Scanner Data, Elementary Price Indexes and the Chain Drift Problem

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Abstract

Scanner data from retail outlets has allowed national statistical agencies to construct superlative indexes at the first stage of aggregation. However if there are strong fluctuations in prices and quantities, chained indexes using scanner data will typically show strong trends which are too large to be credible. To control this chain drift problem, the chapter suggests the use of multilateral index formulae. The chapter compares all of the main multilateral index number formulae both from a theoretical perspective and illustrates the results using a scanner data set on sales of frozen juices for a retail outlet in Chicago. The chapter suggests a new multilateral method that is based on linking observations that have the most similar structure of relative prices and quantities.

Key Words: Multilateral, superlative and similarity linked indexes.

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1. Introduction

The *Consumer Price Index Manual*[2] recommended that the Fisher, Walsh or Törnqvist Theil price index be used as a *target month to month index* in a Consumer Price Index, provided that monthly price and expenditure data for the class of expenditures in scope were available. In recent years, retail chains in several countries (e.g., Australia, Canada, Japan, the Netherlands, Norway and Switzerland) have been willing to donate their sales value and quantity sold information by detailed product to their national statistical agencies so it has become possible to calculate month to month superlative indexes for at least some strata of the country's Consumer Price Index.[3] However, the following issue arises: should the indexes fix a base month (for 12 or 13 months) and calculate Fisher fixed base indexes or should they calculate chained month to month indexes Fisher indexes? The 2004 *CPI Manual* offered the following advice on this choice in the chapter on seasonal commodities:[4]

- Determine the set of commodities that are present in the marketplace in both months of the comparison of prices between the two periods.
- For this maximum overlap set of commodities, calculate one of the three indexes recommended in previous chapters using the chain principle; i.e., calculate the chained Fisher, Walsh or Törnqvist Theil index.

The *CPI Manual* suggested the use of chained superlative indexes as a target index for the following three reasons:[5]

- The set of seasonal commodities which overlaps during two consecutive months is likely to be much larger than the set obtained by comparing the prices of any given month with a fixed base month (like January of a base year). Hence the comparisons made using chained indexes will be more comprehensive and accurate than those made using a fixed base.
- In many economies, on average 2 or 3 percent of price quotes disappear each month due to the introduction of new commodities and the disappearance of older ones. This rapid sample attrition means that fixed base indexes rapidly become unrepresentative and hence it seems preferable to use chained indexes that can more closely follow marketplace developments.
- If prices and quantities are trending relatively smoothly over time, chaining will reduce the spread between the Paasche and Laspeyres indexes.[6] Since these indexes provide reasonable bounds for true cost of living indexes, reducing the spread between these indexes will narrow the zone of uncertainty about the cost of living.

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[2] See paragraph 22.63 in the ILO, Eurostat, IMF, OECD, UN and the World Bank (2004).

[3] Some countries may be able to obtain price and quantity data for individual products from third party data aggregators. This can be a cost effective strategy for a statistical agency. In other cases, price and quantity data for regulated industries can be obtained from regulators.

[4] For more on the economic approach and the assumptions on consumer preferences that can justify month to month maximum overlap indexes, see Diewert (1999a; 51-56).

[5] See the ILO, Eurostat, IMF, OECD, UN and the World Bank (2004; 407).

[6] See Diewert (1978; 895) and Hill (1988) (1993; 387-388). Chaining under these conditions will also reduce the spread between fixed base and chained indexes using P_F, P_W or P_T as the basic bilateral formula.

Thus the 2004 *Manual* recommended the use of chained Fisher, Walsh or Törnqvist Theil indexes as a target index concepts. But, as will be seen in the subsequent text, this advice does not always work out too well.

The problem with the above advice is the assumption of smooth trends in prices and quantities. Hill (1993; 388), drawing on the earlier research of Szulc (1983) (1987) and Hill (1988; 136-137), noted that it is not appropriate to use the chain system when prices oscillate or "bounce" to use Szulc's (1983; 548) term. This phenomenon can occur in the context of regular seasonal fluctuations or in the context of sales. The extent of the *price bouncing problem* or the problem of *chain drift* can be measured if we make use of the following test due to Walsh (1901; 389), (1921b; 540):[7]

Multiperiod Identity Test: $P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)P(p^2,p^0,q^2,q^0) = 1$

where $p^t \equiv [p_{t1},...,p_{tN}]$ and $q^t \equiv [q_{t1},...,q_{tN}]$ are the period t price and quantity vectors and p_{tn} and q_{tn} are the period t price and quantity for commodity n for n = 1,...,N in the class of commodities under consideration. $P(p^0,p^1,q^0,q^1)$ is a bilateral index number formula that is a function of the prices and quantities of periods 0 and 1. Thus price change is calculated over consecutive periods but an artificial final period is introduced as the final period where the prices and quantities revert back to the prices and quantities in the very first period. The test asks that the product of all of these price changes should equal unity. If prices have no definite trends but are simply bouncing up and down in a range, then the above test can be used to evaluate the amount of chain drift that occurs if chained indexes are used under these conditions. *Chain drift* occurs when an index does not return to unity when prices in the current period return to their levels in the base period.[8] Fixed base indexes that satisfy the time reversal test will satisfy Walsh's test and hence will not be subject to chain drift as long as the base period is not changed.

The *Manual* did not take into account how severe the chain drift problem could be in practice.[9] The problem is mostly caused by *sales* (i.e., highly discounted prices) of products.[10] An example will illustrate the problem.

Suppose that we are given the price and quantity data for two commodities for four periods. The data are listed in Table 1 below.[11]

Table 1: Price and Quantity Data for Two Products for Four Periods

Period t	p_1^t	p_2^t	q_1^t	q_2^t
1	1.0	1.0	10	100
2	0.5	1.0	5000	100

[7] Fisher (1922; 293) realized that the chained Carli, Laspeyres and Young indexes could be subject to upward chain drift but for his empirical example, there was no evidence of chain drift for the Fisher formula. However, Persons (1921; 110) came up with an empirical example where the Fisher index exhibited substantial downward chain drift. Frisch (1936; 9) seems to have been the first to use the term "chain drift". Both Frisch (1936; 8-9) and Persons (1928; 100-105) discussed and analyzed the chain drift problem. These indexes will be formally defined later in the chapter.
[8] See the ILO, Eurostat, IMF, OECD, UN and the World Bank (2004; 445).
[9] Szulc (1983) (1987) demonstrated how big the chain drift problem could be using chained Laspeyres indexes but the authors of the 2004 *Manual* did not realize that chain drift could also be a problem with chained *superlative* indexes.
[10] Pronounced fluctuations in the prices and quantities of seasonal commodities can also cause chain drift.
[11] This example is taken from Diewert (2012).

3	1.0	1.0	1	100
4	1.0	1.0	10	100

The first commodity is subject to periodic sales (in period 2), when the price drops to ½ of its normal level of 1. In period 1, we have "normal" off sale demand for commodity 1 which is equal to 10 units. In period 2, the sale takes place and demand explodes to 5000 units.[12] In period 3, the commodity is off sale and the price is back to 1 but many shoppers have stocked up in the previous period so demand falls to only 1 unit. Finally in period 4, the commodity is off sale and we are back to the "normal" demand of 10 units. Commodity 2 exhibits no price or quantity change across periods: its price is 1 in all periods and the quantity sold is 100 units in each period. Note that the only thing that has happened going from period 3 to 4 is that the demand for commodity one has picked up from 1 unit to the "normal" level of 10 units. Also note that, conveniently, the period 4 data are exactly equal to the period 1 data so that for Walsh's test to be satisfied, the product of the period to period chain links must equal one.

Table 2 lists the fixed base Fisher, Laspeyres and Paasche price indexes, $P_{F(FB)}$, $P_{L(FB)}$ and $P_{P(FB)}$ and as expected, they behave perfectly in period 4, returning to the period 1 level of 1. Then the chained Fisher, Törnqvist-Theil, Laspeyres and Paasche price indexes, $P_{F(CH)}$, $P_{T(CH)}$, $P_{L(CH)}$ and $P_{P(CH)}$ are listed. Obviously, the chained Laspeyres and Paasche indexes have chain drift bias that is extraordinary but what is interesting is that the chained Fisher has a 2% downward bias and the chained Törnqvist has a close to 3% downward bias.

Table 2: Fixed Base and Chained Fisher, Törnqvist-Theil, Laspeyres and Paasche Indexes

Period	$P_{F(FB)}$	$P_{L(FB)}$	$P_{P(FB)}$	$P_{F(CH)}$	$P_{T(CH)}$	$P_{L(CH)}$	$P_{P(CH)}$
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.698	0.955	0.510	0.698	0.694	0.955	0.510
3	1.000	1.000	1.000	0.979	0.972	1.872	0.512
4	1.000	1.000	1.000	0.979	0.972	1.872	0.512

What explains the results in the above table? The problem is this: when commodity one comes off sale and goes back to its regular price in period 3, *the corresponding quantity does not return to the level it had in period 1*: the period 3 demand is only 1 unit whereas the "normal" period 1 demand for commodity 1 was 10 units. It is only in period 4, that demand for commodity one recovers to the period 1 level. However, since prices are the same in periods 3 and 4, all of the chain links show no change (even though quantities are changing) and this is what causes the difficulties. If demand for commodity one in period 3 had immediately recovered to its "normal" period 1 level of 10, then there would be no chain drift problem.[13]

[12] This example is based on an actual example that used Dutch scanner data. When the price of a detergent product went on sale in the Netherlands at approximately one half of the regular price, the volume sold shot up approximately one thousand fold; see de Haan (2008; 15) and de Haan and van der Grient (2011). These papers brought home the magnitude of volume fluctuations due to sales and led Ivancic, Diewert and Fox (2009) (2011) to propose the use of rolling window multilateral indexes to mitigate the chain drift problem.

[13] If the economic approach to index number theory is adopted, what causes chain drift in the above example is *inventory stocking behavior* on the part of households. The standard theory for the cost of living index implicitly assumes that all purchased goods are nondurable and used up in the period of purchase. In real life households can stockpile goods when they go on sale and it is this stockpiling phenomenon that leads to downward chain drift for a superlative index. For an example where a chained superlative index has upward chain drift, see section 7 below. Feenstra and Shapiro (2003) also looked at the chain drift

There are at least four possible real time solutions to the chain drift problem:

- Use a fixed base index;
- Use a multilateral index;[14]
- Use annual weights for a past year or
- Give up on the use of weights at the first stage of aggregation and simply use the Jevons index, which does not rely on representative weights.

There are two problems with the first solution: (i) the results depend asymmetrically on the choice of the base period and (ii) with new and disappearing products,[15] the base period prices and quantities may lose their representativeness; i.e., over long periods of time, matching products becomes very difficult.[16]

A problem with the second solution is that as an extra period of data becomes available, the indexes may have to be recomputed. This is not a major problem. A solution to this problem is to use a rolling window of observations and use the results of the current window to update the index to the current period. This methodology was suggested by Ivancic, Diewert and Fox (2009) (2011) and is being used by the Australian Bureau of Statistics (2016). There is the problem of deciding exactly how to link the results of the current rolling window to the indexes generated by the previous rolling window but again, this is not a major problem.[17] However, it is possible to solve these linking problems by making use of a different class of multilateral methods; namely methods that rely on linking the data of the current period with a prior period that has the most similar structure of relative prices. This new class of multilateral methods will be explained in sections 18 and 20 below.

The problem with the third possible solution is that the use of annual weights will inevitably result in some substitution bias, usually in the range of 0.15 to 0.40 percentage points per year.[18]

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problem that was caused by sales and restocking dynamics. Their suggested solution to the chain drift problem was to use fixed base indexes which was also the advice of Persons (1921; 112).

[14] A multilateral price index compares average price levels over multiple periods. A bilateral price index compares price levels over two periods. Multilateral price indexes were originally applied in making *cross country* comparisons of prices. The use of multilateral indexes in the time series context dates back to Persons (1921) and Fisher (1922; 297-308), Gini (1931) and Balk (1980) (1981). Fisher (1922; 305) suggested taking the arithmetic average of the Fisher "star" indexes whereas Gini suggested taking the geometric mean of the star indexes. For additional material on multilateral indexes, see Diewert (1988) (1999b), Balk (1996) (2008) and Diewert and Fox (2020).

[15] We use the term "products" as meaning "goods and services".

[16] Persons (1928; 99-100) has an excellent discussion on the difficulties of matching products over time.

[17] Ivancic, Diewert and Fox (2009) (2011) suggested that the movement of the rolling window indexes for the last two periods in the new window be linked to the last index value generated by the previous window. However Krsinich (2016) suggested that the movement of the indexes generated by the new window be linked to the previous window index value for the second period in the previous window. Krsinich called this a *window splice* as opposed to the IDF *movement splice*. De Haan (2015; 27) suggested that perhaps the linking period should be in the middle of the old window which the Australian Bureau of Statistics (2016; 12) termed a *half splice*. Ivancic, Diewert and Fox (2010) suggested that the *average* of all links for the last period in the new window to the observations in the old window could be used as the linking factor. Diewert and Fox (2020) looked at these alternative methods for linking. *Average* or *mean linking* seems to be the safest strategy.

[18] For retrospective studies on upper level substitution bias for national CPIs, see Diewert, Huwiler and Kohli (2009), Huang, Wimalaratne and Pollard (2015) and Armknecht and Silver (2014). For studies of lower level substitution bias for a Lowe index, see Diewert, Finkel and Artsev (2009) and Diewert (2014).

The problem with the fourth possible solution is that the use of an index that does not use quantity or expenditure weights will give equal weight to the prices of products that may be unimportant in household budgets, which can lead to a biased Consumer Price Index.

There is a possible fifth method to avoid chain drift within a year when using a superlative index and that is to simply compute a sequence of 12 year over year monthly indexes so that say January prices in the previous year would be compared with January prices in the current year and so on. Handbury, Watanabe and Weinstein (2013) used this methodological approach for the construction of year over year monthly superlative Japanese consumer price indexes using the Nikkei point of sale data base. This data base has monthly price and expenditure data covering the years 1988 to 2010 and contains 4.82 billion price and quantity observations. This type of index number was recommended in chapter 22 of the 2004 *Consumer Price Index Manual* as a valid year over year index that would avoid seasonality problems. However, central banks and other users require month to month CPIs in addition to year over year monthly CPIs and so the approach of Handbury, Watanabe and Weinstein does not solve the problems associated with the construction of superlative month to month indexes.

Many national statistical agencies are using web-scraping to collect large numbers of prices as a substitute for selective sampling of prices at the first stage of aggregation. Thus it is of interest to look at elementary indexes that depend only on prices, such as the Carli (1804), Dutot (1738) and Jevons (1865) indexes, and compare these indexes to superlative indexes; i.e., under what conditions will these indexes adequately approximate a superlative index.[19]

The two superlative indexes that we will consider in this chapter are the Fisher (1922) and the Törnqvist[20] indexes. The reasons for singling out these two indexes as preferred bilateral index number formulae are as follows: (i) both indexes can be given a strong justification from the viewpoint of the economic approach to index number theory; (ii) the Fisher index emerges as probably being the "best" index from the viewpoint of the axiomatic or test approach to index number theory;[21] (iii) the Törnqvist index has a strong justification from the viewpoint of the stochastic approach to index number theory.[22] Thus there are strong cases for the use of these two indexes when making comparisons of prices between two periods when detailed price and quantity data are available.

When comparing two indexes, two methods for making the comparisons will be used: (i) use second order Taylor series approximations to the index differences; (ii) the difference between two indexes can frequently be written as a covariance and it is possible in many cases to determine the likely sign of the covariance.[23]

[19] We will also look at the approximation properties of the CES price index with equal weights.
[20] The usual reference is Törnqvist (1936) but the index formula did not actually appear in this paper. It did appear explicitly in Törnqvist and Törnqvist (1937). It was listed as one of Fisher's (1922) many indexes: namely number 123. It was explicitly recommended as one of his top five ideal indexes by Warren Persons (1928; 86) so it probably should be called the *Persons index*. Theil (1967) developed a compelling descriptive statistics justification for the index. Superlative indexes are explained in Diewert (1976) (2021a).
[21] See Diewert (1992).
[22] See Theil (1967; 136-137) or Chapter 4.
[23] This second method for making comparisons can be traced back to Bortkiewicz (1923).

When looking at scanner data from a retail outlet (or price and quantity data from a firm that uses dynamic pricing to price its products or services[24]), a fact emerges: if a product or a service is offered at a highly discounted price (i.e., it goes on sale), then the quantity sold of the product can increase by a very large amount. This empirical observation will allow us to make reasonable guesses about the signs of various covariances that express the difference between two indexes. If we are aggregating products that are close substitutes for each other, then a heavily discounted price may not only increase the *quantities sold* of the product but it may also increase the *expenditure share* of the sales in the list of products or services that are in scope for the index.[25] It turns out that the behavior of shares in response to discounted prices does make a difference in analyzing the differences between various indexes: in the context of highly substitutable products, a heavily discounted price will probably increase the market share of the product but if the products are weak substitutes (which is typically the case at higher levels of aggregation), then a discounted price will typically increase sales of the product but not increase its market share. These two cases (strong or weak substitutes) will play an important role in our analysis.

Sections 2 and 3 look at relationships between the fixed base and chained Carli, Dutot, Jevons and CES (Constant Elasticity of Substitution) elementary indexes that do not use expenditure share or quantity information. These indexes are used by national statistical agencies at the first stage of aggregation when they calculate price indexes for components of their consumer price indexes in the case when quantity or value information is not available. It should be noted that we will start our analysis of various index number formulae by first developing the concept of a *price level*, which is an average of prices pertaining to a given period of time. A bilateral *price index* calulates *price change* between two periods. A price index could be a ratio of two price levels or it could be an average of price ratios, where the price of a good or service in the comparison period is in the numerator and the corresponding price in the base period is in the denominator. Comparing price levels for two periods is quite different from undertaking price comparisons over multiple periods. In the multiple period case, it turns out to be easier to compare price *levels* across periods rather than taking averages of price ratios as is done in the case of bilateral comparisons. Thus from the viewpoint of the economic approach to index number theory, it is simpler to target the estimation of unit cost functions rather than target the estimation of a ratio of unit cost functions. Once we have estimates for period by period price levels, we can easily form ratios of these estimates which will give us “normal” index numbers.

Section 4 looks at the relationships between the Laspeyres, Paasche, Geometric Laspeyres, Geometric Paasche, Fisher and Törnqvist bilateral price indexes. Section 5 investigates how close the unweighted Jevons index is to the Geometric Laspeyres $P_{GL}{}^{t}$, Geometric Paasche $P_{GP}{}^{t}$ and Törnqvist $P_{T}{}^{t}$ price indexes.

Section 6 develops some relationships between the Törnqvist index and geometric indexes that use average *annual* shares as weights.

Section 7 looks at the differences between fixed base and chained Törnqvist indexes.

Multilateral indexes finally make their appearance in section 8: the fixed base Törnqvist index is compared to the GEKS (Gini, Eltetö, Köves and Szulc) and GEKS-Törnqvist or CCDI (Caves, Christensen, Diewert and Inklaar) multilateral indexes.

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[24] Airlines and hotels are increasingly using dynamic pricing; i.e., they change prices frequently.

[25] In the remainder of this chapter, we will speak of products but the same analysis applies to services.

Sections 9 and 10 compare Unit Value and Quality Adjusted Unit Value indexes to the Fisher index. It turns out that some multilateral indexes are actually quality adjusted unit value indexes as will be seen in section 12. Section 11 compares the Lowe index to the Fisher index.

Section 12 looks at the Geary Khamis multilateral index and shows that it is actually a special case of a quality adjusted unit value index.

Sections 13 and 14 introduce Time Product Dummy multilateral indexes. Section 13 assumes that there are no missing products in the window of time periods under consideration while section 14 deals with the case of missing products. Sections 15 and 16 introduce Weighted Time Product Dummy indexes for the case of two periods; the missing products case is considered in section 16. Finally, the Weighted Time Product Dummy multilateral indexes for T periods with missing products is discussed in section 17. Readers who are only interested in the general case can skip sections 13-16 and just consider the general case in section 17.

Section 18 introduces a less familiar multilateral method that is based on linking observations that have the most similar structure of relative prices. This *similarity method* for linking observations has for the most part been used in the context of making cross country comparisons. This class of methods depends on the choice of a measure of *dissimilarity* between the prices of two observations. The dissimilarity measure used in section 18 is Diewert's (2009) asymptotic linear measure of relative price dissimilarity.

A problem with the dissimilarity measure used in section 18 is that it requires positive prices for all products.[26] Thus in section 19, a simple method for constructing imputed prices for missing products is described.

In section 20, a new measure of relative price dissimilarity, the predicted share measure of relative price dissimilarity, is defined that does not require positive prices for all products in the two periods being compared. This new measure can be adapted to measures of dissimilarity between relative quantities. Section 20 also introduces another method for constructing bilateral index number links between pairs of observations that have either proportional price vectors or proportional quantity vectors. This new method has some good axiomatic properties as will be seen in the following section 21.

Section 21 introduces an axiomatic or test approach to evaluate the properties of alternative multilateral methods for generating price and quantity levels cross multiple time periods. However, this section makes only a start on the axiomatic approach to evaluating alternative price levels for many time periods.

Section 22 summarizes some of the more important results in this chapter.

The online Appendix evaluates all of the above indexes for a grocery store scanner data set that is publicly available. This data set had a number of missing prices and quantities. Some of these missing prices may be due to lack of sales or shortages of inventory. A general problem is how should the introduction of new products and the disappearance of (possibly) obsolete products be treated in the context of forming a consumer price index? Hicks (1940; 140) suggested a general approach to this measurement problem in the context of the economic approach to index number

[26] Products that are absent in both periods that are being compared can be ignored. However for products that are present in only one of the two comparison periods, the dissimilarity measure defined in section 18 requires that an imputed price for the missing products be constructed.

theory. His approach was to apply normal index number theory but estimate (or guess at) hypothetical prices that would induce utility maximizing purchasers of a related group of products to demand 0 units of unavailable products. With these virtual (or reservation or imputed) prices in hand, one can just apply normal index number theory using the augmented price data and the observed quantity data. The empirical example discussed in the online Appendix uses the scanner data that was used in Diewert and Feenstra (2017) for frozen juice products for a Dominick's store in Chicago for three years. This data set had 20 observations where $q_{tn} = 0$. For these 0 quantity observations, Diewert and Feenstra estimated positive Hicksian reservation prices for these missing price observations and these imputed prices are used in the empirical example in the Appendix. The Appendix lists the Dominick's data along with the estimated reservation prices. The Appendix also has tables and charts of the various index number formulae that are discussed in the main text of the study.

2. Comparing CES Price Levels and Price Indexes

In this section, we will begin our analysis by considering alternative methods by which the prices for N related products could be aggregated into an *aggregate price level* for the products for a given period.

We introduce some notation that will be used in the rest of the chapter. It is supposed that price and quantity data for N closely related products has been collected for T time periods.[27] Typically, a time period is a month. Denote the price of product n in period t as p_{tn} and the corresponding quantity during period t as q_{tn} for $n = 1,...,N$ and $t = 1,...,T$. Usually, p_{tn} will be the period t *unit value price* for product n in period t; i.e., $p_{tn} \equiv v_{tn}/q_{tn}$ where v_{tn} is the total value of product n that is sold or purchased during period t and q_{tn} is the total quantity of product n that is sold or purchased during period t. We assume that $q_{tn} \geq 0$ and $p_{tn} > 0$ for all t and n.[28] The restriction that all products have positive prices associated with them is a necessary one for much of our analysis since many popular index numbers are constructed using logarithms of prices and the logarithm of a zero price is not well defined. However, our analysis does allow for possible 0 quantities and values for some products for some time periods. Denote the period t strictly positive *price vectors* as $p^t \equiv [p_{t1},...,p_{tN}] >> 0_N$ and nonnegative (and nonzero) *quantity vectors* as $q^t \equiv [q_{t1},...,q_{tN}] > 0_N$ respectively for $t = 1,...,T$ where 0_N is an N dimensional vector of zeros. As usual, the inner product of the vectors p^t and q^t is denoted by $p^t \cdot q^t \equiv \Sigma_{n=1}^N p_{tn}q_{tn} > 0$. Define the period t sales (or expenditure) share for product n as $s_{tn} \equiv p_{tn}q_{tn}/p^t \cdot q^t$ for $n = 1,...,N$ and $t = 1,...,T$. The *period t sales or expenditure share vector* is defined as $s^t \equiv [s_{t1},...,s_{tN}] > 0_N$ for $t = 1,...,T$.

In many applications, the N products will be closely related and they will have common units of measurement (by weight, or by volume or by "standard" package size). In this context, it is useful to define the period t "real" share for product n of total product sales or purchases, $S_{tn} \equiv q_{tn}/1_N \cdot q^t$ for $n = 1,...,N$ and $t = 1,...,T$ where 1_N is an N dimensional vector of ones. Denote the *period t real share vector* as $S^t \equiv [S_{t1},...,S_{tN}]$ for $t = 1,...,T$.

[27] The T periods can be regarded as a window of observations, followed by another window of length T that has dropped the first period from the window and added the data of period T+1 to the window. The literature on how to link the results of one window to the next window was briefly discussed in the introduction and is discussed at length in Diewert and Fox (2020).

[28] In the case where $q_{tn} = 0$, then $v_{tn} = 0$ as well and hence $p_{tn} \equiv v_{tn}/q_{tn}$ is not well defined in this case. In the case where $q_{tn} = 0$, we will assume that p_{tn} is a positive imputed price. Imputed prices will be discussed in section 19 below.

Define a generic *product weighting vector* as $\alpha \equiv [\alpha_1,...,\alpha_N]$. We assume that α has strictly positive components which sum to one; i.e., we assume that α satisfies:

(1) $\alpha\cdot 1_N = 1$; $\alpha >> 0_N$.

Let $p \equiv [p_1,...,p_N] >> 0_N$ be a strictly positive price vector. The corresponding *mean of order r of the prices p (with weights α)* or *CES price level*, $m_{r,\alpha}(p)$ is defined as follows:[29]

(2) $m_{r,\alpha}(p) \equiv [\Sigma_{n=1}^N \alpha_n p_n^r]^{1/r}$; $r \neq 0$;
$\equiv \prod_{n=1}^N (p_n)^{\alpha_n}$; $r = 0$.

It is useful to have a special notation for $m_{r,\alpha}(p)$ when $r = 1$:

(3) $p_\alpha \equiv \Sigma_{n=1}^N \alpha_n p_n = \alpha\cdot p$.

Thus p_α is an α weighted arithmetic mean of the prices $p_1,p_2,...,p_N$ and it can be interpreted as a *weighted Dutot price level*.[30]

From Schlömilch's (1858) Inequality,[31] we know that $m_{r,\alpha}(p) \geq m_{s,\alpha}(p)$ if $r \geq s$ and $m_{r,\alpha}(p) \leq m_{s,\alpha}(p)$ if $r \leq s$. However, we do not know how big the gaps are between these price levels for different r and s. When $r = 0$, $m_{0,\alpha}(p)$ becomes a weighted geometric mean or a *weighted Jevons* (1865) or *Cobb-Douglas price level* and it is of interest to know how much higher the weighted Dutot price level is than the corresponding weighted Jevons price level. Proposition 1 below provides an approximation to the gap between $m_{r,\alpha}(p)$ and $m_{1,\alpha}(p)$ for any r, including $r = 0$.

Define the *α weighted variance* of $p/p_\alpha \equiv [p_1/p_\alpha,...,p_N/p_\alpha]$ where p_α is defined by (3) as follows:[32]

(4) $\mathrm{Var}_\alpha(p/p_\alpha) \equiv \Sigma_{n=1}^N \alpha_n[(p_n/p_\alpha) - 1]^2$.

Proposition 1: Let $p >> 0_N$, $\alpha >> 0_N$ and $\alpha\cdot 1_N = 1$. Then $m_{r,\alpha}(p)/m_{1,\alpha}(p)$ is approximately equal to the following expression for any r:

(5) $m_{r,\alpha}(p)/m_{1,\alpha}(p) \approx 1 + (½)(r - 1)\mathrm{Var}_\alpha(p/p_\alpha)$

where $\mathrm{Var}_\alpha(p/p_\alpha)$ is defined by (4). The expression on the right hand side of (5) uses a second order Taylor series approximation to $m_{r,\alpha}(p)$ around the equal price point $p = p_\alpha 1_N$ where p_α is defined by (3).[33]

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[29] Hardy, Littlewood and Pólya (1934; 12-13) refer to this family of means or averages as *elementary weighted mean values* and study their properties in great detail. The function $m_{r,\alpha}(p)$ can also be interpreted as a *Constant Elasticity of Substitution (CES) unit cost function* if $r \leq 1$. The corresponding utility or production function was introduced into the economics literature by Arrow, Chenery, Minhas and Solow (1961). For additional material on CES functions, see Diewert (2021a), Feenstra (1994) and Diewert and Feenstra (2017).

[30] The ordinary Dutot (1738) price level for the period t prices p^t is defined as $p_D^t \equiv (1/N)\Sigma_{n=1}^N p_{tn}$. Thus it is equal to $m_{1,\alpha}(p^t)$ where $\alpha = (1/N)1_N$.

[31] See Hardy, Littlewood and Pólya (1934; 26) for a proof of this result.

[32] Note that the α weighted mean of p/p_α is equal to $\Sigma_{n=1}^N \alpha_n p_n/p_\alpha = 1$. Thus (4) defines the corresponding weighted variance.

Proof: Straightforward calculations show that the level, vector of first order partial derivatives and matrix of second order partial derivatives of $m_{r,\alpha}(p)$ evaluated at the equal price point $p = p_\alpha 1_N$ are equal to the following expressions: $m_{r,\alpha}(p_\alpha 1_N) = p_\alpha \equiv \alpha \cdot p$; $\nabla_p m_{r,\alpha}(p_\alpha 1_N) = \alpha$; $\nabla^2_{pp} m_{r,\alpha}(p_\alpha 1_N) = (p_\alpha)^{-1}(r-1)(\hat{\alpha} - \alpha\alpha^T)$ where $\hat{\alpha}$ is a diagonal N by N matrix with the elements of the column vector α running down the main diagonal and α^T is the transpose of the column vector α. Thus $\alpha\alpha^T$ is a rank one N by N matrix.

Thus the second order Taylor series approximation to $m_{r,\alpha}(p)$ around the point $p = p_\alpha 1_N$ is given by the following expression:

$$\begin{aligned}
(6)\ m_{r,\alpha}(p) &\approx p_\alpha + \alpha\cdot(p - p_\alpha 1_N) + (½)(p - p_\alpha 1_N)^T(p_\alpha)^{-1}(r-1)(\hat{\alpha} - \alpha\alpha^T)(p - p_\alpha 1_N) \\
&= p_\alpha + (½)(p_\alpha)^{-1}(r-1)(p - p_\alpha 1_N)^T(p_\alpha)^{-1}(\hat{\alpha} - \alpha\alpha^T)(p - p_\alpha 1_N) && \text{using (1) and (3)} \\
&= p_\alpha[1 + (½)(r-1)(p_\alpha)^{-2}(p - p_\alpha 1_N)^T(\hat{\alpha} - \alpha\alpha^T)(p - p_\alpha 1_N)] \\
&= m_{1,\alpha}(p)[1 + (½)(r-1)\mathrm{Var}_\alpha(p/p_\alpha)] && \text{using (2), (3) and (4).}
\end{aligned}$$

Q.E.D.

The approximation (6) also holds if r = 0. In this case, (6) becomes the following approximation:[34]

$$\begin{aligned}
(7)\ m_{0,\alpha}(p) &\equiv \prod_{n=1}^N (p_n)^{\alpha_n} \\
&\approx m_{1,\alpha}(p)[1 - (½)\mathrm{Var}_\alpha(p/p_\alpha)] \\
&= m_{1,\alpha}(p)\{1 - (½)\Sigma_{n=1}^N \alpha_n[(p_n/p_\alpha) - 1]^2\} && \text{using (4)} \\
&= [\Sigma_{n=1}^N \alpha_n p_n]\{1 - (½)\Sigma_{n=1}^N \alpha_n[(p_n/p_\alpha) - 1]^2\} && \text{using (2) for r = 1} \\
&\le \Sigma_{n=1}^N \alpha_n p_n.
\end{aligned}$$

Thus the bigger is the variation in the N prices $p_1,...,p_N$, the bigger will be $\mathrm{Var}_\alpha(p/p_\alpha)$ and the more the weighted arithmetic mean of the prices, $\Sigma_{n=1}^N \alpha_n p_n$, will be greater than the corresponding weighted geometric mean of the prices, $\prod_{n=1}^N (p_n)^{\alpha_n}$. Note that if all of the p_n are equal, then $\mathrm{Var}_\alpha(p/p_\alpha)$ will be equal to 0 and the approximations in (6) and (7) become exact equalities.

At this point, it is useful to define the Jevons (1865) and Dutot (1738) period t *price levels* for the prices in our window of observations, p_J^t and p_D^t, and the corresponding Jevons and Dutot *price indexes*, P_J^t and P_D^t, for t = 1,...,T:

$$(8)\ p_D^t \equiv \Sigma_{n=1}^N (1/N)p_{tn}\ ;$$
$$(9)\ p_J^t \equiv \Pi_{n=1}^N p_{tn}^{1/N}\ ;$$
$$(10)\ P_D^t \equiv p_D^t/p_D^1;$$
$$(11)\ P_J^t \equiv p_J^t/p_J^1 = \Pi_{n=1}^N (p_{tn}/p_{1n})^{1/N}.$$

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[33] For alternative approximations for the differences between mean of order r averages, see Vartia (1978; 278-279). Vartia's approximations involve variances of logarithms of prices, whereas our approximations involve variances of deflated prices. Our analysis is a variation on his pioneering analysis.

