneurs can raise their consumption by increasing x_{1t} by a small amount. Using the borrowing and the resource constraints to express

x_{1t} and x_{2t} in terms of $(1 + r_{t-1})e$ and $i_t - b_t^m$, and then noting that $x_{1t} \geq 0$ and $x_{2t} \geq 0$, one gets

$$\left(\frac{\theta_1 R_1(1 + r_{t-1})}{1 + r_t - \theta_1 R_1}\right) e \leq (i_t - b_t^m) \leq \left(\frac{\theta_2 R_2(1 + r_{t-1})}{1 + r_t - \theta_2 R_2}\right) e \,.$$

Equation 9 implies $i_t - b_t^m = e - b_t$, and so one obtains:

$$\frac{\theta_1 R_1}{1 - \sigma_t}(1 + r_{t-1}) + \theta_1 R_1 \leq 1 + r_t \leq \frac{\theta_2 R_2}{1 - \sigma_t}(1 + r_{t-1}) + \theta_2 R_2 \,.$$

The problem of the middle-aged in II_b is similar to II. The only difference is that entrepreneurs are maximizing with respect to $i_t - b_t^m$ rather than i_t. The net gain of increasing the size of net investment $i_t - b_t^m$ is equal to $\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t)$ as before, and hence the threshold interest rate at which entrepreneurs switch from one type to another is equal to $r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$. Therefore, the law of motion for the interest rate is

$$r_t = \begin{cases} \frac{\theta_2 R_2}{1-\sigma_t}(1 + r_{t-1}) + \theta_2 R_2 - 1 & \text{if } \frac{\theta_2 R_2}{1-\sigma_t}(1 + r_{t-1}) + \theta_2 R_2 - 1 < r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \,, \\ \frac{\theta_1 R_1}{1-\sigma_t}(1 + r_{t-1}) + \theta_1 R_1 - 1 & \text{if } \frac{\theta_1 R_1}{1-\sigma_t}(1 + r_{t-1}) + \theta_1 R_1 - 1 > r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) \,, \\ r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R}) & \text{otherwise} \,. \end{cases}$$

Assumption 2 implies that there exists a $\epsilon > 0$ such that $\sigma_t < 1 - \theta_2 R_2 - \epsilon$ for all t. This implies that $1 - \sigma_t > \theta_2 R_2 + \epsilon$ for all t, which guarantees that the above difference equations are stable and that there is a unique path of interest rates. The rest of the proof is similar to the proof of Proposition 1 and Lemma 2 and so is omitted. ■

Proof of Lemma 7. The first two equations follow from Lemma 6 and market clearing condition 9. The last two equations can be easily derived using II_b at the steady state. ■

Proof of Lemma 8. Using problem II_b and conditions 8 and 9 for $z \in \{\ell, m, i\}$, one observes that

$$V_z^{ss}(\sigma) = \Big((1 - \sigma)\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_z^{ss}(\sigma)) + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_z^{ss}(\sigma)) - \sigma r_z^{ss}(\sigma)\Big) e \,.$$

Using the definitions of Λ and Φ, I can obtain a more explicit form of the objective function:

$$V_z^{ss}(\sigma) = \left(\frac{(1-\sigma)\Big((\theta_2-\theta_1)R_1R_2 - ((1-\theta_1)R_1 - (1-\theta_2)R_2)(1+r_z^{ss}(\sigma))\Big)}{\theta_2R_2 - \theta_1R_1} \right) e$$
$$+ \left(\frac{(1+r_z^{ss}(\sigma))(\theta_2-\theta_1)R_1R_2 - r_z^{ss}(\sigma)(\sigma)(\theta_2R_2-\theta_1R_1)}{\theta_2R_2-\theta_1R_1} \right) e\,.$$

Now using the expressions for $r_z^{ss}(\sigma)$, $z \in \{\ell, m, i\}$, in Lemma 6, I can take the derivative for each $z \in \{\ell, m, i\}$. For $z = \ell$:

$$\frac{d(1+r_\ell^{ss}(\sigma))}{d\sigma} = \left(\frac{1}{1-\sigma}\right)^2 (1+r_\ell^{ss}(\sigma)) = \left(\frac{\theta_2R_2}{(1-\sigma)-\theta_2R_2}\right)^2\,.$$