[34] Note that $m_{0,\alpha}(p)$ can be regarded as a weighted Jevons (1865) price level or a Cobb Douglas (1928) price level. Similarly, $p_\alpha \equiv m_{1,\alpha}(p)$ can be regarded as a weighted Dutot (1738) price level or a Leontief (1936) price level.

Thus the period t price index is simply the period t price level divided by the corresponding period 1 price level. Note that the Jevons price index can also be written as the geometric mean of the long term price ratios (p_{tn}/p_{1n}) between the period t prices relative to the corresponding period 1 prices.

The *weighted Dutot and Jevons period t price levels* using a weight vector α which satisfies the restrictions (1), $p_{D\alpha}{}^t$ and $p_{J\alpha}{}^t$, are defined by (12) and (13) and the corresponding *weighted Dutot and Jevons period t price indexes*, $P_{D\alpha}{}^t$ [35] and $P_{J\alpha}{}^t$,[36] are defined by (14) and (15) for t = 1,...,T:

(12) $p_{D\alpha}{}^t \equiv \Sigma_{n=1}{}^N \alpha_n p_{tn} = m_{1,\alpha}(p^t)$;
(13) $p_{J\alpha}{}^t \equiv \prod_{n=1}{}^N (p_{tn})^{\alpha_n} = m_{0,\alpha}(p^t)$;
(14) $P_{D\alpha}{}^t \equiv p_{D\alpha}{}^t/p_{D\alpha}{}^1 = \alpha \cdot p^t/\alpha \cdot p^1$;
(15) $P_{J\alpha}{}^t \equiv p_{J\alpha}{}^t/p_{J\alpha}{}^1 = \prod_{n=1}{}^N (p_{tn}/p_{1n})^{\alpha_n}$.

Obviously, (12)-(15) reduce to definitions (8)-(11) if $\alpha = (1/N)1_N$. We can use the approximation (7) for $p = p^1$ and $p = p^t$ in order to obtain the following approximate relationship between the weighted Dutot price index for period t, $P_{D\alpha}{}^t$, and the corresponding weighted Jevons index, $P_{J\alpha}{}^t$:

(16) $P_{J\alpha}{}^t \equiv p_{J\alpha}{}^t/p_{J\alpha}{}^1$; t = 1,...,T
$= m_{0,\alpha}(p^t)/m_{0,\alpha}(p^1)$ using (2) and (13)
$\approx m_{1,\alpha}(p^t)\{1 - (½)\Sigma_{n=1}{}^N \alpha_n[(p_{tn}/p_\alpha{}^t) - 1]^2\}/m_{1,\alpha}(p^1)\{1 - (½)\Sigma_{n=1}{}^N \alpha_n[(p_{1n}/p_\alpha{}^1) - 1]^2\}$
using (7) for $p = p^t$ and $p = p^1$ where $p_\alpha{}^t \equiv \alpha \cdot p^t$ and $p_\alpha{}^1 \equiv \alpha \cdot p^1$
$= P_{D\alpha}{}^t\{1 - (½)\Sigma_{n=1}{}^N \alpha_n[(p_{tn}/p_\alpha{}^t) - 1]^2\}/\{1 - (½)\Sigma_{n=1}{}^N \alpha_n[(p_{1n}/p_\alpha{}^1) - 1]^2\}$
$= P_{D\alpha}{}^t\{1 - (½)\mathrm{Var}_\alpha(p^t/p_\alpha{}^t)\}/\{1 - (½)\mathrm{Var}_\alpha(p^1/p_\alpha{}^1)\}$.

In the elementary index context where there are no trends in prices in diverging directions, it is likely that $\mathrm{Var}_\alpha(p^t/p_\alpha{}^t)$ will be approximately equal to $\mathrm{Var}_\alpha(p^1/p_\alpha{}^1)$.[37] Under this condition, *the weighted Jevons price index* $P_{J\alpha}{}^t$ *is likely to be approximately equal to the corresponding weighted Dutot price index*, $P_{D\alpha}{}^t$. Of course, this approximate equality result extends to the case where $\alpha = (1/N)1_N$ and so it is likely that the Dutot price indexes $P_D{}^t$ are approximately equal to their Jevons price index counterparts, $P_J{}^t$.[38] However, if the variance of the deflated period 1 prices is unusually large (small), then there will be a tendency for $P_J{}^t$ to exceed (to be less than) $P_D{}^t$ for t > 1.

At higher levels of aggregation where the products may not be very similar[39], it is likely that there will be *divergent trends in prices* over time. In this case, we can expect $\mathrm{Var}_\alpha(p^t/p_\alpha{}^t)$ to exceed $\mathrm{Var}_\alpha(p^1/p_\alpha{}^1)$. Thus using (16) under these circumstances leads to the likelihood that the weighted index $P_{J\alpha}{}^t$ will be significantly lower than $P_{D\alpha}{}^t$. Similarly, under the *diverging trends in prices*

[35] A weighted Dutot index can also be interpreted as a Lowe (1823) index.
[36] This type of index is frequently called a *Geometric Young index*; see Armknecht and Silver (2014; 4-5).
[37] Note that the vectors $p^t/p_\alpha{}^t$ and $p^1/p_\alpha{}^1$ are price vectors that are divided by their α weighted arithmetic means. Thus these vectors have eliminated general inflation between periods 1 and t.
[38] The same approximate inequalities hold for the weighted case. An approximation result similar to (16) for the equal weights case where $\alpha = (1/N)1_N$ was first obtained by Carruthers, Sellwood and Ward (1980; 25). See Diewert (2021b), equation (16).
[39] If the products are not very similar, then the Dutot index should not be used since it is not invariant to changes in the units of measurement.

hypothesis, we can expect the ordinary Jevons index P_J^t to be lower than the ordinary Dutot index P_D^t.[40]

We conclude this section by finding an approximate relationship between a CES price index and the corresponding weighted Dutot price index $P_{D\alpha}^t$. This approximation result assumes that econometric estimates for the parameters of the CES unit cost function $m_{r,\alpha}(p)$ defined by (2) are available so that we have estimates for the weighting vector α (which we assume satisfies the restrictions (1)) and the parameter r which we assume satisfies $r \leq 1$.[41] The *CES period t price levels* using a weight vector α which satisfies the restrictions (1) and an $r \leq 1$, $p_{CES\alpha,r}^t$, and the corresponding *CES period t price indexes*, $P_{CES\alpha,r}^t$, are defined as follows for t = 1,...,T:

$$(17)\ p_{CES\alpha,r}^t \equiv [\Sigma_{n=1}^N \alpha_n p_{tn}^r]^{1/r} = m_{r,\alpha}(p^t)\ ;$$
$$(18)\ P_{CES\alpha,r}^t \equiv p_{CES\alpha,r}^t/p_{CES\alpha,r}^1 = m_{r,\alpha}(p^t)/m_{r,\alpha}(p^1)\ .$$

Now use the approximation (6) for $p = p^1$ and $p = p^t$ in order to obtain the following approximate relationship between the weighted Dutot price index for period t, $P_{D\alpha}^t$, and the corresponding period t CES index, $P_{CES\alpha,r}^t$ for t = 1,...,T:

$$\begin{aligned}(19)\ P_{CES\alpha,r}^t &\equiv p_{CES\alpha,r}^t/p_{CES\alpha,r}^1\ ; \\ &= m_{r,\alpha}(p^t)/m_{r,\alpha}(p^1) && \text{using (18)} \\ &\approx [m_{1,\alpha}(p^t)/m_{1,\alpha}(p^1)][1 + (½)(r-1)\mathrm{Var}_\alpha(p^t/p_\alpha^t)]/[1 + (½)(r-1)\mathrm{Var}_\alpha(p^1/p_\alpha^1)] \\ &= P_{D\alpha}^t\{1 + (½)(r-1)\Sigma_{n=1}^N \alpha_n[(p_{tn}/p_\alpha^t) - 1]^2\}/\{1 + (½)(r-1)\Sigma_{n=1}^N \alpha_n[(p_{1n}/p_\alpha^1) - 1]^2\}\end{aligned}$$

where we used definitions (4), (12) and (14) to establish the last equality in (19). Again, in the elementary index context with no diverging trends in prices, we could expect $\mathrm{Var}_\alpha(p^t/p_\alpha^t) \approx \mathrm{Var}_\alpha(p^1/p_\alpha^1)$ for t = 2,...,T. Using this assumption about the approximate constancy of the (weighted) variance of the deflated prices over time, and using (16) and (19), we obtain the following approximations for t = 2,3,...,T:

$$(20)\ P_{CES\alpha,r}^t \approx P_{J\alpha}^t \approx P_{D\alpha}^t\ .$$

Thus under the assumption of approximately *constant variances* for deflated prices, the CES, weighted Jevons and weighted Dutot price indexes should approximate each other fairly closely, provided that the same weighting vector α is used in the construction of these indexes.[42]

The parameter r which appears in the definition of the CES unit cost function is related to the *elasticity of substitution* σ; i.e., it turns out that $\sigma = 1 - r$.[43] Thus as r takes on values from 1 to $-\infty$, σ will take on values from 0 to $+\infty$. In the case where the products are closely related, typical

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[40] Furthermore, as we shall see later, the Dutot index can be viewed as a fixed basket index where the basket is a vector of ones. Thus it is subject to substitution bias that will show up under the divergent price trends hypothesis.

[41] These restrictions imply that $m_{r,\alpha}(p)$ is a linearly homogeneous, nondecreasing and concave function of the price vector p. These restrictions must be satisfied if we apply the economic approach to price index theory.

[42] Again, the approximate relationship $P_{CES\alpha,r}^t \approx P_{D\alpha}^t$ may not hold if the variance of the prices in the base period, $\mathrm{Var}_\alpha(p^1/p_\alpha^1)$, is unusually large or small. Also under the diverging trends in prices assumption, $\mathrm{Var}_\alpha(p^t/p_\alpha^t)$ will tend to increase relative to $\mathrm{Var}_\alpha(p^1/p_\alpha^1)$ and the approximate equalities in (20) will become inequalities.

[43] See Feenstra (1994; 158) or equation (115) in Diewert (2021a).

estimates for σ range from 1 to 10. If we substitute $\sigma = 1 - r$ into the approximations (19), we obtain the following approximations for t = 1,...,T:

$$(21)\ P_{CES\alpha,r}^{t} \approx P_{D\alpha}^{t}\,[1 - (½)\sigma Var_{\alpha}(p^{t}/p_{\alpha}^{t})]/[1 - (½)\sigma Var_{\alpha}(p^{1}/p_{\alpha}^{1})].$$

The approximations in (21) break down for large and positive σ (or equivalently, for very negative r); i.e., the expressions in square brackets on the right hand sides of (21) will pass through 0 and become meaningless as σ becomes very large. The approximations become increasingly accurate as σ approaches 0 (or as r approaches 1). Of course, the approximations also become more accurate as the dispersion of prices within a period becomes smaller. For σ between 0 and 1 and with "normal" dispersion of prices, the approximations in (21) should be reasonably good. However, as σ becomes larger, the expressions in square brackets will become closer to 0 and the approximations in (21) will become more volatile and less accurate as σ increases from an initial 0 value.

If the products in the aggregate are not very similar, it is more likely that there will be *divergent trends in prices* over time and in this case, we can expect $Var_{\alpha}(p^{t}/p_{\alpha}^{t})$ to exceed $Var_{\alpha}(p^{1}/p_{\alpha}^{1})$. In this case, the approximate *equalities* (20) will no longer hold. In the case where the elasticity of substitution σ is greater than 1 (so $r < 0$) and $Var_{\alpha}(p^{t}/p_{\alpha}^{t}) > Var_{\alpha}(p^{1}/p_{\alpha}^{1})$, we can expect that $P_{CES\alpha,r}^{t} < P_{D\alpha}^{t}$ and the gaps between these two indexes will grow bigger over time as $Var_{\alpha}(p^{t}/p_{\alpha}^{t})$ grows larger than $Var_{\alpha}(p^{1}/p_{\alpha}^{1})$.

In the following section, we will use the mean of order r function to aggregate the price ratios p_{tn}/p_{1n} into an aggregate price index for period t directly; i.e., we will not construct *price levels* as a preliminary step in the construction of a *price index*.

3. Using Means of Order r to Aggregate Price Ratios

In the previous section, we compared various elementary indexes using approximate relationships between price levels constructed by using means of order r to construct the aggregate *price levels*. In this section, we will develop approximate relationships between price indexes constructed by using means of order r to aggregate over *price ratios*.

In what follows, it is assumed that the weight vector α satisfies conditions (1); i.e., $\alpha >> 0_N$ and $\alpha \cdot 1_N = 1$. Define the *mean of order r price index for period t* (relative to period 1), $P_{r,\alpha}^{t}$, as follows for t = 1,...,T:

$$\begin{aligned}(22)\ P_{r,\alpha}^{t} &\equiv [\Sigma_{n=1}^{N} \alpha_n (p_{tn}/p_{1n})^{r}]^{1/r}\ ; && r \neq 0;\\ &\equiv \textstyle\prod_{n=1}^{N} (p_{tn}/p_{1n})^{\alpha_n} \ ; && r = 0.\end{aligned}$$

When $r = 1$ and $\alpha = (1/N)1_N$, then $P_{r,\alpha}^{t}$ becomes the *fixed base Carli* (1804) *price index* (for period t relative to period 1), P_C^{t}, defined as follows for t = 1,...,T:

$$(23)\ P_C^{t} \equiv \Sigma_{n=1}^{N} (1/N)(p_{tn}/p_{1n}).$$

With a general α and r = 1, $P_{r,\alpha}^t$ becomes the *fixed base weighted Carli price index*, $P_{C\alpha}^t$,[44] defined as follows for t = 1,...,T:

(24) $P_{C\alpha}^t \equiv \Sigma_{n=1}^N \alpha_n(p_{tn}/p_{1n})$.

Using (24), it can be seen that the α weighted mean of the period t long term price ratios p_{tn}/p_{1n} divided by $P_{C\alpha}^t$ is equal to 1; i.e., we have for t = 1,...,T:

(25) $\Sigma_{n=1}^N \alpha_n(p_{tn}/p_{1n}P_{C\alpha}^t) = 1$.

Denote the α weighted variance of the deflated period t price ratios $p_{tn}/p_{1n}P_{C\alpha}^t$ as $Var_\alpha(p^t/p^1P_{C\alpha}^t)$ and define it as follows for t = 1,...,T:

(26) $Var_\alpha(p^t/p^1P_{C\alpha}^t) \equiv \Sigma_{n=1}^N \alpha_n[(p_{tn}/p_{1n}P_{C\alpha}^t) - 1]^2$.

Proposition 2: Let $p >> 0_N$, $\alpha >> 0_N$ and $\alpha\cdot 1_N = 1$. Then $P_{r,\alpha}^t/P_{1,\alpha}^t = P_{r,\alpha}^t/P_{C\alpha}^t$ is approximately equal to the following expression for any r for t = 1,...,T:

(27) $P_{r,\alpha}^t/P_{C\alpha}^t \approx 1 + (½)(r - 1)Var_\alpha(p^t/p^1P_{C\alpha}^t)$

where $P_{r,\alpha}^t$ is the mean of order r price index (with weights α) defined by (22), $P_{C\alpha}^t$ is the α weighted Carli index defined by (24) and $Var_\alpha(p^t/p^1P_{C\alpha}^t)$ is the α weighted variance of the deflated long term price ratios $(p_{tn}/p_{1n})/P_{C\alpha}^t$ defined by (26).

Proof: Replace the vector p in Proposition 1 by the vector $[p_{t1}/p_{11},p_{t2}/p_{12},...,p_{tN}/p_{1N}]$.[45] Then the ratio $m_{r,\alpha}(p)/m_{1,\alpha}(p)$ which appears on the left hand side of (5) becomes the ratio $P_{r,\alpha}^t/P_{1,\alpha}^t = P_{r,\alpha}^t/P_{C\alpha}^t$ using definitions (22) and (24). The terms p_α and $Var_\alpha(p/p_\alpha)$ which appear on the right hand side of (5) become $P_{C\alpha}^t$ and $Var_\alpha(p^t/p^1P_{C\alpha}^t)$ respectively. With these substitutions, (5) becomes (27) and we have established Proposition 2. Q.E.D.

It is useful to look at the special case of (27) when r = 0. In this case, using definitions (22) and (15), we can establish the following equalities for t = 1,...,T:

(28) $P_{0,\alpha}^t \equiv \prod_{n=1}^N (p_{tn}/p_{1n})^{\alpha_n} = P_{J\alpha}^t$

where $P_{J\alpha}^t$ is the period t *weighted Jevons* or *Cobb Douglas price index* defined by (15) in the previous section.[46] Thus when r = 0, the approximations defined by (27) become the following approximations for t = 1,...,T:

(29) $P_{J\alpha}^t/P_{C\alpha}^t \approx 1 - (½)Var_\alpha(p^t/p^1P_{C\alpha}^t)$.

Thus the bigger is the α weighted variance of the deflated period t long term price ratios, $(p_{t1}/p_{11})/P_{C\alpha}^t$,..., $(p_{tN}/p_{1N})/P_{C\alpha}^t$, the more the period t weighted Carli index $P_{C\alpha}^t$ will exceed the corresponding period t weighted Jevons index $P_{J\alpha}^t$.

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[44] This type of index is due to Arthur Young (1812; 72) and so we could call this index the *Young index*, $P_{Y\alpha}^t$.
[45] In Proposition 1, some prices in either period could be 0. However, Proposition 2 requires that all period 1 prices be positive.
[46] Again, recall that Armknecht and Silver (2014; 4) call this index the Geometric Young index.

When $\alpha = (1/N)1_N$, the approximations (29) become the following approximate relationships between the period t *Carli index* P_C^t defined by (23) and the period t *Jevons index* P_J^t defined by (11) for t = 1,...,T:[47]

$$(30)\ P_J^t/P_C^t \approx 1 - (½)\mathrm{Var}_{(1/N)1}(p^t/p^1P_C^t)$$
$$= 1 - (½)\Sigma_{n=1}^N (1/N)[(p_{tn}/p_{1n}P_C^t) - 1]^2.$$

Thus the Carli price indexes P_C^t will exceed their Jevons counterparts P_J^t (unless $p^t = \lambda_t p^1$ in which case prices in period t are proportional to prices in period 1 and in this case, $P_C^t = P_J^t$).[48] This is an important result, since from an axiomatic perspective, the Jevons price index has much better properties than the corresponding Carli indexes[49] and in particular, typically *chaining Carli indexes will lead to large upward biases as compared to their Jevons counterparts*.

The results in this section can be summarized as follows: holding the weight vector α constant, the weighted Jevons price index for period t, $P_{J\alpha}^t$ will lie below the corresponding weighted Carli index, $P_{C\alpha}^t$, (unless all prices move in a proportional manner, in which case $P_{J\alpha}^t$ will equal $P_{C\alpha}^t$) with the gap growing as the α weighted variance of the deflated price ratios, $(p_{t1}/p_{11})/P_{C\alpha}^t$,..., $(p_{tN}/p_{1N})/P_{C\alpha}^t$, increases.[50]

In the following section, we turn our attention to weighted price indexes where the weights are not exogenous constants but depend on observed sales or expenditure shares.

4. Relationships between Some Share Weighted Price Indexes

In this section (and in subsequent sections), we will look at comparisons between price indexes that use information on the observed expenditure or sales shares of products in addition to price information. Recall that $s_{tn} \equiv p_{tn}q_{tn}/p^t{\cdot}q^t$ for n = 1,...,N and t = 1,...,T.

The *fixed base Laspeyres* (1871) *price index* for period t, P_L^t, is defined as the following base period share weighted *arithmetic* average of the price ratios, p_{tn}/p_{1n}, for t = 1,...,T:

$$(31)\ P_L^t \equiv \Sigma_{n=1}^N s_{1n}(p_{tn}/p_{1n}).$$

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[47] Results that are essentially equivalent to (30) were first obtained by Dalén (1992) and Diewert (1995). The approximations in (27) and (29) for weighted indexes are new. Vartia and Suoperä (2018; 5) derived alternative approximations. The analysis in this section is similar to Vartia's (1978; 276-289) analysis of Fisher's (1922) five-tined fork.

[48] From Schlömilch's Inequality, we know that P_C is always equal to or greater than P_J; the approximate result (30) provides an indication of the size of the gap between the two indexes.

[49] See Diewert (1995) (2021b) and Reinsdorf (2007) on the axiomatic approach to equally weighted elementary indexes. The Jevons index emerges as the best index from the viewpoint of the axiomatic approach.

[50] Since the Jevons price index has the best axiomatic properties, this result implies that CPI compilers should avoid the use of the Carli index in the construction of a CPI. This advice goes back to Fisher (1922; 29-30). Since the Dutot index will approximate the corresponding Jevons index provided that the products are similar and there are no systematic divergent trends in prices, Dutot indexes can be satisfactory at the elementary level. If the products are not closely related, Dutot indexes become problematic since they are not invariant to changes in the units of measurement. Moreover, in the case of nonsimilar products, divergent trends in prices become more probable and, using (16), the Dutot index will tend to be above the corresponding Jevons index.

It can be seen that P_L^t is a weighted Carli index $P_{C\alpha}^t$ of the type defined by (24) in the previous section where $\alpha \equiv s^1 \equiv [s_{11},s_{12},...,s_{1N}]$. We will compare P_L^t with its weighted geometric mean counterpart, P_{GL}^t, which is a weighted Jevons index $P_{J\alpha}^t$ where the weight vector is $\alpha = s^1$. Thus the logarithm of the *fixed base Geometric Laspeyres price index* is defined as follows for t = 1,...,T:[51]

$$(32)\ \ln P_{GL}^t \equiv \Sigma_{n=1}^N s_{1n} \ln(p_{tn}/p_{1n}).$$

Since P_{GL}^t and P_L^t are weighted geometric and arithmetic means of the price ratios p_{tn}/p_{1n} (using the weights in the period 1 share vector s^1), Schlömilch's inequality implies that $P_{GL}^t \le P_L^t$ for t = 1,...,T. The inequalities (29), with $\alpha = s^1$, give us approximations to the gaps between the $P_{GL}^t = P_{J\alpha}^t$ and the $P_{C\alpha}^t = P_L^t$. Thus we have the following approximate equalities for $\alpha = s^1$ and t = 1,...,T:

$$(33)\ P_{GL}^t/P_L^t \approx 1 - (½)\mathrm{Var}_\alpha(p^t/p^1P_L^t) = 1 - (½)\Sigma_{n=1}^N s_{1n}[(p_{tn}/p_{1n}P_L^t) - 1]^2.$$

The *fixed base Paasche* (1874) *price index* for period t, P_P^t, is defined as the following period t share weighted *harmonic* average of the price ratios, p_{tn}/p_{1n}, for t = 1,...,T:

$$(34)\ P_P^t \equiv [\Sigma_{n=1}^N s_{tn}(p_{tn}/p_{1n})^{-1}]^{-1}.$$

We will compare P_P^t with its weighted geometric mean counterpart, P_{GP}^t, which is a weighted Jevons index $P_{J\alpha}^t$ where the weight vector is $\alpha = s^t$. The logarithm of the *fixed base Geometric Paasche price index* is defined as follows for t = 1,...,T:

$$(35)\ \ln P_{GP}^t \equiv \Sigma_{n=1}^N s_{tn} \ln(p_{tn}/p_{1n}).$$

Since P_{GP}^t and P_P^t are weighted geometric and harmonic means of the price ratios p_{tn}/p_{1n} (using the weights in the period t share vector s^t), Schlömilch's inequality implies that $P_P^t \le P_{GP}^t$ for t = 1,...,T. However, we cannot apply the inequalities (29) directly to give us an approximation to the *size* of the gap between P_{GP}^t and P_P^t. Viewing definition (34), it can be seen that the reciprocal of P_P^t is a period t share weighted average of the reciprocals of the long term price ratios, p_{11}/p_{t1}, p_{12}/p_{t2}, ..., p_{1N}/p_{tN}. Thus using definition (34), we have the following equations and inequalities for $\alpha = s^t$ and t = 1,...,T:

$$\begin{aligned}(36)\ [P_P^t]^{-1} &= \Sigma_{n=1}^N s_{tn}(p_{1n}/p_{tn}) \\ &\ge \textstyle\prod_{n=1}^N (p_{1n}/p_{tn})^{s_{tn}} \\ &= [P_{GP}^t]^{-1} && \text{using definitions (35)}\end{aligned}$$

where the inequalities in (36) follow from Schlömilch's inequality; i.e., a weighted arithmetic mean is always equal to or greater than the corresponding weighted geometric mean. Note that the first equation in (36) implies that the period t share weighted mean of the reciprocal price ratios, p_{1n}/p_{tn}, is equal to the reciprocal of P_P^t. Now adapt the approximate equalities (29) in order to establish the following approximate equalities for t = 1,...,T:

[51] Vartia (1978; 272) used the terms "geometric Laspeyres" and "geometric Paasche" to describe the indexes defined by (32) and (35).

(37) $[P_{GP}{}^t]^{-1}/[P_P{}^t]^{-1} \approx 1 - (½)\Sigma_{n=1}{}^N s_{tn}[(p_{1n}/p_{tn}[P_P{}^t]^{-1}) - 1]^2$.

The approximate equalities (37) may be rewritten as follows for t = 1,...,T:

(38) $P_{GP}{}^t \approx P_P{}^t/\{1 - (½)\Sigma_{n=1}{}^N s_{tn}[(p_{1n}P_P{}^t/p_{tn}) - 1]^2\}$.

Thus for t = 1,...,T, we have $P_{GP}{}^t \geq P_P{}^t$ (and the approximate equalities (38) measure the gaps between these indexes) and $P_{GL}{}^t \leq P_L{}^t$ (and the approximate equalities (33) measure the gaps between these indexes). Later we will show that the inequalities $P_{GP}{}^t \leq P_{GL}{}^t$ are likely if the N products are close substitutes for each other.

Suppose that prices in period t are proportional to the corresponding prices in period 1 so that $p^t = \lambda_t p^1$ where λ_t is a positive scalar. Then it is straightforward to show that $P_P{}^t = P_{GP}{}^t = P_{GL}{}^t = P_L{}^t = \lambda_t$ and the implicit error terms for equation t in (33) and (38) are equal to 0.

Define the period t fixed base Fisher (1922) and Törnqvist Theil price indexes, $P_F{}^t$ and $P_T{}^t$, as the following geometric means for t = 1,...,T:

(39) $P_F{}^t \equiv [P_L{}^t P_P{}^t]^{1/2}$;
(40) $P_T{}^t \equiv [P_{GL}{}^t P_{GP}{}^t]^{1/2}$.

Thus $P_F{}^t$ is the geometric mean of the period t fixed base Laspeyres and Paasche price indexes while $P_T{}^t$ is the geometric mean of the period t fixed base geometric Laspeyres and geometric Paasche price indexes. Now use the approximate equalities in (33) and (38) and substitute these equalities into (40) in order to obtain the following approximate equalities between $P_T{}^t$ and $P_F{}^t$ for t = 1,...,T:

$$\begin{aligned}(41)\ P_T{}^t &\equiv [P_{GL}{}^t P_{GP}{}^t]^{1/2} \\ &\approx [P_L{}^t P_P{}^t]^{1/2}\,\varepsilon(p^1,p^t,s^1,s^t) \\ &= P_F{}^t\,\varepsilon(p^1,p^t,s^1,s^t)\end{aligned}$$

where the approximation error function $\varepsilon(p^1,p^t,s^1,s^t)$ is defined as follows for t = 1,...,T:

(42) $\varepsilon(p^1,p^t,s^1,s^t) \equiv \{1 - (½)\Sigma_{n=1}{}^N s_{1n}[(p_{tn}/p_{1n}P_L{}^t) - 1]^2\}^{1/2}/\{1 - (½)\Sigma_{n=1}{}^N s_{tn}[(p_{1n}P_P{}^t/p_{tn}) - 1]^2\}^{1/2}$.

Thus $P_T{}^t$ is approximately equal to $P_F{}^t$ for t = 1,...,T. But how good are these approximations? We know from Diewert (1978) that $P_T{}^t = P_T(p^1,p^t,s^1,s^t)$ approximates $P_F{}^t = P_F(p^1,p^t,s^1,s^t)$ to the second order around any point where $p^t = p^1$ and $s^t = s^1$.[52] Since the approximations in (33) and (38) are also second order approximations, it is likely that the approximation given by (41) is fairly good.[53]

[52] This result can be generalized to the case where $p^t = \lambda p^1$ and $s^t = s^1$.
[53] However, the Diewert (1978) second order approximation is different from the present second order approximations that are derived from Proposition 2. Thus the closeness of $\varepsilon(p^1,p^t,s^1,s^t)$ to 1 depends on the closeness of the Diewert second order approximation of $P_T{}^t$ to $P_F{}^t$ and the closeness of the second order approximations that were used in (33) and (38), which use different Taylor series approximations. Vartia and Suoperä (2018) used alternative Taylor series approximations to obtain relationships between various indexes.

In general, if the products are highly substitutable and if prices and shares trend in opposite directions, then we expect that the base period share weighted variance $\Sigma_{n=1}^{N} s_{1n}[(p_{tn}/p_{1n}P_L^t) - 1]^2$ and the current period share weighted variance $\Sigma_{n=1}^{N} s_{tn}[(p_{1n}P_P^t/p_{tn}) - 1]^2$ will increase as t increases. It appears that the second variance expression increases more than the first one because the change in expenditure shares from s_{1n} to s_{tn} tends to magnify the squared differences $[(p_{1n}P_P^t/p_{tn}) - 1]^2$. Thus as say p_{tn} increases and the difference $(p_{1n}P_P^t/p_{tn}) - 1$ decreases, the share s_{tn} will become smaller, and this decreasing share weight s_{tn} will lead to a further shrinkage of the term $s_{tn}[(p_{1n}P_P^t/p_{tn}) - 1]^2$. On the other hand, if p_{tn} decreases substantially, the difference $(p_{1n}P_P^t/p_{tn}) - 1$ will substantially increase and the share s_{tn} will become larger, and this increasing share weight s_{tn} will further magnify the term $s_{tn}[(p_{1n}P_P^t/p_{tn}) - 1]^2$. For large changes in prices, the magnification effects will tend to be more important than the shrinkage effects of changing expenditure shares. This overall *share magnification effect* does not occur for the base period share weighted variance $\Sigma_{n=1}^{N} s_{1n}[(p_{tn}/p_{1n}P_L^t) - 1]^2$. Thus if the products are highly substitutable and there are large divergent trends in prices, P_T will tend to increase relative to P_F as time increases under these conditions. The more substitutable the products are, the greater will be this tendency.

Our tentative conclusion at this point is that the approximations defined by (33), (38) and (41) are good enough to provide rough estimates of the differences in the six price indexes involved in these approximate equalities. In an empirical example using scanner data, Diewert (2018) found that the variance terms on the right hand sides of (38) tended to be larger than the corresponding variances on the right hand sides of (33) and these differences led to a tendency for the fixed base Fisher price indexes P_F^t to be slightly smaller than the corresponding fixed base Törnqvist Theil price indexes P_T^t.[54]

We conclude this section by developing an exact relationship between the geometric Laspeyres and Paasche price indexes. Using definitions (32) and (35) for the logarithms of these indexes, we have the following exact decomposition for the logarithmic difference between these indexes for t = 1,...,T:[55]

$$\begin{aligned}(43)\quad \ln P_{GP}^t - \ln P_{GL}^t &= \Sigma_{n=1}^{N} s_{tn} \ln(p_{tn}/p_{1n}) - \Sigma_{n=1}^{N} s_{1n} \ln(p_{tn}/p_{1n}) \\ &= \Sigma_{n=1}^{N} [s_{tn} - s_{1n}][\ln p_{tn} - \ln p_{1n}].\end{aligned}$$

Define the vectors $\ln p^t \equiv [\ln p_{t1},\ln p_{t2},...,\ln p_{tN}]$ for t = 1,...,T. It can be seen that the right hand side of equation t in (43) is equal to $[s^t - s^1]\cdot[\ln p^t - \ln p^1]$, the inner product of the vectors $x \equiv s^t - s^1$ and $y \equiv \ln p^t - \ln p^1$. Let x^* and y^* denote the arithmetic means of the components of the vectors x and y. Note that $x^* \equiv (1/N)1_N\cdot x = (1/N)1_N\cdot[s^t - s^1] = (1/N)[1 - 1] = 0$. The covariance between x and y is defined as $\mathrm{Cov}(x,y) \equiv (1/N)[x - x^*1_N]\cdot[y - y^*1_N] = (1/N)\, x\cdot y - x^*y^* = (1/N)\, x\cdot y$[56] since x^* is equal to 0. Thus the right hand side of (43) is equal to $N\,\mathrm{Cov}(x,y) = N\,\mathrm{Cov}(s^t - s^1,\ln p^t - \ln p^1)$; i.e., the right hand side of (43) is equal to N times the covariance of the long term share difference vector, $s^t - s^1$, with the long term log price difference vector, $\ln p^t - \ln p^1$. Hence if this covariance is positive, then $\ln P_{GP}^t - \ln P_{GL}^t > 0$ and $P_{GP}^t > P_{GL}^t$. If this covariance is negative, then $P_{GP}^t < P_{GL}^t$.

[54] Vartia and Suoperä (2018) also found a tendency for the Fisher price index to lie slightly below their Törnqvist counterparts in their empirical work.

[55] Vartia and Suoperä (2018; 26) derived this result and noticed that the right hand side of (43) could be interpreted as a covariance. They also developed several alternative exact decompositions for the difference $\ln P_{GP}^t - \ln P_{GL}^t$. Their paper also develops a new theory of "excellent" index numbers.

[56] This equation is the *covariance identity* that was first used by Bortkiewicz (1923) to show that normally the Paasche price index is less than the corresponding Laspeyres index.

We argue below that for the case where the N products are close substitutes, it is likely that the covariances on the right hand side of equations (43) are negative for $t > 1$.

Suppose that the observed price and quantity data are approximately consistent with purchasers having identical Constant Elasticity of Substitution preferences. CES preferences are dual to the CES unit cost function $m_{r,\alpha}(p)$, which is defined by (2) above, where α satisfies (1) and $r \leq 1$. It can be shown[57] that the sales share for product n in a period where purchasers face the strictly positive price vector $p \equiv [p_1,...,p_N]$ is the following share:

(44) $s_n(p) \equiv \alpha_n p_n^r/\Sigma_{i=1}^N \alpha_i p_i^r$; $n = 1,...,N$.

Upon differentiating $s_n(p)$ with respect to p_n, we find that the following relations hold:

(45) $\partial \ln s_n(p)/\partial \ln p_n = r[1-s_n(p)]$; $n = 1,...,N$.

Thus $\partial \ln s_n(p)/\partial \ln p_n < 0$ if $r < 0$ (or equivalently, if the elasticity of substitution $\sigma \equiv 1 - r$ is greater than 1) and $\partial \ln s_n(p)/\partial \ln p_n > 0$ if r satisfies $0 < r < 1$ (or equivalently, if the elasticity of substitution satisfies $0 < \sigma < 1$).[58] If we are aggregating prices at the first stage of aggregation where the products are close substitutes and purchasers have common CES preferences, then it is likely that the elasticity of substitution is greater than 1 and hence as the price of product n decreases, it is likely that the share of that product will increase. Hence we expect the terms $[s_{tn} - s_{1n}][\ln p_{tn} - \ln p_{1n}]$ to be predominantly negative; i.e., if p_{1n} is unusually low, then $\ln p_{tn} - \ln p_{1n}$ is likely to be positive and $s_{tn} - s_{1n}$ is likely to be negative. On the other hand, if p_{tn} is unusually low, then $\ln p_{tn} - \ln p_{1n}$ is likely to be negative and $s_{tn} - s_{1n}$ is likely to be positive. Thus for closely related products, we expect the covariances on the right hand sides of (43) to be negative and for P_{GP}^t to be less than P_{GL}^t. We can combine this inequality with our previously established inequalities to conclude that for closely related products, it is likely that $P_P^t < P_{GP}^t < P_T^t < P_{GL}^t < P_L^t$. On the other hand, if we are aggregating at higher levels of aggregation, then it is likely that the elasticity of substitution is in the range $0 < \sigma < 1$,[59] and in this case, the covariances on the right hand sides of (43) will tend to be positive and hence in this case, it is likely that $P_{GP}^t > P_{GL}^t$. We also have the inequalities $P_P^t < P_{GP}^t$ and $P_{GL}^t < P_L^t$ in this case.[60]

We turn now to some relationships between weighted and unweighted (i.e., equally weighted) geometric price indexes.

5. Relationships between the Jevons, Geometric Laspeyres, Geometric Paasche and Törnqvist Price Indexes

[57] See equations (110) in Diewert (2021a) or Diewert and Feenstra (2017).

[58] Thus define product n to be a *strong substitute* with all other products if $\partial \ln s_n(p)/\partial \ln p_n < 0$ and to be a *weak substitute* if $\partial \ln s_n(p)/\partial \ln p_n > 0$.

[59] See Shapiro and Wilcox (1997) who found that $\sigma = 0.7$ fit the US data well at higher levels of aggregation. See also Armknecht and Silver (2014; 9) who noted that estimates for σ tend to be greater than 1 at the lowest level of aggregation and less than 1 at higher levels of aggregation.

[60] See Vartia (1978; 276-290) for a similar discussion about the relationships between P_L^t, P_P^t, P_F^t, P_{GL}^t, P_{GP}^t and P_T^t. Vartia extended the discussion to include period 1 and period t share weighted harmonic averages of the price ratios, p_{tn}/p_{1n}. See also Armknecht and Silver (2014; 10) for a discussion on how weighted averages of the above indexes could approximate a superlative index at higher levels of aggregation.

In this section, we will investigate how close the unweighted Jevons index P_J^t is to the geometric Laspeyres P_{GL}^t, geometric Paasche P_{GP}^t and Törnqvist P_T^t price indexes.

We first investigate the difference between the logarithms of P_{GL}^t and P_J^t. Using the definitions for these indexes, we have the following log differences for t = 1,...,T:

$$(46)\ \ln P_{GL}^t - \ln P_J^t = \Sigma_{n=1}^N [s_{1n} - (1/N)][\ln p_{tn} - \ln p_{1n}]$$
$$= N\mathrm{Cov}(s^1 - (1/N)1_N, \ln p^t - \ln p^1)$$
$$\equiv \varepsilon_t.$$

In the elementary index context where the N products are close substitutes and product shares in period 1 are close to being equal, it is likely that ε_t is positive; i.e., if ln p_{1n} is unusually low, then s_{1n} is likely to be unusually high and thus it is likely that $s_{1n} - (1/N) > 0$ and $\ln p_{tn} - \ln p_{1n}$ minus the mean of the log ratios $\ln(p_{tn}/p_{1n})$ is likely to be greater than 0 and hence ε_t is likely to be greater than 0, implying that $P_{GL}^t > P_J^t$. However, if N is small and the shares have a high variance and if product n goes on sale in period 1, then we cannot assert that s_{1n} is likely to be greater than 1/N and hence we cannot be confident that ε_t is likely to be greater than 0 and hence we cannot predict with certainty that P_{GL}^t will be greater than P_J^t.

There are three simple sets of conditions that will imply that $P_{GL}^t = P_J^t$: (i) the covariance on the right hand side of (46) equals 0; i.e., $\mathrm{Cov}(s^1 - (1/N)1_N, \ln p^t - \ln p^1) = 0$; (ii) period t price proportionality; i.e., $p^t = \lambda_t p^1$ for some $\lambda_t > 0$; (iii) equal sales shares in period 1; i.e., $s^1 = (1/N)1_N$.

Now look at the difference between the logarithms of P_{GP}^t and P_J^t. Using the definitions for these indexes, for t = 1,...,T, we have:

$$(47)\ \ln P_{GP}^t - \ln P_J^t = \Sigma_{n=1}^N [s_{tn} - (1/N)][\ln p_{tn} - \ln p_{1n}]$$
$$= N\mathrm{Cov}(s^t - (1/N)1_N, \ln p^t - \ln p^1)$$
$$\equiv \eta_t.$$

In the elementary index context where the N products are close substitutes and the shares s^t are close to being equal, then it is likely that η_t is negative; i.e., if ln p_{tn} is unusually low, then s_{tn} is likely to be unusually high and thus it is likely that $s_{tn} - (1/N) > 0$ and $\ln p_{tn} - \ln p_{1n}$ minus the mean of the log ratios $\ln(p_{tn}/p_{1n})$ is likely to be less than 0 and hence η_t is likely to be less than 0 implying that $P_{GP}^t < P_J^t$. However if N is small and the period t shares s^t are not close to being equal, then again, we cannot confidently predict the sign of the covariance in (47).

Again, there are three simple sets of conditions that will imply that $P_{GP}^t = P_J^t$: (i) the covariance on the right hand side of (47) equals 0; i.e., $\mathrm{Cov}(s^t - (1/N)1_N, \ln p^t - \ln p^1) = 0$; (ii) period t price proportionality; i.e., $p^t = \lambda_t p^1$ for some $\lambda_t > 0$; (iii) equal sales shares in period t; i.e., $s^t = (1/N)1_N$.

Using the definitions for P_T^t and P_J^t, the log difference between these indexes is equal to the following expression for t = 1,...,T:

$$(48)\ \ln P_T^t - \ln P_J^t = \Sigma_{n=1}^N [(½)s_{tn} + (½)s_{1n} - (1/N)][\ln p_{tn} - \ln p_{1n}]$$
$$= N\mathrm{Cov}[(½)s^t + (½)s^1 - (1/N)1_N, \ln p^t - \ln p^1]$$
$$= (N/2)\mathrm{Cov}(s^t - (1/N)1_N, \ln p^t - \ln p^1) + (N/2)\mathrm{Cov}(s^1 - (1/N)1_N, \ln p^t - \ln p^1)$$
$$= (½)\varepsilon_t + (½)\eta_t.$$

As usual, there are three simple sets of conditions that will imply that $P_T{}^t = P_J{}^t$: (i) the covariance on the right hand side of (48) equals 0; i.e., $\mathrm{Cov}[(½)s^t + (½)s^1 - (1/N)1_N, \ln p^t - \ln p^1] = 0 = (½)\varepsilon_t + (½)\eta_t$ or equivalently, $\mathrm{Cov}(s^t - (1/N)1_N, \ln p^t - \ln p^1) = - \mathrm{Cov}(s^1 - (1/N)1_N, \ln p^t - \ln p^1)$; (ii) period t price proportionality; i.e., $p^t = \lambda_t p^1$ for some $\lambda_t > 0$; (iii) the arithmetic average of the period 1 and t sales shares are all equal to 1/N; i.e., $(½)s^t + (½)s^1 = (1/N)1_N$.

If the trend deflated prices p_{tn}/λ_t are distributed *independently* across time and *independently* of the sales shares s_{tn}, then it can be seen that the expected values of the ε_t and η_t will be 0 and hence $P_T{}^t \approx P_J{}^t$ for t = 1,...,T. Thus it can be the case that the ordinary Jevons price index is able to provide an adequate approximation to the superlative Törnqvist price index in the elementary price index context. However, if the shares are trending and if prices are trending in divergent directions, then $P_J{}^t$ will not be able to approximate $P_T{}^t$.

In the general case, we expect $P_T{}^t$ to be less than $P_J{}^t$. The mean of the average shares for product n in periods 1 and t, $(½)s_{tn} + (½)s_{1n}$, is 1/N. Define the means of the log prices in period t as $\ln p_t{}^* \equiv (1/N)\Sigma_{n=1}{}^N \ln p_{tn}$ for t = 1,...,T. Note that $p_t{}^*$ is the geometric mean of the period t prices. Thus using the first line of (49) and the covariance identity, we have:

$$(49)\ \ln P_T{}^t - \ln P_J{}^t = \Sigma_{n=1}{}^N [(½)s_{tn} + (½)s_{1n} - (1/N)][\ln p_{tn} - \ln p_{1n}]$$
$$= \Sigma_{n=1}{}^N [(½)s_{tn} + (½)s_{1n} - (1/N)][\ln p_{tn} - \ln p_{1n} - \ln p_t{}^* + \ln p_1{}^*]$$
$$= \Sigma_{n=1}{}^N [(½)s_{tn} + (½)s_{1n} - (1/N)][\ln(p_{tn}/p_t{}^*) - \ln(p_{1n}/p_1{}^*)].$$

The second line in (49) follows from the first line because $\Sigma_{n=1}{}^N [(½)s_{tn} + (½)s_{1n} - (1/N)] = 0$ so if these N terms are multiplied by a constant, the resulting sum of terms will still equal 0. Define the *deflated price* for product n in period t as $p_{tn}/p_t{}^*$ for t = 1,...,T. Assume that the products are highly substitutable. Suppose that the deflated price of product n goes down between periods 1 and t so that $\ln(p_{tn}/p_t{}^*) - \ln(p_{1n}/p_1{}^*)$ is negative. Under these conditions, there will be a tendency for the average expenditure share for product n, $(½)s_{tn} + (½)s_{1n}$, to be greater than the average of these shares, which is 1/N. Thus the term $[(½)s_{tn} + (½)s_{1n} - (1/N)][\ln(p_{tn}/p_t{}^*) - \ln(p_{1n}/p_1{}^*)]$ is *likely* to be negative. Now suppose that the deflated price of product n goes up between periods 1 and t so that $\ln(p_{tn}/p_t{}^*) - \ln(p_{1n}/p_1{}^*)$ is positive. Under these conditions, there will be a tendency for the average expenditure share for product n, $(½)s_{tn} + (½)s_{1n}$, to be less than the average of these shares. Again, the term $[(½)s_{tn} + (½)s_{1n} - (1/N)][\ln(p_{tn}/p_t{}^*) - \ln(p_{1n}/p_1{}^*)]$ is *likely* to be negative. Thus if the products under consideration are highly substitutable, we expect $P_T{}^t$ to be less than $P_J{}^t$.[61] If the products are not highly substitutable, we expect $P_T{}^t$ to be greater than $P_J{}^t$.

The results in this section can be summarized as follows: the unweighted Jevons index, $P_J{}^t$, can provide a reasonable approximation to a fixed base superlative index like $P_T{}^t$ *provided* that the expenditure shares do not systematically trend with time and prices do not systematically grow at diverging rates. If these assumptions are not satisfied, then it is likely that the Jevons index will have some bias relative to a superlative index; $P_J{}^t$ is likely to exceed $P_T{}^t$ as t becomes large if the products are close substitutes and $P_J{}^t$ is likely to be less than $P_T{}^t$ if the products are not close substitutes.

[61] This is perhaps an important result in the context where a statistical agency is collecting web scraped prices for very similar products and using an equally weighted geometric mean of these scraped prices as an estimated elementary price level. The resulting Jevons price index may have an upward bias relative to its superlative counterpart.

6. Relationships between Superlative Fixed Base Indexes and Geometric Indexes that use Average Annual Shares as Weights

We consider the properties of weighted Jevons indexes where the weight vector is an *annual average* of the observed monthly shares in a previous year. Recall that the weighted Jevons (or Cobb Douglas) price index $P_{J\alpha}{}^t$ was defined by (15) in section 2 as $P_{J\alpha}{}^t \equiv \prod_{n=1}^N (p_{tn}/p_{1n})^{\alpha_n}$ where the product weighting vector α satisfied the restrictions $\alpha >> 0_N$ and $\alpha \cdot 1_N = 1$. The following counterparts to the covariance identities (46)-(48) hold for t = 1,...,T where the Geometric Young index or weighted Jevons index $P_{J\alpha}{}^t$ has replaced $P_J{}^t$:[62]

$$
\begin{aligned}
(50)\ \ln P_{GL}{}^t - \ln P_{J\alpha}{}^t &= \Sigma_{n=1}^N [s_{1n} - \alpha_n][\ln p_{tn} - \ln p_{1n}] \\
&= N\mathrm{Cov}(s^1 - \alpha, \ln p^t - \ln p^1); \\
(51)\ \ln P_{GP}{}^t - \ln P_{J\alpha}{}^t &= \Sigma_{n=1}^N [s_{tn} - \alpha_n][\ln p_{tn} - \ln p_{1n}] \\
&= N\mathrm{Cov}(s^t - \alpha, \ln p^t - \ln p^1); \\
(52)\ \ln P_T{}^t - \ln P_{J\alpha}{}^t &= \Sigma_{n=1}^N [(½)s_{tn} + (½)s_{1n} - \alpha_n][\ln p_{tn} - \ln p_{1n}] \\
&= N\mathrm{Cov}[(½)s^t + (½)s^1 - \alpha, \ln p^t - \ln p^1] \\
&= (½)[\ln P_{GL}{}^t - \ln P_{J\alpha}{}^t] + (½)[\ln P_{GP}{}^t - \ln P_{J\alpha}{}^t].
\end{aligned}
$$

Define α as the *arithmetic average of the first T^* observed share vectors* s^t:

$$(53)\ \alpha \equiv \Sigma_{t=1}^{T^*} (1/T^*)s^t.$$

In the context where the data consists of monthly periods, T^* will typically be equal to 12; i.e., the elementary index under consideration is the weighted Jevons index $P_{J\alpha}{}^t$ where the weight vector α is the average of the observed expenditure shares for the first 12 months in the sample.

The decompositions (50)-(52) will hold for the α defined by (53). If the N products are highly substitutable, it is likely that $\mathrm{Cov}(s^1 - \alpha, \ln p^t - \ln p^1) > 0$ and $\mathrm{Cov}(s^t - \alpha, \ln p^t - \ln p^1) < 0$ and hence it is likely that $P_{GL}{}^t > P_{J\alpha}{}^t$ and $P_{GP}{}^t < P_{J\alpha}{}^t$. If the products are not close substitutes, then it is likely that $P_{GL}{}^t < P_{J\alpha}{}^t$ and $P_{GP}{}^t > P_{J\alpha}{}^t$. If there are no divergent trends in prices, then it is possible that the *average share price index* $P_{J\alpha}{}^t$ could provide an adequate approximation to the superlative Törnqvist index $P_T{}^t$.

Note that t takes on the values t = 1,...,T in equations (50)-(52). However, annual share indexes that are implemented by statistical agencies are not constructed in exactly this manner. The practical month to month indexes that are constructed by statistical agencies using annual shares of the type defined by (53) do not choose the reference month for prices to be month 1; rather they chose the reference month for prices to be $T^* + 1$, the month that follows the first year.[63] Thus the reference *year* for share weights precedes the reference *month* for prices. In this case, the logarithm of the month $t \geq T^* + 1$ annual share weighted Jevons index, $\ln P_{J\alpha}{}^t$, is defined as follows where α is the vector of annual average share weights defined by (53):

$$(54)\ \ln P_{J\alpha}{}^t \equiv \Sigma_{n=1}^N \alpha_n[\ln p_{tn} - \ln p_{T^*+1,n}]\ ; \qquad t = T^*+1,T^*+2,...,T.$$

[62] The relationship (52) was obtained by Armknecht and Silver (2014; 9); i.e., take logarithms on both sides of their equation (12) and we obtain the first equation in equations (52).

[63] In actual practice, the reference month for prices can be many months after T^*.

The following counterparts to the identities (50)-(52) hold for $t = T^*+1,T^*+2,...,T$ where α is defined by (53) and $P_{J\alpha}{}^t$ is defined by (54):

$$(55)\ \ln P_{GL}{}^t - \ln P_{J\alpha}{}^t = \Sigma_{n=1}^N [s_{T^*+1,n} - \alpha_n][\ln p_{tn} - \ln p_{T^*+1,n}]$$
$$= N\mathrm{Cov}(s^{T^*+1}-\alpha, \ln p^t - \ln p^{T^*+1});$$
$$(56)\ \ln P_{GP}{}^t - \ln P_{J\alpha}{}^t = \Sigma_{n=1}^N [s_{tn} - \alpha_n][\ln p_{tn} - \ln p_{T^*+1,n}]$$
$$= N\mathrm{Cov}(s^t - \alpha, \ln p^t - \ln p^{T^*+1});$$
$$(57)\ \ln P_T{}^t - \ln P_{J\alpha}{}^t = \Sigma_{n=1}^N [(½)s_{tn} + (½)s_{T^*+1,n} - \alpha_n][\ln p_{tn} - \ln p_{T^*+1,n}]$$
$$= N\mathrm{Cov}[(½)s^t + (½)s^{T^*+1} - \alpha, \ln p^t - \ln p^{T^*+1}]$$
$$= (½)[\ln P_{GL}{}^t - \ln P_{J\alpha}{}^t] + (½)[\ln P_{GP}{}^t - \ln P_{J\alpha}{}^t].$$

If the N products are highly substitutable, it is likely that $\mathrm{Cov}(s^{T^*+1}-\alpha, \ln p^t-\ln p^{T^*+1}) > 0$ so that $P_{GL}{}^t > P_{J\alpha}{}^t$. It is also likely that $\mathrm{Cov}(s^t-\alpha, \ln p^t-\ln p^{T^*+1}) < 0$ and hence it is likely that $P_{GP}{}^t < P_{J\alpha}{}^t$ in the highly substitutable case. If the products are not close substitutes, then it is likely that $P_{GL}{}^t < P_{J\alpha}{}^t$ and $P_{GP}{}^t > P_{J\alpha}{}^t$. If there are no divergent trends in prices, then it is possible that the *average share price index* $P_{J\alpha}{}^t$ could provide an adequate approximation to the superlative Törnqvist index $P_T{}^t$. However, if there are divergent trends in prices and shares and the products are highly substitutable with each other, then we expect the covariance in (56) to be more negative than the covariance in (55) is positive so that $P_T{}^t$ will tend to be less than the annual shares geometric index $P_{J\alpha}{}^t$. Thus $P_{J\alpha}{}^t$ will tend to have a bit of substitution bias if the products are highly substitutable, which is an intuitively plausible result.

As usual, there are three simple sets of conditions that will imply that $P_T{}^t = P_{J\alpha}{}^t$: (i) the covariance on the right hand side of (57) equals 0; i.e., $\mathrm{Cov}[(½)s^t+(½)s^{T^*+1}-\alpha, \ln p^t-\ln p^{T^*+1}] = 0$ or equivalently, $\mathrm{Cov}(s^{T^*+1}-\alpha, \ln p^t-\ln p^{T^*+1}) = -\mathrm{Cov}(s^t-\alpha, \ln p^t-\ln p^{T^*+1})$; (ii) period t price proportionality (to the prices of the price reference period); i.e., $p^t = \lambda_t p^{T^*+1}$ for some $\lambda_t > 0$; (iii) the arithmetic average of the period T^*+1 and t sales shares are all equal to α defined by (53); i.e., $(½)s^t + (½)s^{T^*+1} = \alpha$. This last condition will hold if the shares s^t are constant over all time periods and α is defined by (53).

Suppose that there are *linear trends in shares* and *divergent linear trends in log prices*; i.e., suppose that the following assumptions hold for $t = 2,3,...,T$:

$$(58)\ \ s^t = s^1 + \beta(t-1)\ ;$$
$$(59)\ \ln p^t = \ln p^1 + \gamma(t-1)$$

where $\beta \equiv [\beta_1,...,\beta_N]$ and $\gamma \equiv [\gamma_1,...,\gamma_N]$ are constant vectors and β satisfies the additional restriction:[64]

$$(60)\ \beta\cdot 1_N = 0.$$

In the case where the products are highly substitutable, if the price of product n, p_{tn}, is trending upwards so that γ_n is positive, then we could expect that the corresponding share s_{tn} is trending downward so that β_n is negative. Similarly, if γ_n is negative, then we expect that the corresponding β_n is positive. Thus we expect that $\Sigma_{n=1}^N \beta_n\gamma_n = \beta\cdot\gamma < 0$.

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[64] Since expenditure shares must be nonnegative, if $\beta \neq 0_N$ then some components of β will be negative and thus the linear trends in shares assumption (58) cannot hold forever. Assumptions (58) and (59) will generally be only approximately true and they cannot hold indefinitely.

Substituting (58) into definition (53) gives us the following expression for the annual share weight vector under the linear trends assumption:

$$(61)\ \alpha \equiv \Sigma_{t=1}^{T^*} (1/T^*)s^t$$
$$= \Sigma_{t=1}^{T^*} (1/T^*)[s^1 + \beta(t-1)]$$
$$= s^1 + (½)\beta(T^*-1).$$

Using (57)-(59) and (61), we have the following equations for $t = T^*+1,T^*+2,...,T$:

$$(62)\ \ln P_T^t - \ln P_{J\alpha}^t = [(½)s^t+(½)s^1- \alpha]\cdot[\ln p^t-\ln p^1]$$
$$= (½)\beta\cdot\gamma t(t-T^*-1).$$

Thus if the inner product of the vectors β and γ is not equal to 0, $\ln P_T^t$ and $\ln P_{J\alpha}^t$ will diverge at a *quadratic rate* as t increases. Under these trend assumptions, the average share geometric index $P_{J\alpha}^t$ will be subject to some substitution bias (as compared to P_T^t which controls for substitution bias[65]), which will grow over time.[66] As indicated above, it is likely that $\beta\cdot\gamma < 0$ so that it is likely that P_T^t will be below $P_{J\alpha}^t$ under the assumption of strong substitutability and diverging trends in prices and shares.

Note that in real life, new products appear and existing products disappear. The analysis presented in this section and in previous sections can take this fact into account *in theory* if the price statistician has somehow calculated approximate reservation prices for products that are not available in the current period. Note that product churn means that shares are not constant over time; i.e., *product churn will lead to nonsmooth trends in product shares*. However, superlative indexes like P_F^t and P_T^t can deal with new and disappearing products in a way that is consistent with consumer theory, provided that suitable reservation prices have been either estimated or approximated by suitable rules of thumb.

7. To Chain or Not to Chain

In the above discussions, attention has been focused on direct indexes that compare the prices of period t with the prices of period 1. But it is also possible to move from period 1 prices to period t prices by moving from one period to the next and cumulating the jumps. If the second method is used, the resulting period t price index is called a *chained index*. In this section, we will examine the possible differences between direct and chained Törnqvist price indexes.

It is convenient to introduce some new notation. Denote the Törnqvist price index that compares the prices of period j to the prices of period i (the base period for the comparison) by $P_T(i,j)$. The logarithm of $P_T(i,j)$ is defined as follows for $i,j = 1,...,N$:

$$(63)\ \ln P_T(i,j) \equiv (½)\Sigma_{n=1}^N (s_{in} + s_{jn})(\ln p_{jn} - \ln p_{in})$$
$$= (½)(s^i + s^j)\cdot(\ln p^j - \ln p^i).$$

[65] We regard an index as having some substitution bias if it diverges from a superlative index which controls for substitution bias. See Diewert (1976) for the formal definition of a superlative index.

[66] If all prices grow at the same geometric rate, then it can be verified that $P_{J\alpha}^t = P_{GL}^t = P_{GP}^t = P_T^t$. If in addition, assumptions (58)-(60) hold, then $\gamma = \lambda 1_N$ for some scalar $\lambda > 0$ and using assumption (60), we have $\beta\cdot\gamma = 0$ and thus $P_T^t = P_{J\alpha}^t$ under our assumptions.

The chained Törnqvist price index going from period 1 to T will coincide with the corresponding direct index if the indexes $P_T(i,j)$ satisfy the following *multiperiod identity test*, which is due to Walsh (1901; 389) (1921b; 540):

$$(64)\ P_T(1,2)P_T(2,3)\ldots P_T(T-1,T)P_T(T,1) = 1.$$

The above test can be used to measure the amount that the chained indexes between periods 1 and T differ from the corresponding direct index that compares the prices of period 1 and T; i.e., if the product of indexes on the left hand side of (64) is different from unity, then we say that the index number formula is subject to *chain drift* and the difference between the left and right hand sides of (64) serves to measure the magnitude of the chain drift problem.[67] In order to determine whether the Törnqvist price index formula satisfies the multiperiod identity test (64), take the logarithm of the left hand side of (64) and check whether it is equal to the logarithm of 1 which is 0. Thus substituting definitions (63) into the logarithm of the left hand side of (64) leads to the following expressions:[68]

$$\begin{aligned}(65)\ & \ln P_T(1,2) + \ln P_T(2,3) + \ldots + \ln P_T(T-1,T) + \ln P_T(T,1) \\ &= \tfrac{1}{2}\Sigma_{n=1}^N (s_{1n}+s_{2n})(\ln p_{2n}-\ln p_{1n}) + \tfrac{1}{2}\Sigma_{n=1}^N (s_{2n}+s_{3n})(\ln p_{3n}-\ln p_{2n}) + \ldots \\ &\quad + \tfrac{1}{2}\Sigma_{n=1}^N (s_{T-1,n}+s_{Tn})(\ln p_{Tn}-\ln p_{T-1,n}) + \tfrac{1}{2}\Sigma_{n=1}^N (s_{Tn}+s_{1n})(\ln p_{1n}-\ln p_{Tn}) \\ &= \tfrac{1}{2}\Sigma_{n=1}^N (s_{1n}-s_{3n})\ln p_{2n} + \tfrac{1}{2}\Sigma_{n=1}^N (s_{2n}-s_{4n})\ln p_{3n} + \ldots + \tfrac{1}{2}\Sigma_{n=1}^N (s_{T-2,n}-s_{Tn})\ln p_{T-1,n} \\ &\quad + \tfrac{1}{2}\Sigma_{n=1}^N (s_{Tn}-s_{2n})\ln p_{1n} + \tfrac{1}{2}\Sigma_{n=1}^N (s_{T-1,n}-s_{1n})\ln p_{Tn}.\end{aligned}$$

In general, it can be seen that the Törnqvist price index formula will be subject to some chain drift i.e., the sums of terms on the right hand side of (65) will not equal 0 in general. However there are four sets of conditions where these terms will sum to 0.

The first set of conditions makes use of the first equality on the right hand side of (65). If the prices vary in strict proportion over time, so that $p^t = \lambda_t p^1$ for $t = 2,3,\ldots,T$, then it is straightforward to show that (64) is satisfied.

The second set of conditions makes use of the second equality in equations (65). If the shares s^t are constant over time,[69] then it is obvious that (64) is satisfied.

The third set of conditions also makes use of the second equality in (65). The sum of terms $\Sigma_{n=1}^N (s_{1n}-s_{3n})\ln p_{2n}$ is equal to $(s^1-s^3)\cdot\ln p^2$ which in turn is equal to $(s^1-s^3)\cdot(\ln p^2-\ln p^{2*}) = N\mathrm{Cov}(s^1-s^3,\ln p^2)$ where $\ln p^{2*} \equiv (1/N)\ \Sigma_{n=1}^N \ln p_{2n}$, the mean of the components of $\ln p^2$. Thus the N sets of summations on the right hand side of the second equation in (65) can be interpreted as constants times the covariances of a difference in shares (separated by one or more time periods) with the logarithm of a price vector for a time period that is not equal to either of the time periods involved in the difference in shares. Thus if the covariance equalities $\mathrm{Cov}(s^1-s^3,\ln p^2) = \mathrm{Cov}(s^2-s^4,\ln p^3) = \ldots = \mathrm{Cov}(s^{T-2}-s^T,\ln p^{T-1}) = \mathrm{Cov}(s^T-s^2,\ln p^1) = \mathrm{Cov}(s^{T-1}-s^1,\ln p^T) = 0$, then (64) will be satisfied. These zero covariance conditions will be satisfied if the log prices of one period are uncorrelated with the shares of all other periods. If the time period is long enough and there

[67] Walsh (1901; 401) was the first to propose this methodology to measure chain drift. It was independently proposed later by Persons (1921; 110) and Szulc (1983; 540). Fisher's (1922; 284) circular gap test could also be interpreted as a test for chain drift.
[68] Persons (1928; 101) developed a similar decomposition using the bilateral Fisher formula instead of the Törnqvist formula. See also de Haan and Krsinich (2014) for an alternative decomposition.
[69] If purchasers of the products have Cobb-Douglas preferences, then the sales shares will be constant.

are no trends in log prices and shares, so that prices are merely bouncing around in a random fashion,[70] then these zero covariance conditions are likely to be satisfied to a high degree of approximation and thus under these conditions, the Törnqvist Theil price index is likely to be largely free of chain drift. However, in the elementary index context where retailers have periodic highly discounted prices, the zero correlation conditions are unlikely to hold. Suppose that product n goes on sale during period 2 so that $\ln p_{2n}$ is well below the average price for period 2. Suppose product n is not on sale during periods 1 and 3. If purchasers have stocked up on product n during period 2, it is likely that s_{3n} will be less than s_{1n} and thus it is likely that $\mathrm{Cov}(s^1–s^3,\ln p^2) < 0$. Now suppose that product n is not on sale during period 2. In this case, it is likely that $\ln p_{2n}$ is greater than the average log price during period 2. If product n was on sale during period 1 but not period 3, then s_{1n} will tend to be greater than s_{3n} and thus $\mathrm{Cov}(s^1–s^3,\ln p^2) > 0$. However, if product n was on sale during period 3 but not period 1, then s_{1n} will tend to be less than s_{3n} and thus $\mathrm{Cov}(s^1–s^3,\ln p^2) < 0$. These last two cases should largely offset each other and so we are left with the likelihood that $\mathrm{Cov}(s^1–s^3,\ln p^2) < 0$. Similar arguments apply to the other covariances and so we are left with the expectation that the chained Törnqvist index used in the elementary index context is likely to drift downwards relative to its fixed base counterpart.[71]

Since the Fisher index normally approximates the Törnqvist fairly closely, we expect both the chained Fisher and Törnqvist indexes to exhibit downward chain drift. However, it is not always the case that a superlative index is subject to downward chain drift. Feenstra and Shapiro (2003) found upward chain drift in the Törnqvist formula using a scanner data set. Persons (1928; 100-105) had an extensive discussion of the chain drift problem with the Fisher index and he gave a numerical example on page 102 of his article that showed how upward chain drift could occur. We have adapted his example in Table 3 below.