Using above I get

$$\frac{dV_\ell^{ss}(\sigma)}{d\sigma}|_{\sigma=0} = \left(\frac{\Big(-(\theta_2-\theta_1)R_1R_2 + ((1-\theta_1)R_1 - (1-\theta_2)R_2)(1+r_\ell^{ss})\Big)}{\theta_2R_2-\theta_1R_1} \right) e$$
$$+ \left(\frac{\Big((\theta_2-\theta_1)R_1R_2 - ((1-\theta_1)R_1-(1-\theta_2)R_2)\Big)(1+r_\ell^{ss})^2}{(\theta_2R_2-\theta_1R_1)} \right) e$$
$$- r_\ell^{ss} e\,.$$

I simplify the above to

$$\frac{dV_\ell^{ss}(\sigma)}{d\sigma}|_{\sigma=0} = \left(\frac{\Big((\theta_2-\theta_1)R_1R_2(2+r_\ell^{ss}) - ((1-\theta_1)R_1 - (1-\theta_2)R_2)(1+r_\ell^{ss})\Big)}{\theta_2R_2-\theta_1R_1} \right) r_\ell^{ss} e$$
$$- r_\ell^{ss} e\,.$$

I note that $r_\ell^{ss} = \frac{\theta_2R_2}{1-\theta_2R_2}$, and so I can simplify to get:

$$\frac{dV_\ell^{ss}(\sigma)}{d\sigma}|_{\sigma=0} = \left(\frac{R_2-1}{1-\theta_2R_2}\right) r_\ell^{ss} e\,.$$

The proof for $z = i$ is very similar. For $z = m$, one observes that $r_m^{ss}(\sigma) = r_m^{ss}$ for any small enough σ and so

$$\begin{aligned} \frac{d(1 + r_m^{ss}(\sigma))}{d\sigma} &= 0\,, \\ \Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_m^{ss}(\sigma)) &= 0\,. \end{aligned}$$

Hence

$$\frac{dV_m^{ss}(\sigma)}{d\sigma}|_{\sigma=0} = -r_m^{ss} e\,.$$

■

Proof of Proposition 8. The first part is obvious by Lemma 8, since for small enough $\sigma > 0$ the change in steady-state welfare is strictly negative in inefficient equilibria of the liquid region because the steady-state interest rate is strictly negative. Therefore for small enough $\sigma > 0$, a sequence of bonds Σ with a long-term supply of σ cannot make Pareto improvement.

For the second part, let $T \geq 0$ be such that $r_t = r_\Lambda(\boldsymbol{\theta}, \boldsymbol{R})$ for $t \geq T$. Consider $\Sigma = \{\sigma_t\}_{t=0}^{\infty}$, where $\sigma_t = 0$ for $t \leq T - 1$ and $\sigma_t = \epsilon$ for $t \geq T$ and $\epsilon > 0$ is small enough. Using problem II_b and conditions 8 and 9, one has

$$V_t(\Sigma) = \Big((1 - \sigma_t)\Lambda(\boldsymbol{\theta}, \boldsymbol{R}; r_t(\Sigma)) + \Phi(\boldsymbol{\theta}, \boldsymbol{R}; r_{t-1}(\Sigma)) - ((1 + r_t(\Sigma))\sigma_t - \sigma_{t+1})\Big) e\,.$$

For small enough ϵ, one has $r_t(\Sigma) = r_t$ for all t, where r_t represents the interest rate in the competitive equilibrium without government bonds. Hence, consumption of the middle-aged at $t \leq T - 1$ does not change. For $t \geq T + 1$, consumption increases exactly by $-r_t\epsilon \geq 0$ since competitive equilibrium is inefficient. Finally, the change in consumption of the middle-aged at T is $\epsilon > 0$, and therefore Σ makes a Pareto improvement. ■