Table 3: Prices and Quantities for Two Products and the Fisher Fixed Base and Chained Price Indexes

t	p_1^t	p_2^t	q_1^t	q_2^t	P_F^t	P_{FCh}^t
1	2	1	100	1	1.00000	1.00000
2	10	1	40	40	4.27321	4.27321
3	10	1	25	80	3.55553	4.27321
4	5	2	50	20	2.45676	2.96563

Product 1 is on sale in period 1 and goes back to a relatively high price in periods 2 and 3 and then goes on sale again but the discount is not as steep as the period 1 discount. Product 2 is at its "regular" price for periods 1-3 and then rises steeply in period 4. Products 1 and 2 are close substitutes so when product 1 is steeply discounted, only 1 unit of product 2 is sold in period 1 while 100 units of product 1 are sold. When the price of product 1 increases fivefold in period 2, demand for the product falls and purchasers switch to product 2 but the adjustment to the new higher price of product 1 is not complete in period 2: in period 3 (where prices are unchanged from period 2), purchasers continue to substitute away from product 1 and towards product 2. It is

[70] Szulc (1983) introduced the term "price bouncing" to describe the behavior of soft drink prices in Canada at the elementary level.

[71] Fisher (1922; 284) found little difference in the fixed base and chained Fisher indexes for his particular data set which he used to compare 119 different index number formulae. Fisher noted that the Carli, Laspeyres and share weighted Carli chained indexes showed upward chain drift. However, Persons (1921; 110) showed that the Fisher chained index ended up about 4% lower than its fixed base counterpart for his agricultural data set covering 10 years. This is an early example of the downward chain drift associated with the use of the Fisher index.

this *incomplete adjustment* that causes the chained index to climb above the fixed base index in period 3.[72] Thus it is not always the case that the Fisher index is subject to downward chain drift but we do expect that "normally", this would be the case.

The fourth set of conditions that ensure that there is no chain drift are assumptions (58) and (59); i.e., the assumption that *shares and log prices have linear trends*. To prove this assertion, substitute these equations into either one of the two right hand side equations in (65) and we find that the resulting sum of terms is 0.[73] This result is of some importance at higher levels of aggregation where aggregate prices and quantities are more likely to have smooth trends. If the trends are actually linear, then this result shows that there will be no chain drift if the Törnqvist Theil index number formula is used to aggregate the data.[74] However, when this formula is used at the elementary level when there are frequent fluctuations in prices and quantities, chain drift is likely to occur and thus the use of a fixed base index or a multilateral index is preferred under these conditions.

As was mentioned in the introduction, a main advantage of the chain system is that under conditions where prices and quantities are trending smoothly, chaining will reduce the spread between the Paasche and Laspeyres indexes.[75] These two indexes each provide an asymmetric perspective on the amount of price change that has occurred between the two periods under consideration and it could be expected that a single point estimate of the aggregate price change should lie between these two estimates. Thus at higher levels of aggregation, the use of either a chained Paasche or Laspeyres index will usually lead to a smaller difference between the two and hence to estimates that are closer to the "truth". However, at lower levels of aggregation, smooth changes in prices and quantities are unlikely to occur.

An alternative to the use of a fixed base index is the use of a *multilateral index*. A problem with the use of a fixed base index is that it depends asymmetrically on the choice of the base period. If the structure of prices and quantities for the base period is unusual and fixed base index numbers are used, then the choice of the base period could lead to "unusual" results. Multilateral indexes treat each period symmetrically and thus avoid this problem. In the following section, we will introduce some possible multilateral indexes that are free of chain drift (within our window of T observations).[76]

8. Relationships between the Törnqvist Index and the GEKS and CCDI Multilateral Indexes

It is useful to introduce some additional notation at this point. Denote the Laspeyres, Paasche and Fisher price indexes that compare the prices of period j to the prices of period i (the base period for the comparison) by $P_L(i,j)$, $P_P(i,j)$ and $P_F(i,j)$ respectively. These indexes are defined as follows for r,t = 1,...,N:

[72] Persons (1928; 102) explained that it was *incomplete adjustment* that caused the Fisher chained index to climb above the corresponding fixed base index in his example. Ludwig von Auer (2019) proposed a similar theory.
[73] This result was first established by Alterman, Diewert and Feenstra (1999; 61-65).
[74] This transitivity property carries over to an *approximate* transitivity property for the Fisher and Walsh index number formulae using the fact that these indexes approximate the Törnqvist Theil index to the second order around an equal price and quantity point; see Diewert (1978) on these approximations.
[75] See Diewert (1978; 895) and Hill (1988) for additional discussion on the benefits and costs of chaining.
[76] Ivancic, Diewert and Fox (2009) (2011) advocated the use of multilateral indexes adapted to the time series context in order to control chain drift. Balk (1980) (1981) also advocated the use of multilateral indexes in order to address the problem of seasonal commodities.

(66) $P_L(r,t) \equiv p^t \cdot q^r / p^r \cdot q^r$;
(67) $P_P(r,t) \equiv p^t \cdot q^t / p^r \cdot q^t$;
(68) $P_F(r,t) \equiv [P_L(r,t)P_P(r,t)]^{1/2}$.

The Fisher indexes have very good axiomatic properties and hence are preferred indexes from the viewpoint of the test or axiomatic approach.[77]

Obviously, one could choose period 1 as the base period and form the following sequence of price levels relative to period 1: $P_F(1,1) = 1$, $P_F(1,2)$, $P_F(1,3)$, ..., $P_F(1,T)$. But one could also use period 2 as the base period and use the following sequence of price levels: $P_F(2,1)$, $P_F(2,2) = 1$, $P_F(2,3)$, ..., $P_F(2,T)$. Each period could be chosen as the base period and thus we end up with T alternative series of Fisher price levels. Since each of these sequences of price levels is equally plausible, Gini (1931) suggested that it would be appropriate to take the geometric average of these alternative price levels in order to determine the final set of price levels. Thus the *GEKS price levels*[78] for periods t = 1,2,...,T are defined as follows:

(69) $p_{GEKS}^t \equiv [\prod_{r=1}^T P_F(r,t)]^{1/T}$.

Note that all time periods are treated in a symmetric manner in the above definitions. The GEKS price indexes P_{GEKS}^t are obtained by normalizing the above price levels so that the period 1 index is equal to 1. Thus we have the following definitions for P_{GEKS}^t for t = 1,...,T:

(70) $P_{GEKS}^t \equiv p_{GEKS}^t / p_{GEKS}^1$.

It is straightforward to verify that the GEKS price indexes satisfy Walsh's multiperiod identity test which becomes the following test in the present context:

(71) $[P_{GEKS}^2/P_{GEKS}^1][P_{GEKS}^3/P_{GEKS}^2]...[P_{GEKS}^T/P_{GEKS}^{T-1}][P_{GEKS}^1/P_{GEKS}^T] = 1$.

Thus the GEKS indexes are not subject to chain drift within the window of T periods under consideration.

Recall definition (63) which defined the logarithm of the Törnqvist price index, $\ln P_T(i,j)$, that compared the prices of period j to the prices of period i. The GEKS methodology can be applied using $P_T(r,t)$ in place of the Fisher $P_F(r,t)$ as the basic bilateral index building block. Thus define the *period t GEKS Törnqvist price level*, p_{GEKST}^t, for t = 1,...,T as follows:

(72) $p_{GEKST}^t \equiv [\prod_{r=1}^T P_T(r,t)]^{1/T}$.

The *GEKST price indexes* P_{GEKST}^t are obtained by normalizing the above price levels so that the period 1 index is equal to 1. Thus we have the following definitions for P_{GEKST}^t for t = 1,...,T:

(73) $P_{GEKST}^t \equiv p_{GEKST}^t / p_{GEKST}^1$.

[77] See Diewert (1992) on the axiomatic properties of the Fisher index.
[78] Eltetö and Köves (1964) and Szulc (1964) independently derived the GEKS price indexes by an alternative route. Thus the name GEKS has the initials of all four primary authors of the method. Ivancic, Diewert and Fox (2009) (2011) suggested the use of the GEKS index in the time series context.

Since $P_T(r,t)$ approximates $P_F(r,t)$ to the second order around an equal price and quantity point, the $P_{GEKST}{}^t$ will usually be quite close to the corresponding $P_{GEKS}{}^t$ indexes.

It is possible to provide a very simple alternative approach to the derivation of the GEKS Törnqvist price indexes.[79] Define the *sample average sales share* for product n, $s_{\bullet n}$, and the *sample average log price* for product n, $\ln p_{\bullet n}$, as follows for n = 1,...,N:

(74) $s_{\bullet n} \equiv \Sigma_{t=1}{}^T (1/T)s_{tn}$;
(75) $\ln p_{\bullet n} \equiv \Sigma_{t=1}{}^T (1/T)\ln p_{tn}$.

The logarithm of the *CCDI price level for period t*, $\ln p_{CCDI}{}^t$, is defined by comparing the prices of period t with the sample average prices using the bilateral Törnqvist formula; i.e., for t = 1,...,T, we have the following definitions:

(76) $\ln p_{CCDI}{}^t \equiv \Sigma_{n=1}{}^N \frac{1}{2}(s_{tn} + s_{\bullet n})(\ln p_{tn} - \ln p_{\bullet n})$.

The *CCDI price index for period t*, $P_{CCDI}{}^t$, is defined as the following normalized CCDI price level for t = 1,...,T:

(77) $P_{CCDI}{}^t \equiv p_{CCDI}{}^t/p_{CCDI}{}^1$.

Using the above definitions, the logarithm of the CCDI price index for period t is equal to the following expressions for t = 1,...,T:

$$\begin{aligned}(78)\ \ln P_{CCDI}{}^t &= \ln p_{CCDI}{}^t - \ln p_{CCDI}{}^1 \\ &= \Sigma_{n=1}{}^N (\tfrac{1}{2})(s_{tn} + s_{\bullet n})(\ln p_{tn} - \ln p_{\bullet n}) - \Sigma_{n=1}{}^N (\tfrac{1}{2})(s_{1n} + s_{\bullet n})(\ln p_{1n} - \ln p_{\bullet n}) \\ &= \ln P_T{}^t + \Sigma_{n=1}{}^N (\tfrac{1}{2})(s_{tn} - s_{\bullet n})(\ln p_{1n} - \ln p_{\bullet n}) - \Sigma_{n=1}{}^N (\tfrac{1}{2})(s_{1n} - s_{\bullet n})(\ln p_{tn} - \ln p_{\bullet n}) \\ &= \ln P_{GEKST}{}^t\end{aligned}$$

where the last equality follows by direct computation or by using the computations in Inklaar and Diewert (2016).[80] Thus the CCDI multilateral price indexes are equal to the GEKS Törnqvist multilateral indexes defined by (73). Define $s^\bullet \equiv [s_{\bullet 1},..., s_{\bullet N}]$ as the vector of sample average shares and $\ln p^\bullet \equiv [\ln p_{\bullet 1},...,\ln p_{\bullet N}]$ as the vector of sample average log prices. Then the last two terms on the right hand side of the penultimate equality in (78) can be written as $(\frac{1}{2})N\mathrm{Cov}(s^t - s^\bullet,\ln p^1 - \ln p^\bullet) - (\frac{1}{2})N\mathrm{Cov}(s^1 - s^\bullet,\ln p^t - \ln p^\bullet)$. If the fluctuations in shares and prices are not too violent, it is likely that both covariances are close to 0 and thus $\ln P_{CCDI}{}^t \approx \ln P_T{}^t$ for each t.[81] Thus under these circumstances, it is likely that $\ln P_{CCDI}{}^t \approx \ln P_T{}^t$ for each t. Moreover, under the assumptions of linear trends in log prices and linear trends in shares, assumptions (58) and (59), it was seen in the previous section that the period t bilateral Törnqvist price index, $P_T{}^t$, was equal to its chained counterpart for any t.[82] This result implies that $P_T{}^t = P_{CCDI}{}^t = P_{GEKST}{}^t$ for t = 1,...,T under the linear trends assumption. Thus we expect the period t multilateral index, $P_{GEKST}{}^t$ =

[79] This approach is due to Inklaar and Diewert (2016). It is an adaptation of the distance function approach used by Caves, Christensen and Diewert (1982) to the price index context.
[80] The second from last equality was derived in Diewert and Fox (2020).
[81] For Diewert's (2018) empirical example, the sample average of these two sets of covariance terms turned out to be 0 with variances equal to 0.00024 and 0.00036 respectively.
[82] See the discussion below (65) in the previous section. Note that the assumption of linear trends in shares is not consistent with the existence of new and disappearing products.

$P_{CCDI}{}^t$ to approximate the corresponding fixed base period t Törnqvist price index, $P_T{}^t$, provided that prices and quantities have smooth trends.

Since $P_F{}^t$ approximates $P_T{}^t$, we expect that the following approximate equalities will hold under the smooth trends assumption for t = 1,...,T:

(79) $P_F{}^t \approx P_T{}^t \approx P_{GEKS}{}^t \approx P_{GEKST}{}^t = P_{CCDI}{}^t$.

The above indexes will be free from chain drift within the window of T periods;[83] i.e., if prices and quantities for any two periods in the sample are equal, then the price index will register the same value for these two periods.

Unit values taken over heterogeneous products are often used at the first stage of aggregation. In the following section, bias estimates for unit value price levels will be derived and in the subsequent section, quality adjusted unit value price levels will be studied.

9. Unit Value Price and Quantity Indexes

As was mentioned in section 2, there was a preliminary aggregation over time problem that needed to be addressed; i.e., exactly how should the period t prices and quantities for commodity n, $p_n{}^t$ and $q_n{}^t$, that are used in an index number formula be defined? During any time period t, there will typically be many transactions in a specific commodity n at a number of different prices. Hence, there is a need to provide a more precise definition for the "average" or "representative" price for commodity n in period t, $p_n{}^t$. Starting with Drobisch (1871), many measurement economists and statisticians advocated the use of the *unit value* (total value transacted divided by total quantity) as the appropriate price $p_n{}^t$ for commodity n and the total quantity transacted during period t as the appropriate quantity, $q_n{}^t$; e.g., see Walsh (1901; 96) (1921a; 88), Fisher (1922; 318) and Davies (1924; 183) (1932; 59). If it is desirable to have $q_n{}^t$ be equal to the total quantity of commodity n transacted during period t and also desirable to have the product of the price $p_n{}^t$ times quantity $q_n{}^t$ to be equal the value of period t transactions in commodity n, then one is *forced* to define the aggregate period t price for commodity n, $p_n{}^t$, to be the total value transacted during the period divided by the total quantity transacted, which is the unit value for commodity n.[84]

There is general agreement that a unit value price is an appropriate price concept to be used in an index number formula if the transactions refer to a narrowly defined homogeneous commodity. Our task in this section is to look at the properties of a unit value price index when aggregating over commodities that are *not* completely homogeneous. We will also look at the properties of the companion unit value quantity index in this section.

The period t *unit value price level*, $p_{UV}{}^t$, and the corresponding period t *unit value price index* which compares the price level in period t to that of period 1, $P_{UV}{}^t$, are defined as follows for t = 1,...,T:

[83] See de Haan (2015) and Diewert and Fox (2020) for discussions of the problems associated with linking the results from one rolling window multilateral comparison to a subsequent window of observations. Empirically, there does not appear to be much chain drift when the indexes generated by subsequent windows are linked.

[84] For additional discussion on unit value price indexes, see Balk (2008; 72-74), Diewert and von der Lippe (2010), Silver (2010) (2011) and de Haan and Krsinich (2018).

(80) $p_{UV}^t \equiv p^t \cdot q^t / 1_N \cdot q^t$;
(81) $P_{UV}^t \equiv p_{UV}^t / p_{UV}^1$
$= [p^t \cdot q^t / 1_N \cdot q^t] / [p^1 \cdot q^1 / 1_N \cdot q^1]$
$= [p^t \cdot q^t / p^1 \cdot q^1] / Q_{UV}^t$

where the period t *unit value quantity index*, Q_{UV}^t, is defined as follows for t = 1,...,T:

(82) $Q_{UV}^t \equiv 1_N \cdot q^t / 1_N \cdot q^1$.

It can be seen that the unit value price index satisfies Walsh's multiperiod identity test and thus P_{UV}^t is free from chain drift.

However, there is a big problem in using the unit value price index when the commodities in scope are not homogeneous: *the unit value price index is not invariant to changes in the units of measurement of the individual products in the aggregate*.

We will look at the relationship of the *unit value quantity indexes*, Q_{UV}^t, with the corresponding *Laspeyres, Paasche* and *Fisher fixed base quantity indexes*, Q_L^t, Q_P^t and Q_F^t, defined below for t = 1,...,T:

(83) $Q_L^t \equiv p^1 \cdot q^t / p^1 \cdot q^1 = \Sigma_{n=1}^N s_{1n}(q_{tn}/q_{1n})$;
(84) $Q_P^t \equiv p^t \cdot q^t / p^t \cdot q^1 = [\Sigma_{n=1}^N s_{tn}(q_{tn}/q_{1n})^{-1}]^{-1}$;
(85) $Q_F^t \equiv [Q_L^t Q_P^t]^{1/2}$.

For the second set of equations in (83), we require that $q_{1n} > 0$ for all n and for the second set of equations in (84), we require that all $q_{tn} > 0$. Recall that the period t *sales* or *expenditure share* vector $s^t \equiv [s_{t1},...,s_{tN}]$ was defined at the beginning of section 2. The period t *quantity* share vector $S^t \equiv [S_{t1},...,S_{tN}]$ was also defined in section 2 as follows for t = 1,...,T:

(86) $S^t \equiv q^t / 1_N \cdot q^t$.

Below, we will make use of the following identities (87), which hold for t = 1,...,T:

(87) $\Sigma_{n=1}^N [p_{UV}^t - p_{tn}]q_{tn} = \Sigma_{n=1}^N [(p^t \cdot q^t / 1_N \cdot q^t) - p_{tn}]q_{tn}$ using definitions (80)
$= (p^t \cdot q^t / 1_N \cdot q^t) 1_N \cdot q^t - p^t \cdot q^t$
$= 0.$

The following relationships between Q_{UV}^t and Q_L^t hold for t = 1,...,T:

(88) $Q_{UV}^t - Q_L^t = [1_N \cdot q^t / 1_N \cdot q^1] - [p^1 \cdot q^t / p^1 \cdot q^1]$ using (82) and (83)
$= \Sigma_{n=1}^N S_{1n}(q_{tn}/q_{1n}) - \Sigma_{n=1}^N s_{1n}(q_{tn}/q_{1n})$ using (86) and (83)
$= \Sigma_{n=1}^N [S_{1n} - s_{1n}](q_{tn}/q_{1n})$
$= N\mathrm{Cov}(S^1 - s^1, q^t/q^1)$

where the vector of period t to period 1 relative quantities is defined as $q^t/q^1 \equiv [q_{t1}/q_{11}, q_{t2}/q_{12}...,q_{tN}/q_{1N}]$. As usual, there are three special cases of (88) which will imply that $Q_{UV}^t = Q_L^t$. (i) $S^1 = s^1$ so that the vector of period 1 real quantity shares S^1 is equal to the period 1 sales share vector s^1. This condition is equivalent to $p^1 = \lambda_1 1_N$ so that all period 1 prices are equal.[85] (ii) $q^t = \lambda_t q^1$ for t = 2,3,...,T so that quantities vary in strict proportion over time. (iii) $Cov(S^1 - s^1, q^t/q^1) = 0$.[86]

There are two problems with the above bias formula: (i) it is difficult to form a judgement on the sign of the covariance $Cov(S^1 - s^1, q^t/q^1)$ and (ii) the decomposition given by (88) requires that all components of the period 1 quantity vector be positive.[87] It would be useful to have a decomposition that allowed some quantities (and sales shares) to be equal to 0. Consider the following alternative decomposition to (88) for t = 1,...,T:

$$
\begin{aligned}
(89)\ Q_{UV}^t - Q_L^t &= [1_N\cdot q^t/1_N\cdot q^1] - [p^1\cdot q^t/p^1\cdot q^1] && \text{using (82) and (83)} \\
&= \Sigma_{n=1}^N [(q_{tn}/1_N\cdot q^1) - (p_{1n}q_{tn}/p^1\cdot q^1)] \\
&= \Sigma_{n=1}^N [(1/1_N\cdot q^1) - (p_{1n}/p^1\cdot q^1)]q_{tn} \\
&= \Sigma_{n=1}^N [(p^1\cdot q^1/1_N\cdot q^1) - p_{1n}][q_{tn}/p^1\cdot q^1] \\
&= \Sigma_{n=1}^N [p_{UV}^1 - p_{1n}][q_{tn}/p^1\cdot q^1] && \text{using (80) for t = 1} \\
&= \Sigma_{n=1}^N [p_{UV}^1 - p_{1n}][q_{tn} - q_{1n}Q_{UV}^t]/p^1\cdot q^1 && \text{using (87) for t = 1} \\
&= Q_{UV}^t\, \Sigma_{n=1}^N [p_{UV}^1 - p_{1n}][(q_{tn}/Q_{UV}^t) - q_{1n}]/p^1\cdot q^1 \\
&= Q_{UV}^t\, \Sigma_{n=1}^N s_{1n}[(p_{UV}^1/p_{1n}) - 1][(q_{tn}/q_{1n}Q_{UV}^t) - 1] && \text{if } q_{1n} > 0 \text{ for all n} \\
&= Q_{UV}^t\, \varepsilon_L^t
\end{aligned}
$$

where the *period t error term* ε_L^t is defined for t = 1,...,T as:

$$(90)\ \varepsilon_L^t \equiv \Sigma_{n=1}^N [p_{UV}^1 - p_{1n}][(q_{tn}/Q_{UV}^t) - q_{1n}]/p^1\cdot q^1.$$[88]

If $q_{1n} > 0$ for n = 1,...,N, then ε_L^t is equal to $\Sigma_{n=1}^N s_{1n}[(p_{UV}^1/p_{1n}) - 1][(q_{tn}/q_{1n}Q_{UV}^t) - 1]$.

Note that the terms on the right hand side of (90) can be interpreted as $(N/p^1\cdot q^1)$ times the covariance $Cov(p_{UV}^1 1_N - p^1, q^t - Q_{UV}^t q^1)$ since $1_N\cdot(q^t - Q_{UV}^t q^1) = 0$. If the products are substitutes, it is likely that this covariance is *negative*, since if p_{1n} is unusually low, we would expect that it would be less than the period 1 unit value price level p_{UV}^1 so that $p_{UV}^1 - p_{1n} > 0$. Furthermore, if p_{1n} is unusually low, then we would expect that the corresponding q_{1n} is unusually high, and thus it is likely that q_{1n} is greater than q_{tn}/Q_{UV}^t and so $q_{tn} - q_{1n}Q_{UV}^t < 0$. Thus the N terms in the covariance will tend to be negative provided that there is some degree of substitutability between the products.[89] Looking at formula (90) for ε_L^t, it can be seen that all terms on the right hand side

[85] Consider the case where $p^1 = \lambda 1_N$. Units of measurement for the N commodities can always be chosen so that all prices are equal in period 1. Then $Q_{UV}^t = Q_L^t$ and hence $P_{UV}^t = P_P^t$ where P_{UV}^t is defined by (81) and P_P^t is the fixed base Paasche price index defined by (34). Thus for this particular choice for units of measurement, the unit value price index P_{UV}^t is equal to a fixed base Paasche price index which will typically have a downward bias relative to a superlative index.
[86] For similar bias formulae, see Balk (2008; 73-74) and Diewert and von der Lippe (2010).
[87] We are assuming that all prices are positive in all periods (so if there are missing prices they must be replaced by positive imputed prices) but we are not assuming that all quantities (and expenditure shares) are positive.
[88] Note that this error term is homogeneous of degree 0 in the components of p^1, q^1 and q^t. Hence it is invariant to proportional changes in the components of these vectors.
[89] The results in previous sections looked at responses of product *shares* to changes in prices and with data that are consistent with CES preferences, the results depended on whether the elasticity of substitution was greater or less than unity. In the present section, the results depend on whether the elasticity of substitution is equal to 0 or greater than 0; i.e., it is the response of *quantities* (rather than *shares*) to lower prices that matters.

of (90) do not depend on t, except for the N period t deflated product quantity terms, q_{tn}/Q_{UV}^t for n = 1,...,N. Hence if there is a great deal of variation in the period t quantities q_{tn}, then $q_{tn}/Q_{UV}^t - q_{1n}$ could be positive or negative and thus the tendency for ε_L^t to be negative will be a weak one. Thus our expectation is that the error term ε_L^t is likely to be negative and hence $Q_{UV}^t < Q_L^t$ for $t \geq 2$ but this expectation is a weak one.

It should be noted that P_{UV}^t and Q_{UV}^t *do not depend on the estimated reservation prices* for the missing products; i.e., the definitions of P_{UV}^t and Q_{UV}^t zero out the estimated reservation prices.

As usual, there are 3 special cases of (89) that will imply that $Q_{UV}^t = Q_L^t$: (i) $p^1 = \lambda_1 1_N$ so that all period 1 prices are equal; (ii) $q^t = \lambda_t q^1$ for t = 2,3,...,T so that quantities vary in strict proportion over time; (iii) $\mathrm{Cov}(p_{UV}^1 1_N - p^1, q^t - Q_{UV}^t q^1) = 0$. These conditions are equivalent to our earlier conditions listed below (88).

If we divide both sides of equation t in equations (89) by Q_{UV}^t, we obtain the following system of identities for t = 1,...,T:

(91) $Q_L^t/Q_{UV}^t = 1 - \varepsilon_L^t$

where we expect ε_L^t to be a small negative number in the elementary index context.

The identities in (89) and (91) are valid if we interchange prices and quantities. The quantity counterparts to p_{UV}^t and P_{UV}^t defined by (80) and (81) are the period t *Dutot quantity level* q_D^t and *quantity index* Q_D^t [90] defined as $q_D^t \equiv p^t \cdot q^t/1_N \cdot p^t = \alpha^t \cdot q^t$ (where $\alpha^t \equiv p^t/1_N \cdot p^t$ is a vector of period t price weights for q^t) and $Q_D^t \equiv q_{UV}^t/q_{UV}^1 = [p^t \cdot q^t/p^1 \cdot q^1]/P_D^t$ where we redefine the period t Dutot price level as $p_D^t \equiv 1_N \cdot p^t$ and the period t Dutot price index as $P_D^t \equiv p_D^t/p_D^1 = 1_N \cdot p^t/1_N \cdot p^1$ which coincides with our earlier definition (10) for P_D^t. Using these definitions and interchanging prices and quantities, equations (91) become the following equations for t = 1,...,T:

(92) $P_L^t/P_D^t = 1 - \varepsilon_L^{t*}$

where the period t error term ε_L^{t*} is defined for t = 1,...,T as:

(93) $\varepsilon_L^{t*} \equiv \Sigma_{n=1}^N [q_D^1 - q_{1n}][(p_{tn}/P_D^t) - p_{1n}]/p^1 \cdot q^1$.

If p_{1n} is unusually low, then it is likely that it will be less than p_{tn}/P_D^t and it is also likely that q_{1n} will be unusually high and hence greater than the average period 1 Dutot quantity level, q_D^1. Thus the N terms in the definition of ε_L^{t*} will tend to be negative and thus $1 - \varepsilon_L^{t*}$ will tend to be greater than 1. Thus there will be a tendency for $P_D^t < P_L^t$ for $t \geq 2$ but again, this expectation is a weak one if there are large fluctuations in the deflated period t prices, p_{tn}/P_D^t for n = 1,...,N.

It can be verified that the following identities hold for the period t Laspeyres, Paasche and unit value price and quantity indexes for t = 1,...,T:

(94) $p^t \cdot q^t/p^1 \cdot q^1 = P_{UV}^t Q_{UV}^t = P_P^t Q_L^t = P_L^t Q_P^t$.

Equations (94) imply the following identities for t = 1,...,T:

[90] Balk (2008; 7) called Q_{UV}^t a Dutot-type quantity index.

$$(95)\ P_{UV}^t/P_P^t = Q_L^t/Q_{UV}^t = 1 - \varepsilon_L^t$$

where the last set of equations follow from equations (91). Thus we expect that $P_{UV}^t > P_P^t$ for t = 2,3,...,T if the products are substitutes and ε_L^t is negative.[91]

We now turn our attention to developing an exact relationship between Q_{UV}^t and the Paasche quantity index Q_P^t. Using definitions (82) and (84), we have for t = 1,...,T:

$$\begin{aligned}(96)\ [Q_{UV}^t]^{-1} - [Q_P^t]^{-1} &= [1_N\cdot q^1/1_N\cdot q^t] - [p^t\cdot q^1/p^t\cdot q^t] && \text{using (82) and (84)}\\ &= \Sigma_{n=1}^N [S_{tn} - s_{tn}][q_{1n}/q_{tn}]\\ &= N\mathrm{Cov}(S^t - s^t, q^1/q^t)\end{aligned}$$

where the second set of equalities in (96) follows using (88) and (86), assuming that $q_{tn} > 0$ for n = 1,...,N.

As usual, there are three special cases of (96) that will imply that $Q_{UV}^t = Q_P^t$: (i) $S^t = s^t$ so that the vector of period t real quantity shares S^t is equal to the period t sales share vector s^t. This condition is equivalent to $p^t = \lambda_t 1_N$ which implies that all period t prices are equal.[92] (ii) $q^t = \lambda_t q^1$ for t = 2,3,...,T so that quantities vary in strict proportion over time. (iii) $N\mathrm{Cov}(S^t - s^t, q^1/q^t) = 0$.

Again, there are two problems with the above bias formula: (i) it is difficult to form a judgement on the sign of the covariance $N\mathrm{Cov}(S^t - s^t, q^1/q^t)$ and (ii) the decomposition given by (96) requires that all components of the period t quantity vector be positive. We will proceed to develop a decomposition that does not require the positivity of q^t. The following exact decomposition holds for t = 1,...,T:

$$\begin{aligned}(97)\ [Q_{UV}^t]^{-1} - [Q_P^t]^{-1} &= [1_N\cdot q^1/1_N\cdot q^t] - [p^t\cdot q^1/p^t\cdot q^t]\\ &= \Sigma_{n=1}^N [(q_{1n}/1_N\cdot q^t) - (p_{tn}q_{1n}/p^t\cdot q^t)]\\ &= \Sigma_{n=1}^N [(1/1_N\cdot q^t) - (p_{tn}/p^t\cdot q^t)]q_{1n}\\ &= \Sigma_{n=1}^N [(p^t\cdot q^t/1_N\cdot q^t) - p_{tn}][q_{1n}/p^t\cdot q^t]\\ &= \Sigma_{n=1}^N [p_{UV}^t - p_{tn}][q_{1n}/p^t\cdot q^t] && \text{using (80) for t = t}\\ &= \Sigma_{n=1}^N [p_{UV}^t - p_{tn}][q_{1n} - (q_{tn}/Q_{UV}^t)]/p^t\cdot q^t && \text{using (87) for t = t}\\ &= [Q_{UV}^t]^{-1} \Sigma_{n=1}^N [p_{UV}^t - p_{tn}][(q_{1n}Q_{UV}^t) - q_{tn}]/p^t\cdot q^t\\ &= [Q_{UV}^t]^{-1} \Sigma_{n=1}^N s_{tn}[(p_{UV}^t/p_{tn}) - 1][(q_{1n}Q_{UV}^t/q_{tn}) - 1] && \text{if } q_{tn} > 0 \text{ for all n}\\ &= [Q_{UV}^t]^{-1}\, \varepsilon_P^t\end{aligned}$$

where the period t error term ε_P^t is defined as follows for t = 1,...,T:

$$(98)\ \varepsilon_P^t \equiv \Sigma_{n=1}^N [p_{UV}^t - p_{tn}][(q_{1n}Q_{UV}^t) - q_{tn}]/p^t\cdot q^t.$$[93]

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[91] As was discussed earlier, if all prices are equal in the base period, then $\varepsilon_L^t = 0$ and $P_{UV}^t/P_P^t = Q_L^t/Q_{UV}^t = 0$.

[92] If $p^t = \lambda 1_N$, so that all prices are equal in period t, then it can be shown directly that $P_{UV}^t = P_L^t$. Thus for the particular choice for units of measurement that makes all prices equal in period t, the unit value price index P_{UV}^t is equal to a fixed base Laspeyres price index which will typically have an upward bias relative to a superlative index.

[93] Note that this error term is homogeneous of degree 0 in the components of p^t, q^1 and q^t. Thus for $\lambda > 0$, we have $\varepsilon_P(p^t,q^1,q^t) = \varepsilon_P(\lambda p^t,q^1,q^t) = \varepsilon_P(p^t,\lambda q^1,q^t) = \varepsilon_P(p^t,q^1,\lambda q^t)$. Note also that ε_P^t is well defined if some quantities are equal to 0 and ε_P^t does depend on the reservation prices p_{tn} for products n that are not present

If $q_{tn} > 0$ for n = 1,...,N, then ε_P^t is equal to $\Sigma_{n=1}^N s_{tn}[(p_{UV}^t/p_{tn}) - 1][(q_{1n}Q_{UV}^t/q_{tn}) - 1]$.

Note that the terms on the right hand side of (97) can be interpreted as $(N/p^t \cdot q^t)$ times the covariance $\mathrm{Cov}(p_{UV}^t 1_N - p^t, q^1 - [Q_{UV}^t]^{-1} q^t)$ since $1_N \cdot (q^1 - [Q_{UV}^t]^{-1} q^t) = 0$. If the products are substitutable, it is likely that this covariance is *negative*, since if p_{tn} is unusually low, we would expect that it would be less than the period t unit value price p_{UV}^t so that $p_{UV}^t - p_{tn} > 0$. If p_{tn} is unusually low, then we also expect that the corresponding q_{tn} is unusually high, and thus it is likely that q_{tn} is greater than $q_{1n}Q_{UV}^t$ and so $q_{1n}Q_{UV}^t - q_{tn} < 0$. Thus the N terms in the covariance will tend to be negative. Thus our expectation is that the error term $\varepsilon_P^t < 0$ and $[Q_{UV}^t]^{-1} < [Q_P^t]^{-1}$ or $Q_{UV}^t > Q_P^t$ for $t \geq 2$.[94]

There are three special cases of (97) that will imply that $Q_{UV}^t = Q_P^t$: (i) $p^t = \lambda_t 1_N$ so that all period t prices are equal; (ii) $q^t = \lambda_t q^1$ for t = 2,3,...,T so that quantities vary in strict proportion over time; (iii) $\mathrm{Cov}(p_{UV}^t 1_N - p^t, q^1 - [Q_{UV}^t]^{-1} q^t) = 0$. These conditions are equivalent to our earlier conditions listed below (96).

If we divide both sides of equation t in equations (97) by $[Q_{UV}^t]^{-1}$, we obtain the following system of identities for t = 1,...,T:

(99) $Q_P^t/Q_{UV}^t = [1 - \varepsilon_P^t]^{-1}$

where we expect ε_P^t to be a small negative number if the products are substitutable. Thus we expect $Q_P^t < Q_{UV}^t < Q_L^t$ for t = 2,3,...,T.

Equations (97) and (99) are valid if we interchange prices and quantities. Using the definitions for the Dutot price and quantity levels and indexes t and interchanging prices and quantities, equations (99) become $P_P^t/P_D^t = [1 - \varepsilon_P^{t*}]^{-1}$ where $\varepsilon_P^{t*} \equiv \Sigma_{n=1}^N [q_D^t - q_{tn}][(p_{1n}P_D^t) - p_{tn}]/p^t \cdot q^t$ for t = 1,...,T. If p_{tn} is unusually low, then it is likely that it will be less than p_{tn}/P_D^t and it is also likely that q_{tn} will be unusually high and hence greater than the average period t Dutot quantity level q_D^t. Thus the N terms in the definition of ε_P^{t*} will tend to be negative and hence a tendency for $[1 - \varepsilon_P^{t*}]^{-1}$ to be less than 1. Thus there will be a tendency for $P_P^t < P_D^t$ for $t \geq 2$.

Equations (94) imply the following identities for t = 1,...,T:

$$
\begin{aligned}
(100)\quad P_{UV}^t/P_L^t &= Q_P^t/Q_{UV}^t \\
&= [1 - \varepsilon_P^t]^{-1}
\end{aligned}
$$

where the last set of equations follow from equations (99). Thus we expect that $P_P^t < P_{UV}^t < P_L^t$ for t = 2,3,...,T if the products are substitutes.[95]

in period t. If product n is missing in period t, then it is likely that the reservation price p_{tn} is greater than the unit value price level for period t, p_{UV}^t, and since $q_{tn} = 0$, it can be seen that the nth term on the right hand side of (98) will be negative; i.e., the greater the number of missing products in period t, the greater is the likelihood that ε_P^t is negative.

[94] Our expectation that ε_P^t is negative is more strongly held than our expectation that ε_L^t is negative.

[95] If If $p^t = \lambda 1_N$, then $\varepsilon_P^t = 0$, $P_{UV}^t = P_L^t$ and $Q_{UV}^t = Q_P^t$. Thus if prices in period t are all equal, the period t fixed base unit value index will equal the fixed base Laspeyres price index. Thus the unit value index will tend to have an upward bias relative to a superlative index in this equal period t prices case.

Equations (95) and (100) develop exact relationships for the unit value price index P_{UV}^t with the corresponding fixed base Laspeyres and Paasche price indexes, P_L^t and P_P^t. Taking the square root of the product of these two sets of equations leads to the following exact relationships between the fixed base Fisher price index, P_F^t, and its unit value counterpart period t index, P_{UV}^t, for t = 1,...,T:

$$(101)\ P_{UV}^t = P_F^t\{(1 - \varepsilon_L^t)/(1 - \varepsilon_P^t)\}^{1/2}$$

where ε_L^t and ε_P^t are defined by (90) and (98). If there are no strong (divergent) trends in prices and quantities, then it is likely that ε_L^t is approximately equal to ε_P^t and hence under these conditions, it is likely that $P_{UV}^t \approx P_F^t$; i.e., the unit value price index will provide an adequate approximation to the fixed base Fisher price index under these conditions. However, with diverging trends in prices and quantities (in opposite directions), we would expect the error term ε_P^t defined by (98) to be more negative than the error term ε_L^t defined by (90) and thus under these conditions, we expect the unit value price index P_{UV}^t to have a *downward bias* relative to its Fisher price index counterpart P_F^t.[96]

However, if there are missing products in period 1 so that that some q_{1n} are equal to 0 and the corresponding imputed prices p_{1n} are greater than the unit value price for observation 1, p_{UV}^1, then the nth term in the sum of terms on the right hand side of (90) can become negative and large in magnitude, which can make ε_L^t defined by (90) much more negative than ε_P^t, which in turn means that P_{UV}^t will be greater than unit value price index P_F^t using (101) above. Thus under these circumstances, the unit value price index P_{UV}^t will have an *upward bias* relative to its Fisher price index counterpart P_F^t.

It is possible that unit value price indexes can approximate their Fisher counterparts to some degree in some circumstances but these approximations are not likely to be very accurate. If the products are somewhat heterogeneous and there are some divergent trends in price and quantities, then the approximations are likely to be poor.[97] They are also likely to be poor if there is substantial product turnover.

10. Quality Adjusted Unit Value Price and Quantity Indexes

In the previous section, the period t unit value quantity *level* was defined by $q_{UV}^t \equiv 1_N \cdot q^t = \Sigma_{n=1}^N q_{tn}$ for t = 1,...,T. The corresponding period t unit value quantity *index* was defined by (82) for t = 1,...,T; i.e., $Q_{UV}^t \equiv 1_N \cdot q^t / 1_N \cdot q^1$. In the present section, we will consider *quality adjusted unit value quantity levels*, $q_{UV\alpha}^t$, and the corresponding *quality adjusted unit value quantity indexes*, $Q_{UV\alpha}^t$, defined as follows for t = 1,...,T:

$$(102)\ q_{UV\alpha}^t \equiv \alpha \cdot q^t\ ;$$

[96] The Dutot price index counterparts to the exact relations (101) are $P_F^t = P_D^t\{(1 - \varepsilon_L^{t*})/(1 - \varepsilon_P^{t*})\}^{1/2}$ for t =1,...,T. Thus with diverging trends in prices and quantities (in opposite directions), we would expect the error term ε_P^{t*} to be more negative than the error term ε_L^{t*} and hence we would expect $P_D^t > P_F^t$ for $t \geq 2$. Note that the Dutot price index can be interpreted as a *fixed basket price index* where the basket is proportional to a vector of ones. Thus with divergent trends in prices and quantities in opposite directions, we would expect the Dutot index to exhibit substitution bias and hence we would expect $P_D^t > P_F^t$ for $t \geq 2$.

[97] The problem with unit value price indexes is that they correspond to an additive quantity level. If one takes the economic approach to index number theory, then an additive quantity level corresponds to a linear utility function which implies an infinite elasticity of substitution between products, which is too high in general.

(103) $Q_{UV\alpha}{}^t \equiv q_{UV\alpha}{}^t/q_{UV\alpha}{}^1 = \alpha \cdot q^t/\alpha \cdot q^1$

where $\alpha \equiv [\alpha_1,...,\alpha_N]$ is a vector of positive *quality adjustment factors*. Note that if consumers value their purchases of the N products according to the linear utility function $f(q) \equiv \alpha \cdot q$, then the period t quality adjusted aggregate quantity level $q_{UV\alpha}{}^t = \alpha \cdot q^t$ can be interpreted as the aggregate (sub) *utility* of consumers of the N products. Note that this utility function is linear and thus the products are perfect substitutes, after adjusting for the relative quality of the products. The bigger α_n is, the more consumers will value a unit of product n over other products. The period t *quality adjusted unit value price level* and *price index*, $p_{UV\alpha}{}^t$ and $P_{UV\alpha}{}^t$, are defined as follows for t = 1,...,T:

(104) $p_{UV\alpha}{}^t \equiv p^t \cdot q^t/q_{UV\alpha}{}^t = p^t \cdot q^t/\alpha \cdot q^t$;
(105) $P_{UV\alpha}{}^t \equiv p_{UV\alpha}{}^t/p_{UV\alpha}{}^1 = [p^t \cdot q^t/p^1 \cdot q^1]/Q_{UV\alpha}{}^t$.

It is easy to check that the quality adjusted unit value price index satisfies Walsh's multiperiod identity test and thus is free from chain drift.[98] Note that the $P_{UV\alpha}{}^t$ and $Q_{UV\alpha}{}^t$ *do not depend on any estimated reservation prices*; i.e., the definitions of $P_{UV\alpha}{}^t$ and $Q_{UV\alpha}{}^t$ zero out any reservation prices that are applied to missing products.

Quality adjusted unit value price indexes are consistent with the economic approach to index number theory. If consumers of the N products under consideration all have linear utility functions of the form $f(q) \equiv \alpha \cdot q = \Sigma_{n=1}{}^N \alpha_n q_n$, then $Q_{UV\alpha}{}^t$ defined by (103) accurately represents real welfare growth going from period 1 to t and $P_{UV\alpha}{}^t$ defined by (105) represents consumer inflation over this period. It does not matter if there are new or disappearing products over this period; aggregate welfare or utility for period t is well defined as $\Sigma_{n=1}{}^N \alpha_n q_{tn}$ even if some q_{tn} are equal to 0. If $q_{tn} = 0$, then the contribution of product n to utility in period t is $\alpha_n q_n = 0$. Furthermore, the quality adjusted unit value price and quantity indexes are invariant to changes in the units of measurement if we make the convention that if the units of measurement of q_n are changed to $\lambda_n q_n$ for some positive constant λ_n, then the corresponding α_n is changed to α_n/λ_n.[99] Note that regular unit value price indexes are not invariant to changes in the units of measurement.

From the viewpoint of the economic approach to index number theory, the problem with quality adjusted unit value price and quantity indexes is that the underlying linear utility function assumes that the N products under consideration are perfect substitutes after quality adjustment. Linear preferences are a special case of Constant Elasticity of Substitution preferences and the elasticity of substitution for a linear preferences is equal to plus infinity. Empirical estimates for the elasticity of substitution are far less than plus infinity.[100]

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[98] The term "quality adjusted unit value price index" was introduced by Dalén (2001). Its properties were further studied by de Haan (2004b) (2010) and de Haan and Krsinich (2018). Von Auer (2014) considered a wide variety of choices for the weight vector α (including $\alpha = p^1$ and $\alpha = p^t$) and he looked at the axiomatic properties of the resulting indexes.

[99] Some methods for estimating the α_n are suggested in Diewert and Feenstra (2017) and Diewert (2021c).

[100] Quality adjusted unit value price and quantity levels are also consistent with Leontief (no substitition) preferences. In this case, the dual unit cost function is equal to $c(p) \equiv \Sigma_{n=1}{}^N \beta_n p_n$ where the β_n are positive preference parameters. The period t quantity vector that is consistent with these preferences is $q^t = u_t \beta$ for t = 1,...,T where $\beta \equiv [\beta_1,...,\beta_N]$ and u_t is the period t utility level. Thus the quantity vectors q^t will vary in strict proportion over time. This model of consumer behavior is inconsistent with situations where there are new and disappearing products over the T periods. Moreover, empirically, quantity vectors do not vary in a proportional manner over time.

We will start out by comparing $Q_{UV\alpha}{}^t$ to the corresponding Laspeyres, Paasche and Fisher period t quantity indexes, $Q_L{}^t$, $Q_P{}^t$ and $Q_F{}^t$. The algebra in this section follows the algebra in the preceding section. Thus the counterparts to the identities (87) in the previous section are the following identities for t = 1,...,T:

$$(106)\ \Sigma_{n=1}{}^N [\alpha_n p_{UV\alpha}{}^t - p_{tn}]q_{tn} = \Sigma_{n=1}{}^N [\alpha_n(p^t\cdot q^t/\alpha\cdot q^t) - p_{tn}]q_{tn} \qquad \text{using definitions (104)}$$
$$= (p^t\cdot q^t/\alpha\cdot q^t)\alpha\cdot q^t - p^t\cdot q^t$$
$$= 0.$$

The difference between the quality adjusted unit value quantity index for period t, $Q_{UV\alpha}{}^t$, and the Laspeyres quantity index for period t, $Q_L{}^t$, can be written as follows for t = 1,...,T:

$$(107)\ Q_{UV\alpha}{}^t - Q_L{}^t = [\alpha\cdot q^t/\alpha\cdot q^1] - [p^1\cdot q^t/p^1\cdot q^1] \qquad \text{using (83) and (103)}$$
$$= \Sigma_{n=1}{}^N [(\alpha_n q_{tn}/\alpha\cdot q^1) - (p_{1n}q_{tn}/p^1\cdot q^1)]$$
$$= \Sigma_{n=1}{}^N [(\alpha_n/\alpha\cdot q^1) - (p_{1n}/p^1\cdot q^1)]q_{tn}$$
$$= \Sigma_{n=1}{}^N [(\alpha_n p^1\cdot q^1/\alpha\cdot q^1) - p_{1n}][q_{tn}/p^1\cdot q^1]$$
$$= \Sigma_{n=1}{}^N [\alpha_n p_{UV\alpha}{}^1 - p_{1n}][q_{tn}/p^1\cdot q^1] \qquad \text{using (104) for t = 1}$$
$$= \Sigma_{n=1}{}^N [\alpha_n p_{UV\alpha}{}^1 - p_{1n}][q_{tn} - q_{1n}Q_{UV\alpha}{}^t]/p^1\cdot q^1 \qquad \text{using (106) for t = 1}$$
$$= Q_{UV\alpha}{}^t\, \Sigma_{n=1}{}^N \alpha_n[p_{UV\alpha}{}^1 - (p_{1n}/\alpha_n)][(q_{tn}/Q_{UV\alpha}{}^t) - q_{1n}]/p^1\cdot q^1$$
$$= Q_{UV\alpha}{}^t\, \varepsilon_{L\alpha}{}^t$$

where the period t error term $\varepsilon_{L\alpha}{}^t$ is defined for t = 1,...,T as:

$$(108)\ \varepsilon_{L\alpha}{}^t \equiv \Sigma_{n=1}{}^N \alpha_n[p_{UV\alpha}{}^1 - (p_{1n}/\alpha_n)][(q_{tn}/Q_{UV\alpha}{}^t) - q_{1n}]/p^1\cdot q^1.$$ [101]

Assuming that $\alpha_n > 0$ for n = 1,...,N, the vector of period t *quality adjusted prices* $p_\alpha{}^t$ is defined as follows for t = 1,...,T:

$$(109)\ p_\alpha{}^t \equiv [p_{t1\alpha},..., p_{tN\alpha}] \equiv [p_{t1}/\alpha_1,p_{t2}/\alpha_2,...,p_{tN}/\alpha_N].$$

It can be seen that $p_{UV\alpha}{}^1 - (p_{1n}/\alpha_n)$ is the difference between the period 1 unit value price level, $p_{UV\alpha}{}^1$, and the period 1 quality adjusted price for product n, p_{1n}/α_n. Define the period t *quality adjusted quantity share for product n* (using the vector α of quality adjustment factors) as follows for t = 1,...,T and n = 1,...,N:

$$(110)\ S_{tn\alpha} \equiv \alpha_n q_{tn}/\alpha\cdot q^t.$$

The vector of *period t quality adjusted real product shares* (using the vector α of quality adjustment factors) is defined as $S_\alpha{}^t \equiv [S_{t1\alpha},S_{t2\alpha},...,S_{tN\alpha}]$ for t = 1,...,T. It can be seen that these vectors are share vectors in that their components sum to 1; i.e., we have for t = 1,...,T:

$$(111)\ 1_N\cdot S_\alpha{}^t = 1.$$

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[101] This error term is homogeneous of degree 0 in the components of p^1, q^1 and q^t. Hence it is invariant to proportional changes in the components of these vectors. Definition (108) is only valid if all $\alpha_n > 0$. If this is not the case, redefine $\varepsilon_{L\alpha}{}^t$ as $\Sigma_{n=1}{}^N [\alpha_n p_{UV\alpha}{}^1 - p_{1n}][q_{tn} - q_{1n}Q_{UV\alpha}{}^t]/p^1\cdot q^1$ and with this change, the decomposition defined by the last line of (107) will continue to hold. It should be noted that $\varepsilon_{L\alpha}{}^t$ does not have an interpretation as a *covariance* between a vector of price differences and a vector of quantity differences.

Using the above definitions, we can show that the period t quality adjusted unit value price level, $p_{UV\alpha}{}^t$ defined by (104) is equal to a share weighted average of the period t quality adjusted prices $p_{tn\alpha} = p_{tn}/\alpha_n$ defined by (109); i.e., for t = 1,...,T, we have the following equations:

$$\begin{aligned}(112)\ p_{UV\alpha}{}^t &= p^t\cdot q^t/\alpha\cdot q^t && \text{using (104)}\\ &= \Sigma_{n=1}{}^N (p_{tn}/\alpha_n)(\alpha_n q_{tn})/\alpha\cdot q^t\\ &= \Sigma_{n=1}{}^N S_{tn\alpha}p_{tn\alpha} && \text{using (109) and (110)}\\ &= S_\alpha{}^t\cdot p_\alpha{}^t.\end{aligned}$$

Now we are in a position to determine the likely sign of $\varepsilon_{L\alpha}{}^t$ defined by (108). If the products are substitutable, it is likely that $\varepsilon_{L\alpha}{}^t$ is *negative*, since if p_{1n} is unusually low, then it is likely that the quality adjusted price for product n, p_{1n}/α_n, is below the weighted average of the quality adjusted prices for period 1 which is $p_{UV\alpha}{}^1 = S_\alpha{}^1\cdot p_\alpha{}^1$ using (112) for t = 1. Thus we expect that $p_{UV\alpha}{}^1 - (p_{1n}/\alpha_n) > 0$. If p_{1n} is unusually low, then we would expect that the corresponding q_{1n} is unusually high, and thus it is likely that q_{1n} is greater than $q_{tn}/Q_{UV\alpha}{}^t$ and so $q_{tn}/Q_{UV\alpha}{}^t - q_{1n} < 0$. Thus the sum of the N terms on the right hand side of (108) is likely to be negative. Our expectation[102] is that the error term $\varepsilon_{L\alpha}{}^t < 0$ and hence $Q_{UV\alpha}{}^t < Q_L{}^t$ for $t \geq 2$.

As usual, there are three special cases of (108) that will imply that $Q_{UV\alpha}{}^t = Q_L{}^t$: (i) $p_\alpha{}^1 = \lambda_1 1_N$ so that all period 1 quality adjusted prices are equal;[103] (ii) $q^t = \lambda_t q^1$ for t = 2,3,...,T so that quantities vary in strict proportion over time; (iii) the following sum of price differences times quantity differences equals 0; i.e., $\Sigma_{n=1}{}^N [\alpha_n p_{UV\alpha}{}^1 - p_{1n}][(q_{tn}/Q_{UV\alpha}{}^t) - q_{1n}] = 0$.

If we divide both sides of equation t in equations (108) by $Q_{UV\alpha}{}^t$, we obtain the following system of identities for t = 1,...,T:

$$(113)\ Q_L{}^t/Q_{UV\alpha}{}^t = 1 - \varepsilon_{L\alpha}{}^t$$

where we expect $\varepsilon_{L\alpha}{}^t$ to be a small negative number if the products are substitutes.[104]

The difference between the reciprocal of the quality adjusted unit value quantity index for period t, $[Q_{UV\alpha}{}^t]^{-1}$ and the reciprocal of the Paasche quantity index for period t, $[Q_P{}^t]^{-1}$, can be written as follows for t = 1,...,T:

$$\begin{aligned}(114)\ [Q_{UV\alpha}{}^t]^{-1} - [Q_P{}^t]^{-1} &= [\alpha\cdot q^1/\alpha\cdot q^t] - [p^t\cdot q^1/p^t\cdot q^t] && \text{using (84) and (103)}\\ &= \Sigma_{n=1}{}^N [(\alpha_n q_{1n}/\alpha\cdot q^t) - (p_{tn}q_{1n}/p^t\cdot q^t)]\\ &= \Sigma_{n=1}{}^N [(\alpha_n/\alpha\cdot q^t) - (p_{tn}/p^t\cdot q^t)]q_{1n}\\ &= \Sigma_{n=1}{}^N [(\alpha_n p^t\cdot q^t/\alpha\cdot q^t) - p_{tn}][q_{1n}/p^t\cdot q^t]\\ &= \Sigma_{n=1}{}^N [\alpha_n p_{UV\alpha}{}^t - p_{tn}][q_{1n}/p^t\cdot q^t] && \text{using (104)}\end{aligned}$$

[102] As in the previous section, this expectation is not held with great conviction if the period t quantities have a large variance.

[103] The condition $p_\alpha{}^1 = \lambda_1 1_N$ is equivalent to $p^1 = \lambda_1\alpha$. Thus if we choose α to be proportional to the period 1 price vector p^1, then $Q_{UV\alpha}{}^t = Q_L{}^t$ and $P_{UV\alpha}{}^t = P_P{}^t$, the fixed base Paasche price index. Thus with this choice of α, the quality adjusted unit value index will usually have a downward bias relative to a superlative index. This result requires that p^1 be strictly positive.

[104] If $q_{1n} = 0$ and the period 1 quality adjusted reservation price p_{1n}/α_n is greater than the period 1 unit value price $p_{UV\alpha}{}^1$, then $\varepsilon_{L\alpha}{}^t$ defined by (108) could be a large negative number.

$= \Sigma_{n=1}^{N} [\alpha_n p_{UV\alpha}^t - p_{tn}][q_{1n} - (q_{tn}/Q_{UV\alpha}^t)]/p^t \cdot q^t$ using (106)
$= [Q_{UV\alpha}^t]^{-1} \Sigma_{n=1}^{N} \alpha_n [p_{UV\alpha}^t - (p_{tn}/\alpha_n)][(q_{1n}Q_{UV\alpha}^t) - q_{tn}]/p^t \cdot q^t$
$= [Q_{UV\alpha}^t]^{-1} \varepsilon_{P\alpha}^t$

where the period t error term $\varepsilon_{P\alpha}^t$ is defined for t = 1,...,T as:

(115) $\varepsilon_{P\alpha}^t \equiv \Sigma_{n=1}^{N} \alpha_n [p_{UV\alpha}^t - (p_{tn}/\alpha_n)][(q_{1n}Q_{UV\alpha}^t) - q_{tn}]/p^t \cdot q^t$.[105]

If the products are substitutable, it is likely that $\varepsilon_{P\alpha}^t$ is *negative*, since if p_{tn} is unusually low, then it is likely that the period t quality adjusted price for product n, p_{tn}/α_n, is below the weighted average of the quality adjusted prices for period t which is $p_{UV\alpha}^t = S_\alpha^t \cdot p_\alpha^t$ using (112). Thus we expect that $p_{UV\alpha}^t - (p_{tn}/\alpha_n) > 0$. If p_{tn} is unusually low, then we would expect that the corresponding q_{tn} is unusually high, and thus it is likely that q_{tn} is greater than $q_{1n}Q_{UV\alpha}^t$ and so $q_{1n}Q_{UV\alpha}^t - q_{tn} < 0$. Thus the sum of the N terms on the right hand side of (115) is likely to be negative. Thus our expectation is that the error term $\varepsilon_{P\alpha}^t < 0$ and hence $[Q_{UV\alpha}^t]^{-1} < [Q_P^t]^{-1}$ for $t \geq 2$. Assuming that $\varepsilon_{L\alpha}^t$ is also negative, we have $Q_P^t < Q_{UV\alpha}^t < Q_L^t$ for t = 2,...,T as inequalities that are likely to hold.

As usual, there are three special cases of (114) that will imply that $Q_{UV\alpha}^t = Q_P^t$: (i) $p_\alpha^t = \lambda_t 1_N$ so that all period t quality adjusted prices are equal; (ii) $q^t = \lambda_t q^1$ for t = 2,3,...,T so that quantities vary in strict proportion over time; (iii) the following sum of price differences times quantity differences equals zero: i.e., $\Sigma_{n=1}^{N} [\alpha_n p_{UV\alpha}^t - p_{tn}][(q_{1n}Q_{UV\alpha}^t) - q_{tn}] = 0$.

If we divide both sides of equation t in equations (114) by $[Q_{UV\alpha}^t]^{-1}$, we obtain the following system of identities for t = 1,...,T:

(116) $Q_P^t/Q_{UV\alpha}^t = [1 - \varepsilon_{P\alpha}^t]^{-1}$

where we expect $\varepsilon_{P\alpha}^t$ to be a small negative number if the products are substitutes.

Equations (113) and (116) develop exact relationships for the quality adjusted unit value quantity index $Q_{UV\alpha}^t$ with the corresponding fixed base Laspeyres and Paasche quantity indexes, Q_L^t and Q_P^t. Taking the square root of the product of these two sets of equations leads to the following exact relationships between the fixed base Fisher quantity index, Q_F^t, and its quality adjusted unit value counterpart period t quantity index, $Q_{UV\alpha}^t$, for t = 1,...,T:

(117) $Q_F^t = Q_{UV\alpha}^t\{(1 - \varepsilon_{L\alpha}^t)/(1 - \varepsilon_{P\alpha}^t)\}^{1/2}$

where $\varepsilon_{L\alpha}^t$ and $\varepsilon_{P\alpha}^t$ are defined by (108) and (115). If there are no strong (divergent) trends in prices and quantities, then it is likely that $\varepsilon_{L\alpha}^t$ is approximately equal to $\varepsilon_{P\alpha}^t$ and hence under these conditions, it is likely that $Q_{UV\alpha}^t \approx Q_F^t$; i.e., the quality adjusted unit value quantity index will provide an adequate approximation to the fixed base Fisher price index under these conditions. However, if there are divergent trends in prices and quantities (in opposite directions), then it is likely that $\varepsilon_{P\alpha}^t$ will be more negative than $\varepsilon_{L\alpha}^t$ and hence it is likely that $Q_F^t < Q_{UV\alpha}^t$ for t = 2,...,T;

[105] This error term is homogeneous of degree 0 in the components of p^t, q^1 and q^t. Hence it is invariant to proportional changes in the components of these vectors. Definition (115) is only valid if all $\alpha_n > 0$. If this is not the case, then redefine $\varepsilon_{P\alpha}^t$ as $\Sigma_{n=1}^{N} [\alpha_n p_{UV\alpha}^t - p_{tn}][(q_{1n}Q_{UV\alpha}^t) - q_{tn}]/p^t \cdot q^t$ and with this change, the decomposition defined by the last line of (114) will continue to hold..

i.e., *with divergent trends in prices and quantities, the quality adjusted unit value quantity index is likely to have an upward bias* relative to its Fisher quantity index counterparts.[106]

Using equations (105), we have the following counterparts to equations (94) for t = 1,...,T:

$$(118)\ p^t\cdot q^t/p^1\cdot q^1 = P_{UV\alpha}{}^tQ_{UV\alpha}{}^t = P_P{}^tQ_L{}^t = P_L{}^tQ_P{}^t.$$

Equations (113), (116) and (118) imply the following identities for t = 1,...,T:

$$(119)\ P_{UV\alpha}{}^t/P_P{}^t = Q_L{}^t/Q_{UV\alpha}{}^t = 1 - \varepsilon_{L\alpha}{}^t\ ;$$
$$(120)\ P_{UV\alpha}{}^t/P_L{}^t = Q_P{}^t/Q_{UV\alpha}{}^t = [1 - \varepsilon_{P\alpha}{}^t]^{-1}.$$

We expect that $\varepsilon_L{}^t$ and $\varepsilon_{P\alpha}{}^t$ will be predominantly negative if the products are highly substitutable and thus in this case, the quality adjusted unit value indexes $P_{UV\alpha}{}^t$ should satisfy the inequalities $P_P{}^t < P_{UV\alpha}{}^t < P_L{}^t$ for t = 2,3,...,T.

Taking the square root of the product of equations (119) and (120) leads to the following exact relationships between the fixed base Fisher price index, $P_F{}^t$, and its quality adjusted unit value counterpart period t index, $P_{UV\alpha}{}^t$, for t = 1,...,T:

$$(121)\ P_{UV\alpha}{}^t = P_F{}^t\{(1 - \varepsilon_{L\alpha}{}^t)/(1 - \varepsilon_{P\alpha}{}^t)\}^{1/2}$$

where $\varepsilon_{L\alpha}{}^t$ and $\varepsilon_{P\alpha}{}^t$ are defined by (108) and (115). If there are no strong (divergent) trends in prices and quantities, then it is likely that $\varepsilon_{L\alpha}{}^t$ is approximately equal to $\varepsilon_{P\alpha}{}^t$ and hence under these conditions, it is likely that $P_{UV\alpha}{}^t \approx P_F{}^t$; i.e., the quality adjusted unit value price index will provide an adequate approximation to the fixed base Fisher price index under these conditions. However, if there are divergent trends in prices and quantities, then we expect $\varepsilon_{P\alpha}{}^t$ to be more negative than $\varepsilon_{L\alpha}{}^t$ and hence there is an expectation that $P_{UV\alpha}{}^t < P_F{}^t$ for t = 2,...,T; i.e., we expect that normally $P_{UV\alpha}{}^t$ will have a *downward bias* relative to $P_F{}^t$.[107] However, if there are missing products in period 1, then the bias of $P_{UV\alpha}{}^t$ relative to $P_F{}^t$ is uncertain.

11. Relationships between Lowe and Fisher Indexes

We now consider how a Lowe (1823) price index is related to a fixed base Fisher price index. The framework that we consider is similar to the framework developed in section 6 above for the annual share weighted Jevons index, $P_{J\alpha}{}^t$. In the present section, instead of using the average sales shares for the first year in the sample as weights for a weighted Jevons index, we use annual average quantities sold (or purchased) in the first year as a vector of quantity weights for

[106] As was the case in the previous section, if there are missing products in period 1, the expected inequality $Q_F{}^t < Q_{UV\alpha}{}^t$ may be reversed, because $\varepsilon_{L\alpha}{}^t$ defined by (108) may become significantly negative if some q_{1n} equal 0 while their corresponding reservation prices p_{1n} are positive.

[107] Recall that the weighted unit value quantity level, $q_{UV\alpha}{}^t$ is defined as the linear function of the period t quantity data, $\alpha\cdot q^t$. If $T \geq 3$ and the price and quantity data are consistent with purchasers maximizing a utility function that generates data that is exact for the Fisher price index $Q_F{}^t$, then $Q_{UV\alpha}{}^t$ will tend to be greater than $Q_F{}^t$ (and hence $P_{UV\alpha}{}^t$ will tend to be less than $P_F{}^t$) for $t \geq 2$. See Marris (1984; 52), Diewert (1999b; 49) and Diewert and Fox (2020) on this point.

subsequent periods. Define the *annual average quantity vector* $q^* \equiv [q_1^*,...,q_N^*]$ for the first T^* periods in the sample that make up a year, q^*, as follows:[108]

(122) $q^* \equiv (1/T^*)\Sigma_{t=1}^{T^*} q^t$.

As was the case in section 6, the reference year for the weights precedes the reference month for the product prices. Define the *period t Lowe* (1823) *price level* and *price index*, p_{Lo}^t and P_{Lo}^t by (123) and (124) respectively for $t = T^*+1,T^*+2,...,T$:

(123) $p_{Lo}^t \equiv p^t\cdot\alpha$;
(124) $P_{Lo}^t \equiv p_{Lo}^t/p_{Lo}^{T^*+1} = p^t\cdot\alpha/p^{T^*+1}\cdot\alpha$

where the constant price weights vector α is the annual average weights vector q^* defined by (122); i.e., we have:

(125) $\alpha \equiv q^*$.

The *period t Lowe quantity level*, q_{Lo}^t, and the corresponding *period t Lowe quantity index*, Q_{Lo}^t, are defined as follows for $t = T^*+1,T^*+2,...,T$:

(126) $q_{Lo}^t \equiv p^t\cdot q^t/p_{Lo}^t = p^t\cdot q^t/p^t\cdot\alpha = \Sigma_{n=1}^N (p_{tn}\alpha_n/p^t\cdot\alpha)(q_{tn}/\alpha_n)$;[109]
(127) $Q_{Lo}^t \equiv q_{Lo}^t/q_{Lo}^{T^*+1} = [p^t\cdot q^t/p^{T^*+1}\cdot q^{T^*+1}]/P_{Lo}^t$.

It can be seen that the Lowe price index defined by (124) is equal to a *weighted Dutot price index*; see definition (14) above. It is also structurally identical to the quality adjusted unit value quantity index $Q_{UV\alpha}^t$ defined in the previous section, except the role of prices and quantities has been reversed. Thus the identity (107) in the previous section will be valid if we replace $Q_{UV\alpha}^t$ by P_{Lo}^t, replace Q_L^t by P_L^t and interchange prices and quantities on the right hand side of (107).[110] The resulting identities are the following ones for $t = T^*+1,T^*+2,...,T$:

$$
\begin{aligned}
(128)\ P_{Lo}^t - P_L^t &= \Sigma_{n=1}^N [(\alpha_n p_{tn}/\alpha\cdot p^{T^*+1}) - (p_{tn}q_{T^*+1,n}/p^{T^*+1}\cdot q^{T^*+1})] \\
&= \Sigma_{n=1}^N [(\alpha_n/\alpha\cdot p^{T^*+1}) - (q_{T^*+1,n}/p^{T^*+1}\cdot q^{T^*+1})]p_{tn} \\
&= \Sigma_{n=1}^N [(\alpha_n p^{T^*+1}\cdot q^{T^*+1}/\alpha\cdot p^{T^*+1}) - q_{T^*+1,n}][p_{tn}/p^{T^*+1}\cdot q^{T^*+1}] \\
&= \Sigma_{n=1}^N [\alpha_n q_{Lo}^{T^*+1} - q_{T^*+1,n}][p_{tn}/p^{T^*+1}\cdot q^{T^*+1}] \qquad \text{using (126) for } t = T^*+1 \\
&= \Sigma_{n=1}^N [\alpha_n q_{Lo}^{T^*+1} - q_{T^*+1,n}][p_{tn} - p_{T^*+1,n}P_{Lo}^t]/p^{T^*+1}\cdot q^{T^*+1} \text{ [111]} \\
&= P_{Lo}^t\, \Sigma_{n=1}^N [\alpha_n q_{Lo}^{T^*+1} - q_{T^*+1,n}][(p_{tn}/P_{Lo}^t) - p_{T^*+1,n}]/p^{T^*+1}\cdot q^{T^*+1}
\end{aligned}
$$

---

[108] If product n was not available in the first year of the sample, then the nth component of q^*, q_n^*, will equal 0 and hence the nth component of the weight vector α defined by (125) will also equal 0. If product n was also not available in periods $t \geq T^* + 1$, then looking at definitions (123) and (124), it can be seen that P_{LO}^t will not depend on the reservation prices p_{nt} for these subsequent periods where product n is not available. Thus under these circumstances, the Lowe index cannot be consistent with the (Hicksian) economic approach to index number theory since Konüs (1924) true cost of living price indexes will depend on the reservation prices. However, if the products in the elementary aggregate are indeed highly substitutable, then the assumption of a linear utility function will provide an adequate approximation to the "truth" and the estimation of reservation prices becomes unimportant.

[109] This last inequality is only valid if all $\alpha_n > 0$. It can be seen that the Lowe quantity level for period t, q_{Lo}^t, is a share weighted sum of the period t quality adjusted quantities, q_{tn}/α_n.

[110] We also replace period 1 by period T^*+1.

[111] This step follows using the following counterpart to (106): $\Sigma_{n=1}^N [\alpha_n q_{Lo}^{T^*+1} - q_{T^*+1,n}]p_{T^*+1,n} = 0$.

$= P_{Lo}^t \Sigma_{n=1}^N \alpha_n[q_{Lo}^{T*+1} - (q_{T*+1,n}/\alpha_n)][(p_{tn}/P_{Lo}^t) - p_{T*+1,n}]/p^{T*+1}\cdot q^{T*+1}$
$= P_{Lo}^t \varepsilon_{L\alpha}^t$

where the period t error term $\varepsilon_{L\alpha}^t$ is now defined for $t = T^*+1,...,T$ as follows:

(129) $\varepsilon_{L\alpha}^t \equiv \Sigma_{n=1}^N \alpha_n[q_{Lo}^{T*+1} - (q_{T*+1,n}/\alpha_n)][(p_{tn}/P_{Lo}^t) - p_{T*+1,n}]/p^{T*+1}\cdot q^{T*+1}$.[112]

If the products are substitutable, it is likely that $\varepsilon_{L\alpha}^t$ is *negative*, since if $p_{T*+1,n}$ is unusually low, then it is likely that $(p_{tn}/P_{Lo}^t) - p_{T*+1,n} > 0$ and that $q_{T*+1,n}/\alpha_n$ is unusually large and hence is greater than q_{Lo}^{T*+1}, which is a weighted average of the period T^*+1 quantity ratios, $q_{T*+1,1}/\alpha_1$, $q_{T*+1,2}/\alpha_2$,..., $q_{T*+1,N}/\alpha_N$ using definition (126) for $t = T^*+1$. Thus the sum of the N terms on the right hand side of (129) is likely to be negative. Thus our expectation[113] is that the error term $\varepsilon_{L\alpha}^t < 0$ and hence $P_{Lo}^t < P_L^t$ for $t > T^* + 1$.

The α_n can be interpreted as *inverse quality indicators* of the utility provided by one unit of the nth product. Suppose purchasers of the N commodities have Leontief preferences with the utility function $f(q_1,q_2,...,q_N) \equiv \min_n \{q_n/\alpha_n : n = 1,2,...,N\}$. Then the dual unit cost function that corresponds to this functional form is $c(p_1,p_2,...,p_N) \equiv \Sigma_{n=1}^N p_n\alpha_n = p\cdot\alpha$. If we evaluate the unit cost function at the prices of period t, p^t, we obtain the Lowe price level for period t defined by (123); i.e., $p_{Lo}^t \equiv p^t\cdot\alpha$. Thus the bigger α_n is, the more units of q_n it will take for purchasers of the N commodities to attain one unit of utility. Thus the α_n can be interpreted as inverse indicators of the relative utility of each product.

As usual, there are three special cases of (128) that will imply that $P_{Lo}^t = P_L^t$: (i) $q^{T*+1} = \lambda q^*$ for some $\lambda > 0$ so that the period T^*+1 quantity vector q^{T*+1} is proportional to the annual average quantity vector q^* for the base year; (ii) $p^t = \lambda_t p^{T*+1}$ for some $\lambda_t > 0$ for $t = T^*+1,...,T$ so that prices vary in strict proportion over time; (iii) the sum of terms $\Sigma_{n=1}^N [\alpha_n q_{Lo}^{T*+1} - q_{T*+1,n}][(p_{tn}/P_{Lo}^t) - p_{T*+1,n}] = 0$.

If we divide both sides of equation t in equations (128) by P_{Lo}^t, we obtain the following system of identities for $t = T^*+1,...,T$:

(130) $P_L^t/P_{Lo}^t = 1 - \varepsilon_{L\alpha}^t$

where we expect $\varepsilon_{L\alpha}^t$ to be a small negative number.

We turn now to developing a relationship between the Lowe and Paasche price indexes. The difference between reciprocal of the Lowe price index for period t, $[P_{Lo}^t]^{-1}$ and the reciprocal of the Paasche price index for period t, $[P_P^t]^{-1}$, can be written as follows for $t = T^*+1,...,T$:

(131) $[P_{Lo}^t]^{-1} - [P_P^t]^{-1} = [\alpha\cdot p^{T*+1}/\alpha\cdot p^t] - [q^t\cdot p^{T*+1}/q^t\cdot p^t]$
$= \Sigma_{n=1}^N [(\alpha_n p_{T*+1,n}/\alpha\cdot p^t) - (q_{tn}p_{T*+1,n}/p^t\cdot q^t)]$
$= \Sigma_{n=1}^N [(\alpha_n/\alpha\cdot p^t) - (q_{tn}/p^t\cdot q^t)]p_{T*+1,n}$

---

[112] Note that this error term is homogeneous of degree 0 in the components of p^{T*+1}, q^{T*+1} and p^t. Hence it is invariant to proportional changes in the components of these vectors. Definition (129) is only valid if all $\alpha_n > 0$. If this is not the case, redefine $\varepsilon_{L\alpha}^t$ as $\Sigma_{n=1}^N [\alpha_n q_{Lo}^{T*+1} - q_{T*+1,n}][(p_{tn}/P_{Lo}^t) - p_{T*+1,n}]/p^{T*+1}\cdot q^{T*+1}$ and with this change, the decomposition defined by the last line of (128) will continue to hold.
[113] This expectation is not held with great conviction if the period t prices have a large variance.

$$
\begin{aligned}
&= \Sigma_{n=1}^{N} [(\alpha_n p^t \cdot q^t/\alpha \cdot p^t) - q_{tn}][p_{T^*+1,n}/p^t \cdot q^t] \\
&= \Sigma_{n=1}^{N} [\alpha_n q_{Lo}^t - q_{tn}][p_{T^*+1,n}/p^t \cdot q^t] && \text{using (126)} \\
&= \Sigma_{n=1}^{N} [\alpha_n q_{Lo}^t - q_{tn}][p_{T^*+1,n} - (p_{tn}/P_{Lo}^t)]/p^t \cdot q^t \text{ [114]} \\
&= [P_{Lo}^t]^{-1} \Sigma_{n=1}^{N} [\alpha_n q_{Lo}^t - q_{tn}][p_{T^*+1,n} P_{Lo}^t - p_{tn}]/p^t \cdot q^t \\
&= [P_{Lo}^t]^{-1} \Sigma_{n=1}^{N} \alpha_n [q_{Lo}^t - (q_{tn}/\alpha_n)][p_{T^*+1,n} P_{Lo}^t - p_{tn}]/p^t \cdot q^t && \text{if all } \alpha_n > 0 \\
&= [P_{Lo}^t]^{-1} \varepsilon_{P\alpha}^t
\end{aligned}
$$

where the period t error term $\varepsilon_{P\alpha}^t$ is defined for $t = T^*+1,...,T$ as:

(132) $\varepsilon_{P\alpha}^t \equiv \Sigma_{n=1}^{N} \alpha_n[q_{Lo}^t - (q_{tn}/\alpha_n)][p_{T^*+1,n}P_{Lo}^t - p_{tn}]/p^t \cdot q^t$.[115]

If the products are substitutable, it is likely that $\varepsilon_{P\alpha}^t$ is *negative*, since if p_{tn} is unusually low, then it is likely that it will be less than the inflation adjusted nth component of the period T^*+1 price, $p_{T^*+1,n}P_{LO}^t$. If p_{tn} is unusually low, then it is also likely that the period t quality adjusted quantity for product n, q_{tn}/α_n, is above the weighted average of the quality adjusted quantities for period t which is q_{Lo}^t. Thus the sum of the N terms on the right hand side of (132) is likely to be negative. Thus our expectation is that the error term $\varepsilon_{P\alpha}^t < 0$ and hence $[P_{Lo}^t]^{-1} < [P_P^t]^{-1}$ for $T^*+2,...,T$. Assuming that $\varepsilon_{L\alpha}^t$ is also negative, we have $P_P^t < P_{Lo}^t < P_L^t$ for $t = T^*+2,T^*+3,...,T$ as inequalities that are likely to hold.

As usual, there are three special cases of (131) that will imply that $P_{Lo}^t = P_P^t$: (i) $q^t = \lambda q^*$ for some $\lambda > 0$ so that the period t quantity vector q^t is proportional to the annual average quantity vector q^* for the reference year prior to the reference month; (ii) $p^t = \lambda_t p^{T^*+1}$ for $t = T^*+2,T^*+3,...,T$ so that prices vary in strict proportion over time; (iii) the sum of terms $\Sigma_{n=1}^{N} [\alpha_n q_{Lo}^t - q_{tn}][p_{T^*+1,n}P_{Lo}^t - p_{tn}] = 0$.

If we divide both sides of equation t in equations (131) by $[P_{Lo}^t]^{-1}$, we obtain the following system of identities for $t = T^*+1,...,T$:

(133) $P_P^t/P_{Lo}^t = [1 - \varepsilon_{P\alpha}^t]^{-1}$

where we expect $\varepsilon_{P\alpha}^t$ to be a negative number.

Equations (130) and (133) develop exact relationships for the Lowe price index P_{Lo}^t with the corresponding fixed base Laspeyres and Paasche price indexes, P_L^t and P_P^t. Taking the square root of the product of these two sets of equations leads to the following exact relationships between the fixed base Fisher price index, P_F^t, and the corresponding Lowe period t price index, P_{Lo}^t, for $t = T^*+1,...,T$:

(134) $P_F^t = P_{Lo}^t\{(1 - \varepsilon_{L\alpha}^t)/(1 - \varepsilon_{P\alpha}^t)\}^{1/2}$

where $\varepsilon_{L\alpha}^t$ and $\varepsilon_{P\alpha}^t$ are defined by (129) and (132). If there are no strong (divergent) trends in prices and quantities, then it is likely that $\varepsilon_{L\alpha}^t$ is approximately equal to $\varepsilon_{P\alpha}^t$ and hence under these

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[114] This step follows using the following counterpart to (106): $\Sigma_{n=1}^{N} [\alpha_n q_{Lo}^t - q_{tn}]p_{tn} = 0$.

[115] This error term is homogeneous of degree 0 in the components of q^t, p^{T^*+1} and p^t. Hence it is invariant to proportional changes in the components of these vectors. Definition (132) is only valid if all $\alpha_n > 0$. If this is not the case, redefine $\varepsilon_{P\alpha}^t$ as $\Sigma_{n=1}^{N} [\alpha_n q_{Lo}^t - q_{tn}][p_{T^*+1,n}P_{Lo}^t - p_{tn}]/p^t \cdot q^t$ and with this change, the decomposition defined by the last line of (131) will continue to hold.

conditions, it is likely that $P_{Lo}^t \approx P_F^t$; i.e., the Lowe price index will provide an adequate approximation to the fixed base Fisher price index under these conditions. However, if there are divergent trends in prices and quantities (in diverging directions), then it is likely that $\varepsilon_{P\alpha}^t$ will be more negative than $\varepsilon_{L\alpha}^t$ and hence it is likely that $P_F^t < P_{Lo}^t$ for $t = T^*+2,...,T$; i.e., *with divergent trends in prices and quantities, the Lowe price index is likely to have an upward bias* relative to its Fisher Price index counterpart. This is an intuitively plausible result since the Lowe index is a fixed basket type index and hence will be subject to some upward substitution bias relative to the Fisher index which is able to control for substitution bias.

In the following section, we show that the Geary Khamis multilateral indexes can be regarded as quality adjusted unit value price indexes and hence the analysis in section 10 on quality adjusted unit value price indexes can be applied to GK multilateral indexes.

12. Geary Khamis Multilateral Indexes

The GK multilateral method was introduced by Geary (1958) in the context of making international comparisons of prices. Khamis (1970) showed that the equations that define the method have a positive solution under certain conditions. A modification of this method has been adapted to the time series context and is being used to construct some components of the Dutch CPI; see Chessa (2016). The GK index was the multilateral index chosen by the Dutch to avoid the chain drift problem for the segments of their CPI that use scanner data.

The GK system of equations for T time periods involves T *price levels* $p_{GK}^1,...,p_{GK}^T$ and N *quality adjustment factors* $\alpha_1,...,\alpha_N$.[116] Let p^t and q^t denote the N dimensional price and quantity vectors for period t (with components p_{tn} and q_{tn} as usual). Define the total consumption (or sales) vector q over the entire window of observations as the following simple sum of the period by period consumption vectors:

(135) $q \equiv \sum_{t=1}^T q^t$

where $q \equiv [q_1,q_2,...,q_N]$. The equations which determine the *GK price levels* $p_{GK}^1,...,p_{GK}^T$ and *quality adjustment factors* $\alpha_1,...,\alpha_N$ (up to a scalar multiple) are the following ones:

(136) $\alpha_n = \sum_{t=1}^T [q_{tn}/q_n][p_{tn}/p_{GK}^t]$; $\quad n = 1,...,N;$
(137) $p_{GK}^t = p^t\cdot q^t/\alpha\cdot q^t = \Sigma_{n=1}^N [\alpha_n q_{tn}/\alpha\cdot q^t][p_{tn}/\alpha_n]$; $\quad t = 1,...,T$

where $\alpha \equiv [\alpha_1,...,\alpha_N]$ is the vector of GK quality adjustment factors. The sample share of period t's purchases of commodity n in total sales of commodity n over all T periods can be defined as $S_{tn} \equiv q_{tn}/q_n$ for n = 1,...,N and t = 1,...,T. Thus $\alpha_n \equiv \Sigma_{t=1}^T S_{tn}[p_{tn}/p_{GK}^t]$ is a (real) share weighted average of the period t inflation adjusted prices p_{tn}/p_{GK}^t for product n over all T periods. The period t quality adjusted sum of quantities sold is defined as the *period t GK quantity level*, $q_{GK}^t \equiv \alpha\cdot q^t = \Sigma_{n=1}^N \alpha_n q_{tn}$.[117] This period t quantity level is divided into the value of period t sales, $p^t\cdot q^t = \Sigma_{n=1}^N p_{tn}q_{tn}$, in order to obtain the period t GK price level, p_{GK}^t. Thus the GK price level for period t can be interpreted as a *quality adjusted unit value index* where the α_n act as the quality adjustment factors.

[116] In the international context, the α_n are interpreted as international commodity reference prices.
[117] Khamis (1972; 101) also derived this equation in the time series context.

Note that the GK price level, p_{GK}^t defined by (137) *does not depend on the estimated reservation prices*; i.e., the definition of p_{GK}^t zeros out any reservation prices that are applied to missing products and thus $P_{GK}^t \equiv p_{GK}^t/p_{GK}^1$ also does not depend on reservation prices.[118] A related property of the GK price levels is the following one: if a product n^* is only available in a single period t^*, then the GK price levels p_{GK}^t do not depend on $p_{n^*t^*}$ or $q_{n^*t^*}$.[119]

It can be seen that if a solution to equations (136) and (137) exists, then if all of the period price levels p_{GK}^t are multiplied by a positive scalar λ say and all of the quality adjustment factors α_n are divided by the same λ, then another solution to (136) and (137) is obtained. Hence, the α_n and p_{GK}^t are only determined up to a scalar multiple and an additional normalization is required such as $p_{GK}^1 = 1$ or $\alpha_1 = 1$ is required to determine a unique solution to the system of equations defined by (136) and (137).[120] It can also be shown that only N + T − 1 of the N + T equations in (136) and (137) are independent.

Using the normalization $p_{GK}^1 = 1$, it is straightforward to show that the GK price levels, p_{GK}^t, are invariant to changes in the units of measurement. Suppose we have a solution p_{GK}^t and α_n for t = 1,...,T and n = 1,...,N with $p_{GK}^1 \equiv 1$. Let $\lambda_n > 0$ for n = 1,...,N. Use these λ_n to measure prices and quantities in new units of measurement; i.e., define $p_{tn}^* \equiv \lambda_n p_{tn}$ and $q_{tn}^* \equiv (\lambda_n)^{-1} q_{tn}$ for t = 1,...,T and n = 1,...,N. Now substitute these transformed prices and quantities into equations (135)-(137). It is straightforward to show that the initial solution GK price levels, p_{GK}^t, along with new $\alpha_n^* \equiv \lambda_n\alpha_n$ also statisfy the new GK equations (135)-(137).

A traditional method for obtaining a solution to (136) and (137) is to iterate between these equations. Thus set $\alpha = 1_N$, a vector of ones, and use equations (137) to obtain an initial sequence for the p_{GK}^t. Substitute these p_{GK}^t estimates into equations (136) and obtain α_n estimates. Substitute these α_n estimates into equations (137) and obtain a new sequence of p_{GK}^t estimates. Continue iterating between the two systems until convergence is achieved.

An alternative method is more efficient. Following Diewert (1999b; 26),[121] substitute equations (137) into equations (136) and after some simplification, obtain the following system of equations that will determine the components of the α vector:

$$(138)\quad [I_N - C]\alpha = 0_N$$

where I_N is the N by N identity matrix, 0_N is a vector of zeros of dimension N and the C matrix is defined as follows:

$$(139)\quad C \equiv \hat{q}^{-1}\Sigma_{t=1}^T s^t q^{tT}$$

[118] In equations (136) and (137), each price p_{tn} always appears with the multiplicative factor q_{tn}. Thus if p_{tn} is an imputed price, it will always be multiplied by $q_{tn} = 0$ and thus any imputed price will have no impact on the α_n and p_{GK}^t. Thus this method fails Test 9 in section 21 below.

[119] Let product n^* be available only in period t^*. Using (136) for $n = n^*$, we have: (i) $\alpha_{n^*} = p_{t^*n^*}/p_{GK}^{t^*}$. Equations (137) can be rewritten as follows: (ii) $p_{GK}^t\,\alpha\cdot q^t = p^t\cdot q^t$; t = 1,...,T. Note that for $t \neq t^*$, these equations do not depend directly on α_{n^*}, $p_{t^*n^*}$ or $q_{t^*n^*}$. For period $t = t^*$, equation t^* in (137) can be written as: (iii) $p_{GK}^{t^*}(\Sigma_{n\neq n^*}\alpha_n q_{t^*n} + \alpha_{n^*}q_{t^*n^*}) = (\Sigma_{n\neq n^*} p_{t^*n}q_{t^*n} + p_{t^*n^*}q_{t^*n^*})$. Substitute (i) into (iii) and after some simplification, we find that $p_{GK}^{t^*} = \Sigma_{n\neq n^*} p_{t^*n}q_{t^*n}/\Sigma_{n\neq n^*}\alpha_n q_{t^*n}$. This proof is due to Claude Lamboray. Thus this method fails Test 8 in section 21 below.

[120] See Diewert and Fox (2017) for various solution methods.

[121] See also Diewert and Fox (2020) for additional discussion on this solution method.

where $\hat{q}$ is an N by N diagonal matrix with the elements of the total window purchase vector q running down the main diagonal and $\hat{q}^{-1}$ denotes the inverse of this matrix, s^t is the period t expenditure share column vector, q^t is the column vector of quantities purchased during period t and q_n is the nth element of the sample total q defined by (135).

The matrix $I_N - C$ is singular which implies that the N equations in (138) are not all independent. In particular, if the first N−1 equations in (138) are satisfied, then the last equation in (138) will also be satisfied. It can also be seen that the N equations in (138) are homogeneous of degree one in the components of the vector α. Thus to obtain a unique b solution to (138), set α_N equal to 1, drop the last equation in (138) and solve the remaining N−1 equations for $\alpha_1,\alpha_2,...,\alpha_{N-1}$. Once the α_n are known, equations (137) can be used to determine the GK price levels, $p_{GK}{}^t = p^t{\cdot}q^t/\alpha{\cdot}q^t$ for t = 1,...,T.

Using equations (137), it can be seen that the *GK price index for period t* (relative to period 1) is equal to $P_{GK}{}^t \equiv p_{GK}{}^t/p_{GK}{}^1 = [p^t{\cdot}q^t/\alpha{\cdot}q^t]/[p^1{\cdot}q^1/\alpha{\cdot}q^1]$ for t = 1,...,T and thus these indexes are *quality adjusted unit value price indexes* with a particular choice for the vector of quality adjustment factors α. Thus these indexes lead to corresponding *additive quantity levels* $q_{GK}{}^t$ that correspond to the linear utility function, $f(q) \equiv \alpha{\cdot}q$.[122] As we saw in section 10, this type of index can approximate the corresponding fixed base Fisher price index provided that there are no systematic divergent trends in prices and quantities. However, if there are diverging trends in prices and quantities (in opposite directions), then we expect the GK price indexes to be subject to some *substitution bias* with the expectation that the GK price index for period $t \geq 2$ to be somewhat *below* the corresponding Fisher fixed base price index. Thus we expect GK and quality adjusted unit value price indexes to *normally* have a downward bias relative to their Fisher and Törnqvist counterparts, provided that there are no missing products, the products are highly substitutable and there are divergent trends in prices and quantities. However, if there are missing products in period 1, then it is quite possible for the GK price indexes to have an upward bias relative to their Fisher fixed base counterparts, which, in principle, use reservation prices for the missing products.[123]

In the following five sections, we will study in some detail another popular method for making price level comparisons over multiple periods: the Weighted Time Product Dummy Multilateral Indexes. The general case with missing observations will be studied in Section 17. It proves to be useful to consider simpler special cases of the method in sections 13-16 below.

13. Time Product Dummy Regressions: The Case of No Missing Observations

[122] Using the economic approach to index number theory, it can be seen that the GK price indexes will be exactly the correct price indexes to use if purchasers maximize utility using a common linear utility function. Diewert (1999b; 27) and Diewert and Fox (2020) show that the GK price indexes will also be exactly correct if purchasers maximize a Leontief no substitution utility function. These extreme cases are empirically unlikely. As was noted earlier in section 10 , Leontief preferences are not consistent with new and disappearing products.

[123] New products appear with some degree of regularity and so it is likely that there will be missing products in period 1 and this may reverse the "normal" inequality, $P_{GK}{}^t < P_F{}^t$, as was the case for Diewert's (2018) scanner data set. This data set is used in the Appendix to this chapter. The GK index, like all indexes based on quality adjusted unit values, zeros out the effects of reservation prices for the missing products, whereas Fisher indexes can include the effects of reservation prices.

In this section, it is assumed that price and quantity data for N products are available for T periods. As usual, let $p^t \equiv [p_{t1},...,p_{tN}]$ and $q^t \equiv [q_{t1},...,q_{tN}]$ denote the price and quantity vectors for time periods t = 1,...,T. In this section, it is assumed that there are no missing prices or quantities so that all NT prices and quantities are positive. We assume initially that purchasers of the N products maximize the following linear utility function f(q) defined as follows:

$$(140)\ f(q) = f(q_1,q_2,...,q_N) \equiv \Sigma_{n=1}^{N} \alpha_n q_n = \alpha \cdot q$$

where the α_n are positive parameters, which can be interpreted as quality adjustment parameters. Under the assumption of maximizing behavior on the part of purchasers of the N commodities, Wold's Identity[124] applied to a linearly homogeneous utility function tells us that the purchasers' system of *inverse demand functions* should satisfy the following equations:

$$\begin{aligned}(141)\ p^t &= v^t \nabla f(q^t)/f(q^t)\ ; \qquad t = 1,...,T\\ &= [v^t/f(q^t)]\nabla f(q^t)\\ &= P^t \nabla f(q^t)\end{aligned}$$

where $v^t \equiv p^t \cdot q^t$ is period t expenditure on the N commodities, P^t is the *period t aggregate price level* defined as $v^t/f(q^t) = v^t/Q^t$ and $Q^t \equiv f(q^t)$ is the corresponding *period t aggregate quantity level* for t = 1,...,T.

Since f(q) is defined by (140), $\nabla f(q^t) = \alpha \equiv [\alpha_1,...,\alpha_N]$ for t = 1,...,T. Substitute these equations into equations (141) and we obtain the following equations which should hold exactly under our assumptions:

$$(142)\ p_{tn} = \pi_t \alpha_n\ ; \qquad n = 1,...,N;\ t = 1,...,T$$

where we have redefined the period t price levels P^t in equations (141) as the parameters π_t for t = 1,...,T.

Note that equations (142) form the basis for the *time dummy hedonic regression model*, which is due to Court (1939).[125]

At this point, it is necessary to point out that our consumer theory derivation of equations (142) is not accepted by all economists. Rosen (1974), Triplett (1987) (2004) and Pakes (2001)[126] have argued for a more general approach to the derivation of hedonic regression models that is based on supply conditions as well as on demand conditions. The present approach is obviously based on consumer demands and preferences only. This consumer oriented approach was endorsed by Griliches (1971; 14-15), Muellbauer (1974; 988) and Diewert (2003a) (2003b).[127] Of course, the

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[124] See section 4 in Diewert (2021a).

[125] This was Court's (1939; 109-111) hedonic suggestion number two. He transformed the underlying equations (142) by taking logarithms of both sides of these equations (which will be done below). He chose to transform the prices by the log transformation because the resulting regression model fit his data on automobiles better. Diewert (2003b) also recommended the log transformation on the grounds that multiplicative errors were more plausible than additive errors.

[126] "The derivatives of a hedonic price function should not be interpreted as either willingness to pay derivatives or cost derivatives; rather they are formed from a complex equilibrium process." Ariel Pakes (2001; 14).

[127] Diewert (2003b; 97) justified the consumer demand approach as follows: "After all, the purpose of the hedonic exercise is to find how demanders (and not suppliers) of the product value alternative models in a

assumption that purchasers have the same linear utility function is quite restrictive but nevertheless, it is useful to imbed hedonic regression models in a traditional consumer demand setting.

Empirically, equations (142) are unlikely to hold exactly. Thus we assume that the exact model defined by (142) holds only to some degree of approximation and so error terms, e_{tn}, are added to the right hand sides of equations (142). The unknown price level parameters, $\pi \equiv [\pi_1,...,\pi_T]$ and quality adjustment parameters $\alpha \equiv [\alpha_1,...,\alpha_N]$, can be estimated as solutions to the following (nonlinear) least squares minimization problem:

$$(143)\ \min_{\alpha,\pi} \{\Sigma_{n=1}^N \Sigma_{t=1}^T [p_{tn} - \pi_t\alpha_n]^2\}.$$

Our approach to the specification of the error terms will not be very precise. Throughout this chapter, we will obtain estimators for the aggregate price levels π_t and the quality adjustment parameters α_n as solutions to least squares minimization problems like those defined by (143) or as solutions to weighted least squares minimization problems that will be considered in subsequent sections. Our focus will not be on the distributional aspects of our estimators; rather, our focus will be on the *axiomatic* or *test properties* of the price levels that are solutions to the various least squares minimization problems.[128] Basically, the approach taken here is a descriptive statistics approach: we consider simple models that aggregate price and quantity information for a given period over a set of specified commodities into scalar measures of aggregate price and quantity that summarize the detailed price and quantity information in a "sensible" way.[129]

The first order necessary (and sufficient) conditions for $\pi \equiv [\pi_1,...,\pi_T]$ and $\alpha \equiv [\alpha_1,...,\alpha_N]$ to solve the minimization problem defined by (143) are equivalent to the following N + T equations:

$$(144)\ \alpha_n = \Sigma_{t=1}^T \pi_t p_{tn}/\Sigma_{t=1}^T \pi_t^2 = \Sigma_{t=1}^T \pi_t^2 (p_{tn}/\pi_t)/\Sigma_{t=1}^T \pi_t^2\ ; \qquad n = 1,...,N$$

$$(145)\ \pi_t = \Sigma_{n=1}^N \alpha_n p_{tn}/\Sigma_{n=1}^N \alpha_n^2 = \Sigma_{n=1}^N \alpha_n^2 (p_{tn}/\alpha_n)/\Sigma_{n=1}^N \alpha_n^2. \qquad t = 1,...,T$$

given period. Thus for the present purpose, it is the preferences of consumers that should be decisive, and not the technology and market power of producers. The situation is similar to ordinary general equilibrium theory where an equilibrium price and quantity for each commodity is determined by the interaction of consumer preferences and producer's technology sets and market power. However, there is a big branch of applied econometrics that ignores this complex interaction and simply uses information on the prices that consumers face, the quantities that they demand and perhaps demographic information in order to estimate systems of consumer demand functions. Then these estimated demand functions are used to form estimated consumer utility functions and these functions are often used in applied welfare economics. What producers are doing is entirely irrelevant to these exercises in applied econometrics with the exception of the prices that they are offering to sell at. In other words, we do not need information on producer marginal costs and markups in order to estimate consumer preferences: all we need are selling prices." Footnote 25 on page 82 of Diewert (2003b) explained how the present hedonic model can be derived from Diewert's (2003a) consumer based model by strengthening the assumptions in the 2003a paper.

[128] For rigorous econometric approaches to the stochastic approach to index number theory, see Rao and Hajargasht (2016) and Gorajek (2018). These papers consider many transformations of the fundamental hedonic equations (143) and many methods for constructing averages of prices.

[129] Our approach here is broadly similar to Theil's (1967; 136-137) descriptive statistics approach to index number theory.

Solutions to the two sets of equations can readily be obtained by iterating between the two sets of equations. Thus set $\alpha^{(1)} = 1_N$ (a vector of ones of dimension N) in equations (145) and calculate the resulting $\pi^{(1)} = [\pi_1^{(1)},...,\pi_T^{(1)}]$. Then substitute $\pi^{(1)}$ into the right hand sides of equations (144) to calculate $\alpha^{(2)} \equiv [\alpha_1^{(2)},...,\alpha_N^{(2)}]$. And so on until convergence is achieved.

If $\pi^* \equiv [\pi_1^*,...,\pi_T^*]$ and $\alpha^* \equiv [\alpha_1^*,...,\alpha_N^*]$ is a solution to (144) and (145), then $\lambda\pi^*$ and $\lambda^{-1}\alpha^*$ is also a solution for any $\lambda > 0$. Thus to obtain a unique solution we impose the normalization $\pi_1^* = 1$. Then $1,\pi_2^*,...,\pi_T^*$ is the sequence of fixed base aggregate price levels that is generated by the least squares minimization problem defined by (143).

If quantity data are available, then aggregate quantity levels for the t periods can be obtained as $Q^{t*} \equiv \alpha^*\cdot q^t = \Sigma_{n=1}^N \alpha_n^* q_{tn}$ for t = 1,...,T. Estimated aggregate price levels can be obtained directly from the solution to (143); i.e., set $P^{t*} = \pi_t^*$ for t = 1,...,T. Alternative price levels can be *indirectly* obtained as $P^{t**} \equiv p^t\cdot q^t/Q^{t*} = p^t\cdot q^t/\alpha^*\cdot q^t$ for t = 1,...,T. If the optimized objective function in (143) is 0 (so that all errors $e_{tn}^* \equiv p_{tn} - \pi_t^*\alpha_n^*$ equal 0 for t = 1,...,T and n = 1,...,N), then P^{t*} will equal P^{t**} for all t. However, usually nonzero errors will occur and so a choice between the two sets of estimators must be made.[130]

From (144), it can be seen that α_n^*, the quality adjustment parameter for product n, is a weighted average of the T inflation adjusted prices for product n, the p_{tn}/π_t^*, where the weight for p_{tn}/π_t^* is $\pi_t^{*2}/\Sigma_{\tau=1}^T \pi_\tau^{*2}$. This means that the weight for p_{tn}/π_t^* will be very high for periods t where general inflation is high, which seems rather arbitrary. From (145), it can be seen that π_t^*, the period t price level (and fixed base price index), is weighted average of the N quality adjusted prices for period t, the p_{tn}/α_n^*, where the weight for p_{tn}/α_n^* is $\alpha_n^{*2}/\Sigma_{i=1}^N \alpha_i^{*2}$. It is a positive feature of the method that π_t^* is a weighted average of the quality adjusted prices for period t but the quadratic nature of the weights is not an attractive feature.

In addition to having unattractive weighting properties, the estimates generated by solving the least squares minimization problem (143) suffer from a fatal flaw: *the estimates are not invariant to changes in the units of measurement*. In order to remedy this defect, we turn to an alternative error specification.

Instead of adding approximation errors to the exact equations (142), we could append multiplicative approximation errors. Thus the exact equations become $p_{tn} = \pi_t\alpha_n e_{tn}$ for n = 1,...,N and t = 1,...,T. Upon taking logarithms of both sides of these equations, we obtain the following system of estimating equations:

$$(146)\quad \ln p_{tn} = \ln\pi_t + \ln\alpha_n + \ln e_{tn}\,; \qquad n = 1,...,N;\ t = 1,...,T$$
$$= \rho_t + \beta_n + \varepsilon_{tn}$$

where $\rho_t \equiv \ln\pi_t$ for t = 1,...,T and $\beta_n \equiv \ln\alpha_n$ for n = 1,...,N. The model defined by (146) is the basic *Time Product Dummy regression model* with no missing observations.[131] Now choose the ρ_t and

[130] Usually, the direct estimates for the price levels will be used in hedonic regression studies or in applications of the time product dummy method; i.e., the $P^{t*} = \pi_t^*$ estimates will be used. For statistical agencies, an advantage of the direct estimates is that they can be calculated without the use of quantity information. However, later in this chapter, we will note some advantages of the indirect method if quantity information is available.

[131] In the statistics literature, this type of model is known as a fixed effects model. A generalized version of this model (with missing observations) was proposed by Summers (1973) in the international comparison

β_n to minimize the sum of squared residuals, $\Sigma_{n=1}^{N} \Sigma_{t=1}^{T} \varepsilon_{tn}^{2}$. Thus let $\rho \equiv [\rho_1,...,\rho_T]$ and $\beta \equiv [\beta_1,...,\beta_N]$ be a solution to the following least squares minimization problem:

(147) $\min_{\rho,\,\beta} \{\Sigma_{n=1}^{N} \Sigma_{t=1}^{T} [\ln p_{tn} - \rho_t - \beta_n]^2\}$.

The first order necessary conditions for $\rho_1,...,\rho_T$ and $\beta_1,...,\beta_N$ to solve (147) are the following T + N equations:

(148) $N\rho_t + \Sigma_{n=1}^{N} \beta_n = \Sigma_{n=1}^{N} \ln p_{tn}$; $\quad t = 1,...,T;$
(149) $\Sigma_{t=1}^{T} \rho_t + T\beta_n = \Sigma_{t=1}^{T} \ln p_{tn}$; $\quad n = 1,...,N.$

Replace the ρ_t and β_n in equations (148) and (149) by $\ln\pi_t$ and $\ln\alpha_n$ respectively for t = 1,...,T and n = 1,...,N. After some rearrangement, the resulting equations become:

(150) $\pi_t = \Pi_{n=1}^{N} (p_{tn}/\alpha_n)^{1/N}$; $\quad t = 1,...,T;$
(151) $\alpha_n = \Pi_{t=1}^{T} (p_{tn}/\pi_t)^{1/T}$; $\quad n = 1,...,N.$

Thus the period t aggregate price level, π_t, is equal to the geometric average of the N quality adjusted prices for period t, p_{t1}/α_1, ..., p_{tN}/α_N, while the quality adjustment factor for product n, α_n, is equal to the geometric average of the T inflation adjusted prices for product n, p_{1n}/π_1, ..., p_{Tn}/π_T. These estimators look very reasonable (if quantity weights are not available).

Solutions to (150) and (151) can readily be obtained by iterating between the two sets of equations. Thus set $\alpha^{(1)} = 1_N$ (a vector of ones of dimension N) in equations (150) and calculate the resulting $\pi^{(1)} = [\pi_1^{(1)},...,\pi_T^{(1)}]$. Then substitute $\pi^{(1)}$ into the right hand sides of equations (151) to calculate $\alpha^{(2)} \equiv [\alpha_1^{(2)},...,\alpha_N^{(2)}]$. And so on until convergence is achieved. Alternatively, equations (148) and (149) are linear in the unknown parameters and can be solved (after normalizing one parameter) by a simple matrix inversion. A final method of obtaining a solution to (148) and (149) is to apply a simple linear regression model to equations (146).[132]

If $\pi^* \equiv [\pi_1^*,...,\pi_T^*]$ and $\alpha^* \equiv [\alpha_1^*,...,\alpha_N^*]$ is a solution to (148) and (149), then $\lambda\pi^*$ and $\lambda^{-1}\alpha^*$ is also a solution for any $\lambda > 0$. Thus to obtain a unique solution we impose the normalization $\pi_1^* = 1$ (which corresponds to $\rho_1 = 0$). Then $1,\pi_2^*,...,\pi_T^*$ is the sequence of fixed base index numbers that is generated by the least squares minimization problem defined by (147).

Once we have the unique solution $1,\pi_2^*,...,\pi_T^*$ for the T price levels that are generated by solving (147) along with the normalization $\pi_1 = 1$, the *price index* between period t relative to period s can be defined as π_t^*/π_s^*. Using equations (150) for π_t^* and π_s^*, we have the following expression for these price indexes:

$$\begin{aligned}(152)\ \pi_t^*/\pi_s^* &= \Pi_{n=1}^{N} (p_{tn}/\alpha_n^*)^{1/N} / \Pi_{n=1}^{N} (p_{sn}/\alpha_n^*)^{1/N} \\ &= \Pi_{n=1}^{N} (p_{tn}/p_{sn})^{1/N}.\end{aligned}$$

Thus if there are no missing observations, the Time Product Dummy price indexes between any two periods in the window of T period under consideration is equal to the *Jevons index* between

context where it is known as the Country Product Dummy regression model. A weighted version of this model (with missing observations) was proposed by Aizcorbe, Corrado and Doms (2000).

[132] Again we require one normalization on the parameters such as $\rho_1 = 0$.

the two periods (the simple geometric mean of the price ratios, p_{tn}/p_{sn}).[133] This is a somewhat disappointing result since an equally weighted average of the price ratios is not necessarily a representative average of the prices; i.e., unimportant products to purchasers (in the sense that they spend very little on these products) are given the same weight in the Jevons measure of inflation between the two periods as is given to high expenditure products.[134]

Since there are no missing observations, then it can be seen using equations (151) that the ratio of the quality adjustment factor for product n relative to product m is equal to the following sensible expression:

$$(153)\quad \alpha_n^*/\alpha_m^* = \Pi_{t=1}^T (p_{tn}/\pi_t^*)^{1/T} / \Pi_{t=1}^T (p_{tm}/\pi_t^*)^{1/T} = \Pi_{t=1}^T (p_{tn}/p_{tm})^{1/T}.$$

If quantity data are available, then aggregate quantity levels for the t periods can be obtained as $Q^{t*} \equiv \alpha^*\cdot q^t = \Sigma_{n=1}^N \alpha_n^* q_{tn}$ for t = 1,...,T. Estimated aggregate price levels can be obtained directly from the solution to (147); i.e., set $P^{t*} = \pi_t^*$ for t = 1,...,T. Alternative price levels can be obtained *indirectly* as $P^{t**} \equiv p^t\cdot q^t/Q^{t*} = p^t\cdot q^t/\alpha^*\cdot q^t$ for t = 1,...,T.[135] If the optimized objective function in (147) is 0 (so that all errors $e_{tn}^* \equiv \ln p_{tn} - \rho_t^* - \beta_n^*$ equal 0 for t = 1,...,T and n = 1,...,N), then P^{t*} will equal P^{t**} for all t. If the estimated residuals are not all equal to 0, then the two estimates for the period t price level P^t will differ in general. The two alternative estimates for P^t will generate different estimates for the companion aggregate quantity levels.

Note that the underlying exact model ($p_{tn} = \pi_t\alpha_n$ for all t and n) is the same for both least squares minimization problems, (143) and (147). However, different error specifications and different transformations of both sides of the equations $p_{tn} = \pi_t\alpha_n$ can lead to very different estimators for the π_t and α_n. Our strategy in this section and in the following sections will be to choose specifications of the least squares minimization problem that lead to estimators for the price levels π_t that have good axiomatic properties.[136] From this perspective, it is clear that (147) leads to "better" estimates than (143).

In the following section, we allow for missing observations.

14. Time Product Dummy Regressions: The Case of Missing Observations

In this section, the least squares minimization problem defined by (147) is generalized to allow for missing observations. In order to make this generalization, it is first necessary to make some definitions. As in the previous section, there are N products and T time periods but not all products are purchased (or sold) in all time periods. For each period t, define the set of products n that are present in period t as $S(t) \equiv \{n: p_{tn} > 0\}$ for t = 1,2,...,T. It is assumed that these sets are not empty; i.e., at least one product is purchased in each period. For each product n, define the set of periods t where product n is present as $S^*(n) \equiv \{t: p_{tn} > 0\}$. Again, assume that these sets are

[133] This result is a special case of a more general result obtained by Triplett and McDonald (1977; 150).
[134] However, if quantity data are not available, the Jevons index has the strongest axiomatic properties; see Diewert (2021b).
[135] The fact that a time dummy hedonic regression model generates two alternative decompositions of the value aggregate into price and quantity aggregates was first noted in de Haan and Krsinich (2018).
[136] From the perspective of the economic approach to index number theory, the minimization problems (143) and (147) have exactly the same justification; i.e., they are based on the same economic model of consumer behavior.

not empty; i.e., each product is sold in at least one time period. Define the integers N(t) and T(n) as follows:

(154) $N(t) \equiv \Sigma_{n\in S(t)}\, 1$; $\quad t = 1,...,T;$
(155) $T(n) \equiv \Sigma_{t\in S^*(n)}\, 1$; $\quad n = 1,...,N.$

If all N products are present in period t, then N(t) = N; if product n is present in all T periods, then T(n) = T.

The multilateral methods studied in previous sections assumed that reservation prices were available for missing products in any period. Thus the methods discussed up until the present section assumed that there were no missing product prices: p_{tn} was either an actual period t price for product n or an estimated price for the product if it was missing in period t. When discussing the time product dummy multilateral price levels and indexes, we do *not* assume that reservation prices for missing products have been estimated. Instead, the method generates estimated prices for the missing products.

Using the above notation for missing products, the counterpart to (147) when there are missing products is the following least squares minimization problem:

(156) $\min_{\rho,\beta} \{\Sigma_{t=1}^{T} \Sigma_{n\in S(t)} [\ln p_{tn} - \rho_t - \beta_n]^2\} = \min_{\rho,\beta} \{\Sigma_{n=1}^{N} \Sigma_{t\in S^*(n)} [\ln p_{tn} - \rho_t - \beta_n]^2\}.$

Note that there are two equivalent ways of writing the least squares minimization problem.[137] The first order necessary conditions for $\rho_1,...,\rho_T$ and $\beta_1,...,\beta_N$ to solve (156) are the following counterparts to (148) and (149):

(157) $\Sigma_{n\in S(t)} [\rho_t + \beta_n] = \Sigma_{n\in S(t)} \ln p_{tn}$; $\quad t = 1,...,T;$
(158) $\Sigma_{t\in S^*(n)} [\rho_t + \beta_n] = \Sigma_{t\in S^*(n)} \ln p_{tn}$; $\quad n = 1,...,N.$

As in the previous section, let $\rho_t \equiv \ln\pi_t$ for t = 1,...,T and let $\beta_n \equiv \ln\alpha_n$ for n = 1,...,N. Substitute these definitions into equations (157) and (158). After some rearrangement and using definitions (154) and (155), equations (157) and (158) become the following ones:

(159) $\pi_t = \Pi_{n\in S(t)} [p_{tn}/\alpha_n]^{1/N(t)}$; $\quad t = 1,...,T;$
(160) $\alpha_n = \Pi_{t\in S^*(n)} [p_{tn}/\pi_t]^{1/T(n)}$; $\quad n = 1,...,N.$

The same iterative procedure that was explained in the previous section will work to generate a solution to equations (159) and (160).[138] As was the case in the previous section, solutions to (159) and (160) are not unique; if π^*, α^* is a solution to (159) and (160), then $\lambda\pi^*$ and $\lambda^{-1}\alpha^*$ is also a solution for any $\lambda > 0$. Thus to obtain a unique solution we impose the normalization π_1^* =

[137] The first expression is used when (156) is differentiated with respect to ρ_t and the second expression is used when differentiating (156) with respect to β_n.

[138] Of course, it is not necessary to use the iterative procedure to find a solution to equations (157) and (158). After setting $\rho_1 = 0$ and dropping the first equation in (157), matrix algebra can be used to find a solution to the remaining equations. Alternatively, after setting $\rho_1 = 0$, use the equations $\ln p_{tn} = \rho_t + \beta_n + \varepsilon_{tn}$ for t = 1,...,T and $n\in S(t)$ to set up a linear regression model with time and product dummy variables and use a standard ordinary least squares econometric software package to obtain the solution $\rho_2^*,...,\rho_T^*$, $\beta_1^*,...,\beta_N^*$ to the linear regression model $\ln p_{tn} = \rho_t + \beta_n + \varepsilon_{tn}$ for t = 1,...,T and $n\in S(t)$. We need to assume that the X matrix for this linear regression model has full column rank.

1 (which corresponds to $\rho_1 = 0$). Then $1,\pi_2^*,...,\pi_T^*$ is the sequence of (normalized) price levels that is generated by the least squares minimization problem defined by (156).[139] In this case, $\pi_t^* = \Pi_{n\in S(t)}[p_{tn}/\alpha_n^*]^{1/N(t)}$ is the equally weighted geometric mean of all of the quality adjusted prices for the products that are available in period t for t = 2,3,...,T and the quality adjustment factors are normalized so that $\pi_1^* = \Pi_{n\in S(1)}[p_{1n}/\alpha_n^*]^{1/N(1)} = 1$. From (160), we can deduce that α_n^* will be larger for products that are relatively expensive and will be smaller for cheaper products.

Once we have the unique solution $1,\pi_2^*,...,\pi_T^*$ for the T price levels that are generated by solving (156), the *price index* between period t relative to period r can be defined as π_t^*/π_r^*. Using equations (159) and (160), we have the following expressions for π_t^*/π_r^* and α_n^*/α_m^*:

$$(161)\ \pi_t^*/\pi_r^* = \Pi_{n\in S(t)}[p_{tn}/\alpha_n^*]^{1/N(t)} / \Pi_{n\in S(r)}[p_{rn}/\alpha_n^*]^{1/N(r)}\,; \qquad 1 \leq t, r \leq T;$$
$$(162)\ \alpha_n^*/\alpha_m^* = \Pi_{t\in S^*(n)}[p_{tn}/\pi_t^*]^{1/T(n)} / \Pi_{t\in S^*(m)}[p_{tm}/\pi_t^*]^{1/T(m)}\,; \qquad 1 \leq n, m \leq N.$$

Note that, in general, the quality adjustment factors α_n^* do not cancel out for the indexes π_t^*/π_r^* defined by (161) as they did in the previous section. However, these price indexes do have some good axiomatic properties.[140] If the set of available products is the same in periods r and t, then the quality adjustment factors do cancel and the price index for period t relative to period r is $\pi_t^*/\pi_r^* = \Pi_{n\in S(t)}[p_{tn}/p_{rn}]^{1/N(t)}$, which is the Jevons index between periods r and t. Again, while this index is an excellent one if quantity information is not available, it is not satisfactory when quantity information is available due to its equal weighting of economically important and unimportant price ratios.[141]

There is another problematic property of the estimated price levels that are generated by solving the time product dummy hedonic model that is defined by (156): a product that is available only in one period out of the T periods has no influence on the aggregate price levels π_t^*.[142] To see this, consider equations (157) and (158) and suppose that product n^* was available only in period t^*.[143] Equation n^* in the N equations in (158) becomes the equation: $[\rho_{t^*} + \beta_{n^*}] = \ln p_{t^*n^*}$. Thus once ρ_{t^*} has been determined, β_{n^*} can be defined as $\beta_{n^*} \equiv \ln p_{t^*n^*} - \rho_{t^*}$. Subtract the equation $[\rho_{t^*} + \beta_{n^*}] = \ln p_{t^*n^*}$ from equation t^* and the resulting equations in (157) can be written as equations (163) below. Dropping equation n^* in equations (158) leads to equations (164) below:

$$(163)\ \Sigma_{n\in S(t),\, n\neq n^*}[\rho_t + \beta_n] = \Sigma_{n\in S(t),\, n\neq n^*} \ln p_{tn}\,; \qquad t = 1,...,T;$$
$$(164)\quad \Sigma_{t\in S^*(n)}\, [\rho_t + \beta_n] = \Sigma_{t\in S^*(n)} \ln p_{tn}\,; \qquad n = 1,...,n^*-1,n^*+1,...,N.$$

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[139] We need enough observations on products that are present so that a full rank condition is satisfied for equations (157) and (158) after dropping one equation and setting $\rho_1 = 0$. If there is a rapid proliferation of new and disappearing products, then it may not be possible to invert the coefficient matrix that is associated with the modified equations (157) and (158). In subsequent models with missing observations, we will assume that a similar full rank condition is satisfied.

[140] The index π_t^*/π_r^* satisfies the identity test (if prices are the same in periods r and t, then the index is equal to 1) and it is invariant to changes in the units of measurement. It is also homogeneous of degree one in the prices of period t and homogeneous of degree minus one in the prices of period r.

[141] However, if the estimated squared residuals are small in magnitude for periods τ and t, then the index π_t^*/π_r^* defined by (161) will be satisfactory, since in this case $p^\tau \approx \pi_\tau^*\alpha^*$ and $p^t \approx \pi_t^*\alpha^*$ so that prices are approximately proportional for these two periods and π_t^*/π_r^* defined by (161) will be approximately correct. Any missing prices for any period t and product n are defined as $p_{tn}^* \equiv \pi_t^*\alpha_n^*$.

[142] This property of the Time Product Dummy model was first noticed by Diewert (2004) (in the context of the Country Product Dummy model).

[143] We assume that products other than product n^* are available in period t^*.

Equations (163) and (164) are T+N−1 equations that do not involve $p_{t^*n^*}$. After making the normalization $\rho_1^* = 0$, these equations can be solved for $\rho_2^*,..., \rho_T^*, \beta_1^*,..., \beta_{n^*-1}^*, \beta_{n^*+1}^*,..., \beta_N^*$. Now define $\beta_{n^*}^* \equiv \ln p_{t^*n^*} - \rho_{t^*}$ and we have the (normalized) solution for (156). Since the ρ_t^* do not involve $p_{t^*n^*}$, the resulting $\pi_t^* \equiv \exp[\rho_t^*]$ for t = 1,...,T also do not depend on the isolated price $p_{t^*n^*}$. This proof can be repeated for any number of isolated prices. This property of the time product dummy model is unfortunate because it means that when a new product enters the marketplace in period T, it has no influence on the price levels $1,\pi_2^*,...,\pi_T^*$ that are generated by solving the least squares minimization problem defined by (156). In other words, an expansion in the choice of products available to consumers will have no effect on price levels.

If quantity data are available, then aggregate quantity levels for the t periods can be obtained as $Q^{t*} \equiv \Sigma_{n\in S(t)}\, \alpha_n^* q_{tn}$ for t = 1,...,T.[144] Estimated aggregate price levels can be obtained directly from the solution to (42); i.e., set $P^{t*} = \pi_t^*$ for t = 1,...,T. Alternative price levels can be obtained indirectly as $P^{t**} \equiv \Sigma_{n\in S(t)}\, p_n q_{tn}/Q^{t*} = \Sigma_{n\in S(t)}\, p_n q_{tn}/\Sigma_{n\in S(t)}\, \alpha_n^* q_{tn}$ for t = 1,...,T.[145] If the optimized objective function in (156) is 0, so that all errors $\varepsilon_{tn}^* \equiv \ln p_{tn} - \rho_t^* - \beta_n^*$ equal 0 for t = 1,...,T and $n\in S(t)$, then P^{t*} will equal P^{t**} for all t. If the estimated residuals are not all equal to 0, then the two estimates for the period t price level P^t will differ. The two estimates for P^t will generate different estimates for the companion aggregate quantity levels.

15. Weighted Time Product Dummy Regressions: The Bilateral Case

A major problem with the indexes discussed in the previous 2 sections is the fact that they do not weight the individual product prices by their economic importance. The first serious index number economist to stress the importance of weighting was Walsh (1901).[146] Keynes was quick

[144] Note that each $\alpha_n^* > 0$ since $\alpha_n^* \equiv \exp[\beta_n^*]$ for n = 1,...,N.

[145] Note that $P^{t**} \equiv \Sigma_{n\in S(t)}\, p_{tn}q_{tn}/\Sigma_{n\in S(t)}\, \alpha_n^* q_{tn}$ is a period t *quality adjusted unit value price level*; see section 10 above. The corresponding quantity level is $Q^{t**} \equiv \Sigma_{n\in S(t)}\, p_{tn}q_{tn}/P^{t**} = \Sigma_{n\in S(t)}\, \alpha_n^* q_{tn}$, which is the level generated by a *linear aggregator function*. By looking at (156), it can be seen that if prices are identical in periods t and r so that $p^t = p^r$, then $P^{t*} = P^{r*}$; i.e., an identity test for the direct hedonic price levels will be satisfied. However, the corresponding Q^{t*} will not satisfy the identity test for quantity levels; i.e., if quantities q_{tn} and q_{rn} are equal in periods t and r for all n, it is not the case that $Q^{t*} \equiv \Sigma_{n=1}^N p_{tn}q_{tn}/\pi_t^*$ will equal $Q^{r*} \equiv \Sigma_{n=1}^N p_{rn}q_{rn}/\pi_r^*$ for $r \neq t$ unless prices are also equal for the two periods. On the other hand, it can be seen that $Q^{t**} = \Sigma_{n\in S(t)}\, \alpha_n^* q_{tn} = \Sigma_{n\in S(t)}\, \alpha_n^* q_{rn} = Q^{r*}$ if $q_{tn} = q_{rn}$ for all n even if prices are not identical for the two periods. Thus the choice between using P^{t*} or P^{t**} could be made on the basis of choosing which identity test is more important to satisfy. The analysis here follows that of de Haan and Krsinich (2018; 763-764)

[146] See Walsh (1901). This book laid the groundwork for the test or axiomatic approach to index number theory that was further developed by Fisher (1922). In his second book on index number theory, Walsh made the case for weighting by economic importance as follows: “It might seem at first sight as if simply every price quotation were a single item, and since every commodity (any kind of commodity) has one price-quotation attached to it, it would seem as if price-variations of every kind of commodity were the single item in question. This is the way the question struck the first inquirers into price-variations, wherefore they used simple averaging with even weighting. But a price-quotation is the quotation of the price of a generic name for many articles; and one such generic name covers a few articles, and another covers many. … A single price-quotation, therefore, may be the quotation of the price of a hundred, a thousand, or a million dollar’s worths, of the articles that make up the commodity named. Its weight in the averaging, therefore, ought to be according to these money-unit’s worth.” Correa Moylan Walsh (1921a; 82-83).

to follow up on the importance of weighting[147] and Fisher emphatically endorsed weighting.[148] Griliches also endorsed weighting in the hedonic regression context.[149]

In this section, we will discuss some alternative methods for weighting by economic importance in the context of a bilateral time product dummy regression model.[150] We also assume that there are no missing observations in this section.

Recall the least squares minimization problem defined by (147) in section 13 above. The squared residuals, $[\ln p_{tn} - \rho_t - \beta_n]^2$, appear in this problem without any weighting. Thus products, which have a high volume of sales in any period, are given the same weight in the least squares minimization problem as products that have very few sales. In order to take economic importance into account, for the case of two time periods, replace (147) by the following *weighted least squares minimization problem*:

$$(165)\ \min_{\rho,\,\beta}\ \{\Sigma_{n=1}^{N} q_{1n}[\ln p_{1n} - \beta_n]^2 + \Sigma_{n=1}^{N} q_{2n}[\ln p_{2n} - \rho_2 - \beta_n]^2\}$$

where we have set $\rho_1 = 0$. The squared error for product n in period t is repeated q_{tn} times to reflect the sales of the product in period t. Thus the new problem (165) takes into account the popularity of each product.[151]

The first order necessary conditions for the minimization problem defined by (165) are the following N + 1 equations:

$$(166)\ (q_{1n}+q_{2n})\beta_n = q_{1n}\ln p_{1n} + q_{2n}(\ln p_{2n} - \rho_2)\ ; \qquad n = 1,...,N;$$
$$(167)\ (\Sigma_{n=1}^{N} q_{2n})\rho_2 = \Sigma_{n=1}^{N} q_{2n}(\ln p_{2n} - \beta_n).$$

The solution to (166) and (167) is the following one:[152]

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[147] "It is also clear that the so-called unweighted index numbers, usually employed by practical statisticians, are the worst of all and are liable to large errors which could have been easily avoided." J.M. Keynes (1909; 79). This paper won the Cambridge University Adam Smith Prize for that year. Keynes (1930; 76-77) again stressed the importance of weighting in a later paper which drew heavily on his 1909 paper.
[148] "It has already been observed that the purpose of any index number is to strike a fair average of the price movements or movements of other groups of magnitudes. At first a simple average seemed fair, just because it treated all terms alike. And, in the absence of any knowledge of the relative importance of the various commodities included in the average, the simple average is fair. But it was early recognized that there are enormous differences in importance. Everyone knows that pork is more important than coffee and wheat than quinine. Thus the quest for fairness led to the introduction of weighting." Irving Fisher (1922; 43).
[149] "But even here, we should use a weighted regression approach, since we are interested in an estimate of a weighted average of the pure price change, rather than just an unweighted average over all possible models, no matter how peculiar or rare." Zvi Griliches (1971; 8).
[150] The approach taken in this section is based on Rao (1995) (2004) (2005) and Diewert (2003b), (2005a) (2005b). Diewert (2005a) considered all four forms of weighting that will be discussed in this section while Rao (1995) (2005) discussed mainly the third form of weighting.
[151] One can think of repeating the term $[\ln p_{1n} - \beta_n]^2$ for each unit of product n sold in period 1. The result is the term $q_{1n}[\ln p_{1n} - \beta_n]^2$. A similar justification based on repeating the price according to its sales can also be made. This repetition methodology makes the stochastic specification of the error terms somewhat complicated. However, as indicated in the introduction, we leave these difficult distributional problems to other more capable econometricians.
[152] See Diewert (2005a).

(168) $\rho_2^* \equiv \Sigma_{n=1}^N q_{1n}q_{2n}(q_{1n} + q_{2n})^{-1} \ln(p_{2n}/p_{1n})/\Sigma_{i=1}^N q_{1i}q_{2i}(q_{1i} + q_{2i})^{-1}$;
(169) $\beta_n^* \equiv q_{1n}(q_{1n} + q_{2n})^{-1} \ln(p_{1n}) + q_{2n}(q_{1n} + q_{2n})^{-1} \ln(p_{2n}/\pi_2^*)$; $n = 1,...,N$

where $\pi_2^* \equiv \exp[\rho_2^*]$. Note that the weight for the term $\ln(p_{2n}/p_{1n})$ in (168) can be written as follows:

(170) $q_n^* \equiv \Sigma_{n=1}^N q_{1n}q_{2n}(q_{1n} + q_{2n})^{-1} /\Sigma_{i=1}^N q_{1i}q_{2i}(q_{1i} + q_{2i})^{-1}$; $n = 1,...,N$
$= h(q_{1n},q_{2n})/\Sigma_{i=1}^N h(q_{1i},q_{2i})$

where $h(a,b) \equiv 2ab/(a+b) = [\frac{1}{2} a^{-1} + \frac{1}{2} b^{-1}]^{-1}$ is the *harmonic mean* of a and b.[153]

Note that the q_n^* sum to 1 and thus ρ_2^* is a weighted average of the logarithmic price ratios $\ln(p_{2n}/p_{1n})$. Using $\pi_2^* = \exp[\rho_2^*]$ and $\pi_1^* = \exp[\rho_1^*] = \exp[0] = 1$, the bilateral price index that is generated by the solution to (165) is

(171) $\pi_2^*/\pi_1^* = \exp[\rho_2^*] = \exp[\Sigma_{n=1}^N q_n^* \ln(p_{2n}/p_{1n})]$.

Thus π_2^*/π_1^* is a weighted geometric mean of the price ratios p_{2n}/p_{1n} with weights q_n^* defined by (170). Although this seems to be a reasonable bilateral index number formula, it must be rejected for practical use on the grounds that *the index is not invariant to changes in the units of measurement*.

Since values are invariant to changes in the units of measurement, the lack of invariance problem can be solved if we replace the quantity weights in (165) with expenditure or sales weights.[154] This leads to the following weighted least squares minimization problem where the weights v_{tn} are defined as $p_{tn}q_{tn}$ for t = 1,2 and n = 1,...,N:

(172) $\min_{\rho,\beta} \{\Sigma_{n=1}^N v_{1n}[\ln p_{1n} - \beta_n]^2 + \Sigma_{n=1}^N v_{2n}[\ln p_{2n} - \rho_2 - \beta_n]^2\}$.

It can be seen that problem (172) has exactly the same mathematical form as problem (165) except that v_{tn} has replaced q_{tn} and so the solutions (168) and (169) will be valid in the present context if v_{tn} replaces q_{tn} in these formulae. Thus the solution to (172) is:

(173) $\rho_2^* \equiv \Sigma_{n=1}^N v_{1n}v_{2n}(v_{1n} + v_{2n})^{-1} \ln(p_{2n}/p_{1n})/\Sigma_{i=1}^N v_{1i}v_{2i}(v_{1i} + v_{2i})^{-1}$;
(174) $\beta_n^* \equiv v_{1n}(v_{1n} + v_{2n})^{-1} \ln(p_{1n}) + v_{2n}(v_{1n} + v_{2n})^{-1} \ln(p_{2n}/\pi_2^*)$; $n = 1,...,N$

where $\pi_2^* \equiv \exp[\rho_2^*]$.

The resulting price index, $\pi_2^*/\pi_1^* = \pi_2^* = \exp[\rho_2^*]$ is indeed invariant to changes in the units of measurement. However, if we regard π_2^* as a function of the price and quantity vectors for the two periods, say $P(p^1,p^2,q^1,q^2)$, then another problem emerges for the price index defined by the

[153] h(a,b) is well defined by ab/(a+b) if a and b are nonnegative and at least one of these numbers is positive. In order to write h(a,b) as $[\frac{1}{2} a^{-1} + \frac{1}{2} b^{-1}]^{-1}$, we require $a > 0$ and $b > 0$.
[154] "But on what principle shall we weight the terms? Arthur Young's guess and other guesses at weighting represent, consciously or unconsciously, the idea that relative money values of the various commodities should determine their weights. A value is, of course, the product of a price per unit, multiplied by the number of units taken. Such values afford the only common measure for comparing the streams of commodities produced, exchanged, or consumed, and afford almost the only basis of weighting which has ever been seriously proposed." Irving Fisher (1922; 45).

solution to (172): $P(p^1,p^2,q^1,q^2)$ is not homogeneous of degree 0 in the components of q^1 or in the components of q^2. These properties are important because it is desirable that the companion implicit quantity index defined as $Q(p^1,p^2,q^1,q^2) \equiv [p^2\cdot q^2/p^1\cdot q^1]/P(p^1,p^2,q^1,q^2)$ be homogeneous of degree 1 in the components of q^2 and homogeneous of degree minus 1 in the components of q^1.[155] We also want $P(p^1,p^2,q^1,q^2)$ to be homogeneous of degree 1 in the components of p^2 and homogeneous of degree minus 1 in the components of p^1 and these properties are also not satisfied. Thus we conclude that the solution to the weighted least squares problem defined by (172) does not generate a satisfactory price index formula.

The above deficiencies can be remedied if the *expenditure amounts* v_{tn} in (172) are replaced by *expenditure shares*, s_{tn}, where $v_t \equiv \Sigma_{n=1}^N v_{tn}$ for t = 1,2 and $s_{tn} \equiv v_{tn}/v_t$ for t = 1,2 and n = 1,...,N. This replacement leads to the following weighted least squares minimization problem:[156]

(175) $\min_{\rho,\beta} \{\Sigma_{n=1}^N s_{1n}[\ln p_{1n} - \beta_n]^2 + \Sigma_{n=1}^N s_{2n}[\ln p_{2n} - \rho_2 - \beta_n]^2\}$.

Again, it can be seen that problem (175) has exactly the same mathematical form as problem (165) except that s_{tn} has replaced q_{tn} and so the solutions (168) and (169) will be valid in the present context if s_{tn} replaces q_{tn} in these formulae. Thus the solution to (175) is:

(176) $\rho_2^* \equiv \Sigma_{n=1}^N s_{1n}s_{2n}(s_{1n} + s_{2n})^{-1} \ln(p_{2n}/p_{1n})/\Sigma_{i=1}^N s_{1i}s_{2i}(s_{1i} + s_{2i})^{-1}$;
(177) $\beta_n^* \equiv s_{1n}(s_{1n} + s_{2n})^{-1} \ln(p_{1n}) + s_{2n}(s_{1n} + s_{2n})^{-1} \ln(p_{2n}/\pi_2^*)$; n = 1,...,N

where $\pi_2^* \equiv \exp[\rho_2^*]$. Define the *normalized harmonic mean share weights* as $s_n^* \equiv h(s_{1n},s_{2n})/\Sigma_{i=1}^N h(s_{1i},s_{2i})$ for n = 1,...,N. Then the weighted time product dummy bilateral price index, $P_{WTPD}(p^1,p^2,q^1,q^2) \equiv \pi_2^*/\pi_1^* = \pi_2^*$, has the following logarithm:

(178) $\ln P_{WTPD}(p^1,p^2,q^1,q^2) \equiv \Sigma_{n=1}^N s_n^* \ln(p_{2n}/p_{1n})$.

Thus $P_{WTPD}(p^1,p^2,q^1,q^2)$ is equal to a share weighted geometric mean of the price ratios, p_{2n}/p_{1n}.[157] This index is a satisfactory one from the viewpoint of the test approach to index number theory. It can be shown that $P_{WTPD}(p^1,p^2,q^1,q^2)$ satisfies the following tests:

(i) the *identity test*; i.e., $P_{WTPD}(p^1,p^2,q^1,q^2) = 1$ if $p^1 = p^2$;
(ii) the *time reversal test*; i.e., $P_{WTPD}(p^2,p^1,q^2,q^1) = 1/P_{WTPD}(p^1,p^2,q^1,q^2)$;[158]
(iii) *homogeneity of degree 1 in period 2 prices*; i.e., $P_{WTPD}(p^1,\lambda p^2,q^1,q^2) = \lambda P_{WTPD}(p^1,p^2,q^1,q^2)$;
(iv) *homogeneity of degree –1 in period 1 prices*; i.e., $P_{WTPD}(\lambda p^1,p^2,q^1,q^2) = \lambda^{-1} P_{WTPD}(p^1,p^2,q^1,q^2)$;

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155 Thus we want Q to have the following properties: $Q(p^1,p^2,q^1,\lambda q^2) = \lambda Q(p^1,p^2,q^1,q^2)$ and $Q(p^1,p^2,\lambda q^1,q^2) = \lambda^{-1} Q(p^1,p^2,q^1,q^2)$ for all $\lambda > 0$.

156 Note that the minimization problem defined by (175) is equivalent to the problem of minimizing $\Sigma_{n=1}^N e_{1n}^2 + \Sigma_{n=1}^N e_{2n}^2$ with respect to $\rho_2, \beta_1, ..., \beta_N$ where the error terms e_{tn} are defined by the equations $s_{1n}^{1/2}\ln p_{1n} = s_{1n}^{1/2}\beta_n + e_{1n}$ for n = 1,...,N and $s_{2n}^{1/2}\ln p_{2n} = s_{2n}^{1/2}\rho_2 + s_{2n}^{1/2}\beta_n + e_{2n}$ for n = 1,...,N. Thus the solution to (175) can be found by running a linear regression using the above two sets of estimating equations. The numerical equivalence of the least squares estimates obtained by repeating multiple observations or by using the square root of the weight transformation was noticed long ago as the following quotation indicates: "It is evident that an observation of weight w enters into the equations exactly as if it were w separate observations each of weight unity. The best practical method of accounting for the weight is, however, to prepare the equations of condition by multiplying each equation throughout by the square root of its weight." E. T. Whittaker and G. Robinson (1940; 224).

157 See Diewert (2002) (2005a).

158 See Diewert (2003b) (2005b).

(v) *homogeneity of degree 0 in period 1 quantities*; i.e., $P_{WTPD}(p^1,p^2,\lambda q^1,q^2) = P_{WTPD}(p^1,p^2,q^1,q^2)$;
(vi) *homogeneity of degree 0 in period 2 quantities*; i.e., $P_{WTPD}(p^1,p^2,q^1,\lambda q^2) = P_{WTPD}(p^1,p^2,q^1,q^2)$;
(vii) *invariance to changes in the units of measurement*;
(viii) the *min-max test*; i.e.,

$$\min_n\{p_{2n}/p_{1n} : n = 1,...,N\} \leq P_{WTPD}(p^1,p^2,q^1,q^2) \leq \max_n\{p_{2n}/p_{1n} : n = 1,...,N\}; \text{ and}$$

(ix) the *invariance to the ordering of the products test*.

Moreover, it can be shown that $P_{WTPD}(p^1,p^2,q^1,q^2)$ approximates the superlative Törnqvist Theil index to the second order around an equal price and quantity point where $p^1 = p^2$ and $q^1 = q^2$.[159] Thus if changes in prices and quantities going from one period to the next are not too large and there are no missing products, P_{WTPD} should be close to the superlative Fisher (1922) and Törnqvist Theil indexes.[160]

Recall the results from section 13 above for the unweighted time product dummy model. From equation (152), it can be seen that the unweighted bilateral time product dummy regression model generated the Jevons index as the solution to the unweighted least squares minimization problem that is a counterpart to the weighted problem defined by (175) above. Thus appropriate weighting of the squared errors has changed the solution index dramatically: the index defined by (178) weights products by their economic importance and has good test properties whereas the Jevons index can generate very problematic results due to its lack of weighting according to economic importance. Note that both models have the same underlying structure; i.e., they assume that p_{tn} is approximately equal to $\pi_t\alpha_n$ for t = 1,2 and n = 1,...,N. *Thus weighting by economic importance has converted a least squares minimization problem that generates a rather poor price index into a problem that generates a rather good index*.

There is one more weighting scheme that generates an even better index in the bilateral context where we are running a time product dummy hedonic regression using the price and quantity data for only two periods. Consider the following weighted least squares minimization problem:

$$(179)\ \min_{\rho,\beta}\ \{\Sigma_{n=1}^N (½)(s_{1n}+s_{2n})[\ln p_{1n} - \beta_n]^2 + \Sigma_{n=1}^N (½)(s_{1n}+s_{2n})[\ln p_{2n} - \rho_2 - \beta_n]^2\}.$$

As usual, it can be seen that problem (179) has exactly the same mathematical form as problem (165) except that $(½)(s_{1n}+s_{2n})$ has replaced q_{tn} and so the solutions (168) and (169) will be valid in the present context if $(½)(s_{1n}+s_{2n})$ replaces q_{tn} in these formulae. Thus the solution to (179) simplifies to the following solution:

$$(180)\ \rho_2^* \equiv \Sigma_{n=1}^N (½)(s_{1n}+s_{2n})\ln(p_{2n}/p_{1n});$$

$$(181)\ \beta_n^* \equiv (½)\ln(p_{1n}) + (½)\ln(p_{2n}/\pi_2^*)\ ; \qquad n = 1,...,N$$

where $\pi_2^* \equiv \exp[\rho_2^*]$ and $\pi_1^* \equiv \exp[\rho_1^*] = \exp[0] = 1$ since we have set $\rho_1^* = 0$. Thus the bilateral index number formula which emerges from the solution to (179) is $\pi_2^*/\pi_1^* = \exp[\Sigma_{n=1}^N (½)(s_{1n}+s_{2n})\ln(p_{2n}/p_{1n})] \equiv P_T(p^1,p^2,q^1,q^2)$, which is the Törnqvist Theil (1967; 137-138) bilateral index number formula. Thus the use of the weights in (179) has generated an even better bilateral index number formula than the formula that resulted from the use of the weights in (175). This

[159] Diewert (2005a; 564) noted this result. Thus P_{WTPD} is a pseudo-superlative index. For the definition of a superlative index, see Diewert (1976) (2021a). A pseudo-superlative index approximates a superlative index to the second order around any point where $p^1 = p^2$ and $q^1 = q^2$; see Diewert (1978).
[160] However, with large changes in price and quantities going from period 1 to 2, P_{WTPD} will tend to lie below its superlative counterparts; see Diewert (2018; 53) and an example in Diewert and Fox (2020).

result reinforces the case for using appropriately weighted versions of the basic time product dummy hedonic regression model.[161] However, if the implied residuals in the original unweighted minimization problem (147) are small (or equivalently, if the fit in the linear regression model that can be associated with (147) is high so that predicted values for log prices are close to actual log prices), then *weighting will not matter very much* and the unweighted model (147) will give results that are similar to the results generated by the weighted model defined by (179). This comment applies to all of the weighted hedonic regression models that are considered in this paper.[162]

The aggregate quantity levels for the t periods can be obtained as $Q^{t*} \equiv \alpha^*\cdot q^t = \Sigma_{n=1}^N \alpha_n^* q_{tn}$ for t = 1,2 where the α_n^* are defined as the exponentials of the β_n^* defined by (181). Estimated aggregate price levels can be obtained directly from the solution to (179); i.e., set $P^{t*} = \pi_t^*$ for t = 1,2.[163] Alternative price levels can be obtained indirectly as $P^{t**} \equiv p^t\cdot q^t/Q^{t*} = p^t\cdot q^t/\alpha^*\cdot q^t$ for t = 1,2. If the optimized objective function in (179) is 0, so that all errors equal 0, then P^{t*} will equal P^{t**} for t = 1,2. If the estimated residuals are not all equal to 0, then the two estimates for the period t price level P^t will differ and the alternative estimates for P^t will generate different estimates for the companion aggregate quantity levels.

It should be noted that we have not made any bias corrections due to the fact that our model estimates the logarithm of π_t instead of π_t itself. This is due to our perspective that simply tries to fit an exact model by transforming it in a way that leads to solutions π_t^* to a least squares minimization problem where the π_t^* have good axiomatic properties. [164] There is more work to be done in working out the distributional properties of the above estimators for the price levels.

16. Weighted Time Product Dummy Regressions: The Bilateral Case with Missing Observations

[161] Note that the bilateral regression model defined by the minimization problem (175) is readily generalized to the case of T periods whereas the bilateral regression model defined by the minimization problem (179) cannot be generalized to the case of T periods. These facts were noted by de Haan and Krsinich (2014).

[162] If the residuals are small for (147), then prices will vary almost proportionally over time and all reasonable index number formulae will register price levels that are close to the estimated π_t^*; i.e., we will have $p^t \approx \pi_t^* p^1$ for t = 2,3,...,T if the residuals are small for (147).

[163] In this case, alternative period t quantity levels are defined as $Q^{1**} \equiv p^1\cdot q^1$ and $Q^{2**} \equiv p^2\cdot q^2/\pi_2^* = [v_2/v_1]/P_T(p^1,p^2,q^1,q^2)$. If the squared errors in (179) are all 0, then the alternative quantity estimates are equal to each other and the model $\ln p_{tn} = \rho_t + \beta_n$ holds exactly for each t and n, which means that prices are proportional across the two periods; i.e., we have $p^t = \pi_t^*\alpha^*$ for t = 1,2 where $\alpha^* \equiv [\alpha_1^*,...,\alpha_N^*]$. In the case where the squared errors are nonzero, the π_t^*,Q^{t**} aggregates are preferred since $P_T(p^1,p^2,q^1,q^2)$ is a superlative index and thus has a strong economic justification.

[164] We note that de Haan and Krsinich (2018; 769-770) make the following comments on possible biases that result from the use of a weighted least squares model to generate price indexes: "Finally, we will elaborate on a few econometric issues. The estimated quality adjusted prices ... are biased as taking exponentials is a non-linear transformation. The time dummy index is similarly biased. It is questionable whether bias adjustments would be appropriate, though, at least from an index number point of view. For instance, recall the two-period case with only matched items, where Diewert's (2004) choice of regression weights ensures that the time dummy index is equal to the superlative Törnqvist price index. Correcting for the "bias" would mean that this useful property does no longer hold, and so there is a tension between econometrics and index number theory."

In this section, we will generalize the last two models in the previous section to cover the case where there are missing observations.[165] Thus we assume that there are products that are missing in period 2 that were present in period 1 and some new products that appear in period 2. As in section 14 above, S(t) denotes the set of products n that are present in period t for t = 1,2. It is assumed that $S(1)\cap S(2)$ is not the empty set; i.e., there are one or more products that are present in both periods. We need some new notation to deal with missing prices and quantities. For the present, if product n is not present in period t, define p_{tn} and q_{tn} to equal 0. This enables us to define the N dimensional period t price and quantity vectors as $p^t \equiv [p_{t1},...,p_{tN}]$ and $q^t \equiv [q_{t1},...,q_{tN}]$ for t = 1,2. Thus the missing prices and quantities are simply set equal to 0. The period t share of sales or expenditures for product n is defined in the usual case as $s_{tn} \equiv p_{tn}q_{tn}/p^t \cdot q^t$ for n = 1,...,N and t = 1,2. With these notational conventions, the new weighted least squares minimization problem that generalizes (175) is the following minimization problem:[166]

$$(182)\ \min_{\rho,\beta}\ \{\Sigma_{n\in S(1)}\, s_{1n}[\ln p_{1n} - \beta_n]^2 + \Sigma_{n\in S(2)}\, s_{2n}[\ln p_{2n} - \rho_2 - \beta_n]^2\}.$$

The first order conditions for ρ_2^*, β_1^*,..., β_N^* to solve (182) are equivalent to the following equations:

$$(183)\ \Sigma_{n\in S(2)}\, s_{2n}\rho_2^* + \Sigma_{n\in S(2)}\, s_{2n}\beta_n^* = \Sigma_{n\in S(2)}\, s_{2n}\ln p_{2n}\ ;$$
$$(184)\quad s_{2n}\rho_2^* + (s_{1n} + s_{2n})\beta_n^* = s_{1n}\ln p_{1n} + s_{2n}\ln p_{1n}\ ; \qquad n\in S(1)\cap S(2);$$
$$(185)\quad \beta_n^* = \ln p_{1n}\ ; \qquad n\in S(1),\ n\notin S(2);$$
$$(186)\quad \rho_2^* + \beta_n^* = \ln p_{2n}\ ; \qquad n\in S(2),\ n\notin S(1).$$

Define the intersection set of products S^* as follows:

$$(187)\ S^* \equiv S(1)\cap S(2).$$

Substituting equations (186) into equation (183) leads to the following equation:

$$(188)\ \Sigma_{n\in S^*}\, s_{2n}[\ln p_{2n} - \rho_2^* - \beta_n^*] = 0.$$

Consider the following least squares minimization problem that is defined over the set of products that are present in both periods:

$$(189)\ \min_{\rho,\beta}\ \{\Sigma_{n\in S^*}\, s_{1n}[\ln p_{1n} - \beta_n]^2 + \Sigma_{n\in S^*}\, s_{2n}[\ln p_{2n} - \rho_2 - \beta_n]^2\}.$$

The first order conditions for this problem are (188) and (184). Once we find the solution to this problem, define β_n^* for the products that are not present in both periods by equations (185) and (186). This augmented solution will solve problem (182). The solution to (189) can be found by adapting the solution to (175) to the current situation. Recall equations (176) and (177) from the previous section. Replacing the entire set of product indices n = 1,...,N by the intersection set S^* defined by (187) leads to the following solution to (189):

$$(190)\ \rho_2^* \equiv [\Sigma_{n\in S^*}\, s_{1n}s_{2n}(s_{1n} + s_{2n})^{-1}\ln(p_{2n}/p_{1n})]/[\Sigma_{i\in S^*}\, s_{1i}s_{2i}(s_{1i} + s_{2i})^{-1}]\ ;$$

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165 The results in this section are closely related to the results derived by de Haan (2004a), Silver and Heravi (2005) and de Haan and Krsinich (2014) (2018). However, our method of derivation is somewhat different.

166 This form of weighting was suggested by Rao (1995) (2004) (2005), Diewert (2002) (2004) (2005a) and de Haan (2004a).

(191) $\beta_n^* \equiv s_{1n}(s_{1n} + s_{2n})^{-1} \ln(p_{1n}) + s_{2n}(s_{1n} + s_{2n})^{-1} \ln(p_{2n}/\pi_2^*)$; $n \in S^*$

where $\pi_2^* \equiv \exp[\rho_2^*]$. Define the *normalized harmonic mean share weights* for the always present products as follows as $s_n^* \equiv h(s_{1n},s_{2n})/\Sigma_{i\in S^*} h(s_{1i},s_{2i})$ for $n \in S^*$. Using these definitions for the shares s_n^*, the *weighted time product dummy bilateral price index with missing observations*, $P_{WTPD}(p^1,p^2,q^1,q^2) \equiv \pi_2^*/\pi_1^* = \pi_2^*$, has the following logarithm:

(192) $\ln P_{WTPD}(p^1,p^2,q^1,q^2) \equiv \Sigma_{n\in S^*} s_n^* \ln(p_{2n}/p_{1n})$.

Note that $P_{WTPD} \equiv \pi_2^*/\pi_1^*$ depends directly on the price ratios for the products *that are present in both periods*. However, it also depends on the shares s_{tn}, which in turn depend on all of the price and quantity information for both periods. It can be seen that $P_{WTPD}(p^1,p^2,q^1,q^2)$ is a weighted geometric mean of the matched prices p_{2n}/p_{1n} for products n that are present in both periods. Thus if matched product prices are equal in the two periods, then $P_{WTPD}(p^1,p^2,q^1,q^2)$ will equal unity even if there is an expanding or contracting choice set over the two periods; i.e., alternative reservation prices for any missing products will not affect the estimated price levels and price indexes.

However, the hedonic regression model that is generated by solving (189) can be used to impute (neutral) reservation prices for missing observations. Thus define $\alpha_n^* \equiv \exp[\beta_n^*]$ for n = 1,...,N. Then the missing prices p_{tn}^* can be defined as follows:

(193) $p_{2n}^* \equiv \pi_2^*\alpha_n^* = \pi_2^* p_{1n}$ $n\in S(1), n \notin S(2)$;
(194) $p_{1n}^* \equiv \pi_1^*\alpha_n^* = p_{2n}/\pi_2^*$ $n\in S(2), n \notin S(1)$.

Thus the missing prices for period 2, p_{2n}^*, are the corresponding *inflation adjusted carry forward prices* from period 1, p_{1n} times π_2^* and the missing prices for period 1, p_{1n}^*, are the corresponding *inflation adjusted carry backward prices* from period 2, p_{2n} deflated by π_2^*, where π_2^* is the weighted time product dummy price index $P_{WTPDM}(p^1,p^2,q^1,q^2)$ defined as $\pi_2^* \equiv \exp[\rho_2^*]$ where ρ_2^* is defined by (190).[167] As noted above, these reservation prices are neutral in the sense that they do not affect the definition of ρ_2^* and hence they do not affect the definition of $P_{WTPDM}(p^1,p^2,q^1,q^2)$.

Estimated aggregate price levels can be obtained directly from the solution to (189); i.e., set $P^{1*} = 1$ and $P^{2*} = \pi_2^*$. The corresponding quantity levels are defined as $Q^{1*} \equiv p^1\cdot q^1$ and $Q^{2*} \equiv p^2\cdot q^2/\pi_2^*$. Alternative price and quantity levels can be obtained as $Q^{t**} \equiv \alpha^*\cdot q^t$ and $P^{t**} \equiv p^t\cdot q^t/Q^{t**}$ for t = 1,2. If the optimized objective function in (189) is 0, so that all errors equal 0, then P^{t*} will equal P^{t**} for all t. If the estimated residuals are not all equal to 0, then the two estimates for the period 2 price level P^2 will differ and, as usual, the alternative estimates for P^2 will generate different estimates for the companion aggregate quantity levels.

The above analysis is not quite the end of the story. The expenditure shares s_{1n} and s_{2n} which appear in (182) are not the expenditure shares that characterize the always present products; they are the original expenditure shares defined over all N products. It is of interest to compare $P_{WTPD}(p^1,p^2,q^1,q^2)$ defined implicitly by (192) with the weighted time product dummy index, $P_{WTPDM}(p^{1*},p^{2*},q^{1*},q^{2*})$, that is defined over the common set of products, S^*;[168] i.e., P_{WTPDM} is the

[167] The corresponding imputed values for the missing quantities in each period are set equal to 0.
[168] Define p^{t*} and q^{t*} as the period t price and quantity vectors that include only products that are present in both periods.

weighted time product dummy regression model that is defined over the set of *matched products* for the two periods under consideration.

Define $v_t^* \equiv \Sigma_{n\in S^*} v_{tn}$ as the total expenditure on always present products for t = 1,2 and define the corresponding *restricted expenditure shares* as:[169]

$$(195)\ s_{tn}^* \equiv v_{tn}/v_t^* ; \qquad t = 1,2;\ n\in S^*.$$

The matched model version of (189) is the following weighted least squares minimization problem:

$$(196)\ \min_{\rho,\beta} \{\Sigma_{n\in S^*} s_{1n}^*[\ln p_{1n} - \beta_n]^2 + \Sigma_{n\in S^*} s_{2n}^*[\ln p_{2n} - \rho_2 - \beta_n]^2\}.$$

The ρ_2 solution to (196) is the following one:

$$(197)\ \rho_2^{**} \equiv [\Sigma_{n\in S^*} s_{1n}^* s_{2n}^* (s_{1n}^* + s_{2n}^*)^{-1} \ln(p_{2n}/p_{1n})]/[\Sigma_{i\in S^*} s_{1i}^* s_{2i}^* (s_{1i}^* + s_{2i}^*)^{-1}]$$
$$= [\Sigma_{n\in S^*} h(s_{1n}^*,s_{2n}^*) \ln(p_{2n}/p_{1n})]/[\Sigma_{i\in S^*} h(s_{1i}^*,s_{2i}^*)]$$

where $h(s_{1n}^*,s_{2n}^*)$ is the harmonic mean of the restricted shares s_{1n}^* and s_{2n}^*. Thus $P_{WTPDM}(p^{1*},p^{2*},q^{1*},q^{2*}) \equiv \exp[\rho_2^{**}]$ where ρ_2^{**} is defined by (197).

The relationship between the *true shares*, the s_{tn}, and the *restricted shares*, the s_{tn}^*, for the always present products is given by the following equations:

$$(198)\ s_{tn} \equiv v_{tn}/v_t = [v_{tn}/v_t^*][v_t^*/v_t] = s_{tn}^* f_t ; \qquad t = 1,2 ;\ n\in S^*$$

where the *fraction* of expenditures on always available commodities compared to expenditures on all commodities during period t is $f_t \equiv v_t^*/v_t$ for t = 1,2. Using definitions (190) and (198), it can be seen that the logarithm of $P_{WTPD}(p^1,p^2,q^1,q^2)$ defined by (192) is equal to the following expression:

$$(199)\ \rho_2^* \equiv [\Sigma_{n\in S^*} h(s_{1n},s_{2n}) \ln(p_{2n}/p_{1n})]/[\Sigma_{i\in S^*} h(s_{1i},s_{2i})]$$
$$= [\Sigma_{n\in S^*} h(f_1 s_{1n}^*,f_2 s_{2n}^*) \ln(p_{2n}/p_{1n})]/[\Sigma_{i\in S^*} h(f_1 s_{1i}^*,f_2 s_{2i}^*)].$$

Now compare (197) and (199). If either: (i) $p_{2n} = \lambda p_{1n}$ for all $n\in S^*$ so that we have price proportionality for the always present products or (ii) $f_1 = f_2$ so that the ratio of expenditures on always present products to total expenditure in each period is constant across the two periods, then $\rho_2^{**} = \rho_2^*$. However, if these conditions are not satisfied and there is considerable variation in prices and quantities across periods, then ρ_2^{**} could differ substantially from ρ_2^*. Since neither index is superlative, it is difficult to recommend one of these indexes over the other as the

[169] The matched product expenditure shares defined by (195), $s_{tn}^* \equiv v_{tn}/v_t^*$, differ from the original "true" expenditure shares defined as $s_{tn} \equiv v_{tn}/v_t$ because the true period t expenditures v_t include expenditures on "isolated" products that are present in only one of the two periods under consideration. Thus if there are isolated products in both periods, v^t will be greater than v^{t*} for t = 1,2 and thus the two sets of shares will be different.

"optimal" carry forward and backward inflation rate that could be used to construct the inflation adjusted carry forward and backward estimates for the missing prices.[170]

In the following section, we define weighted time dummy regression models for the general case of T periods and missing observations.

17. Weighted Time Product Dummy Regressions: The General Case

We first consider the case of T periods and no missing observations. The generalization of the two period weighted least squares minimization problem that was defined by (175) in section 15 to the case of T > 2 periods is (200) below:[171]

$$(200)\ \min_{\rho,\,\beta}\ \{\Sigma_{n=1}^{N}\Sigma_{t=1}^{T} s_{tn}[\ln p_{tn} - \rho_t - \beta_n]^2\}.$$

The first order necessary conditions for $\rho^* \equiv [\rho_1^*,...,\rho_T^*]$ and $\beta^* \equiv [\beta_1^*,...,\beta_N^*]$ to solve (200) are the following T equations (201) and N equations (202):

$$(201)\ \rho_t^* = \Sigma_{n=1}^{N} s_{tn}[\ln p_{tn}^* - \beta_n^*]\ ; \qquad t = 1,...,T;$$
$$(202)\ \beta_n^* = \Sigma_{t=1}^{T} s_{tn}[\ln p_{tn}^* - \rho_t^*]/(\Sigma_{t=1}^{T} s_{tn})\ ; \qquad n = 1,...,N.$$

As usual, the solution to (200) given by (201) and (202) is not unique: if $\rho^* \equiv [\rho_1^*,...,\rho_T^*]$ and $\beta^* \equiv [\beta_1^*,...,\beta_N^*]$ solve (201) and (202), then so do $[\rho_1^*+\lambda,...,\rho_T^*+\lambda]$ and $[\beta_1^*-\lambda,...,\beta_N^*-\lambda]$ for all λ. Thus we can set $\rho_1^* = 0$ in equations (201) and drop the first equation in (201) and use linear algebra to find a unique solution for the resulting equations.[172] Once the solution is found, define the estimated *price levels* π_t^* and *quality adjustment factors* α_n^* as follows:

$$(203)\ \pi_t^* \equiv \exp[\rho_t^*]\ ;\ t = 2,3,...,T;\ \alpha_n^* \equiv \exp[\beta_n^*]\ ;\ n = 1,...,N.$$

Note that the resulting *price index* between periods t and τ is equal to the following expression:

$$(204)\ \pi_t^*/\pi_\tau^* = \prod_{n=1}^{N} \exp[s_{tn}\ln(p_{tn}/\alpha_n^*)]/\prod_{n=1}^{N} \exp[s_{\tau n}\ln(p_{\tau n}/\alpha_n^*)]\ ; \qquad 1 \le t, \tau \le T.$$

If $s_{tn} = s_{\tau n}$ for n = 1,...,N, then π_t^*/π_r^* will equal a weighted geometric mean of the price ratios $p_{tn}/p_{\tau n}$ where the weight for $p_{tn}/p_{\tau n}$ is the common expenditure share $s_{tn} = s_{\tau n}$. Thus π_t^*/π_τ^* will not depend on the α_n^* in this case.[173]

The price levels π_t^* defined by (203) are functions of the T price vectors, $p^1,...,p^T$ and the T quantity vectors $q^1,...,q^T$. These price level functions have some good axiomatic properties: (i) the π_t^* are invariant to changes in the units of measurement; (ii) π_t^* regarded as a function of the period t price vector p^t is linearly homogeneous in the components of p^t; i.e., $\pi_t^*(\lambda p^t) = \lambda\pi_t^*(p^t)$ for

[170] For another alternative weighting scheme for a bilateral time product dummy model in the case of two periods that generalizes the model defined by (179) to the case of missing observations, see de Haan (2004a).
[171] Rao (1995) (2004) (2005; 574) was the first to consider this model using expenditure share weights. However, Balk (1980; 70) suggested this class of models much earlier using somewhat different weights.
[172] Alternatively, one can set up the linear regression model defined by $(s_{tn})^{1/2}\ln p_{tn} = (s_{tn})^{1/2}\rho_t + (s_{tn})^{1/2}\beta_n + e_{tn}$ for t = 1,...,T and n = 1,...,N where we set $\rho_1 = 0$ to avoid exact multicollinearity. Iterating between equations (201) and (202) will also generate a solution to these equations and the solution can be normalized so that $\rho_1 = 0$.
[173] This case is consistent with utility maximizing purchasers having common Cobb Douglas preferences.

all $p^t >> 0_N$ and $\lambda > 0$; (iii) π_t^* regarded as a function of the period t quantity vector q^t is homogeneous of degree 0 in the components of q^t; i.e., $\pi_t^*(\lambda q^t) = \pi_t^*(q^t)$ for all $q^t >> 0_N$ and $\lambda > 0$;[174] (iv) the π_t^* satisfy a version of Walsh's (1901; 389) (1921b; 540) *multiperiod identity test*; i.e., if $p^t = p^\tau$ and $q^t = q^\tau$, then $\pi_t^* = \pi_\tau^*$.[175]

Once the estimates for the π_t and α_n have been computed, we have the usual two methods for constructing period by period price and quantity levels, P^t and Q^t for t = 1,...,T. The π_t^* estimates can be used to form the aggregates using equations (205) or the α_n^* estimates can be used to form the aggregates using equations (206):[176]

$$(205)\ P^{t*} \equiv \pi_t^*\ ; \qquad Q^{t*} \equiv p^t\cdot q^t/\pi_t^*\ ; \qquad\qquad t = 1,...,T;$$
$$(206)\ Q^{t**} \equiv \alpha^*\cdot q^t\ ;\ P^{t**} \equiv p^t\cdot q^t/\alpha^*\cdot q^t\ ; \qquad\qquad t = 1,...,T.$$

Define the error terms $e_{tn} \equiv \ln p_{tn} - \ln\pi_t^* - \ln\alpha_n^*$ for t = 1,...,T and n = 1,...,N. If all $e_{tn} = 0$, then P^{t*} will equal P^{t**} and Q^{t*} will equal Q^{t**} for t = 1,...,T. However, if the error terms are not all equal to zero, then the statistical agency will have to decide on pragmatic grounds on which option to choose.

It is straightforward to generalize the weighted least squares minimization problem (200) to the case where there are missing prices and quantities. As in section 14 we assume that there are N products and T time periods but not all products are purchased (or sold) in all time periods. For each period t, define the set of products n that are present in period t as $S(t) \equiv \{n: p_{tn} > 0\}$ for t = 1,2,...,T. It is assumed that these sets are not empty; i.e., at least one product is purchased in each period. For each product n, define the set of periods t where product n is present as $S^*(n) \equiv \{t: p_{tn} > 0\}$. Again, assume that these sets are not empty; i.e., each product is sold in at least one time period. The generalization of (200) to the case of missing products is the following weighted least squares minimization problem:

$$(207)\ \min_{\rho,\beta} \Sigma_{t=1}^T \Sigma_{n\in S(t)} s_{tn}[\ln p_{tn} - \rho_t - \beta_n]^2 = \min_{\rho,\beta} \Sigma_{n=1}^N \Sigma_{t\in S^*(n)} s_{tn}[\ln p_{tn} - \rho_t - \beta_n]^2.$$

Note that there are two equivalent ways of writing the least squares minimization problem. The first order necessary conditions for $\rho_1,...,\rho_T$ and $\beta_1,...,\beta_N$ to solve (207) are the following counterparts to (201) and (202):[177]

$$(208)\ \Sigma_{n\in S(t)} s_{tn}[\rho_t^* + \beta_n^*] = \Sigma_{n\in S(t)} s_{tn}\ln p_{tn}\ ; \qquad\qquad t = 1,...,T;$$
$$(209)\ \Sigma_{t\in S^*(n)} s_{tn}[\rho_t^* + \beta_n^*] = \Sigma_{t\in S^*(n)} s_{tn}\ln p_{tn}\ ; \qquad\qquad n = 1,...,N.$$

[174] By looking at the minimization problem defined by (200), it is also straightforward to show that $\pi_t^*(\lambda q^\tau) = \pi_t^*(q^\tau)$ for all $q^\tau >> 0_N$ and $\lambda > 0$ for τ = 1,....,T.
[175] We would like the π_t^* to satisfy the usual (strong) identity test, which is: if $p^t = p^\tau$, then $\pi_t^* = \pi_\tau^*$. However, if the share weights for the two periods are different, then this test no longer holds. However, if we define the period t price and quantity levels using definitions (206), it can be seen that the resulting Q^{t**} will satisfy the usual (strong) identity test for quantities. If our perspective is one of measuring economic welfare, then we may want to choose (206) over (205).
[176] Note that the price level P^{t**} defined in (206) is a quality adjusted unit value index of the type studied by de Haan (2004b).
[177] Equations (208) and (209) show that the solution to (207) does not depend on any independently determined reservation prices p_{tn} for products n that are missing in period t.

As usual, the solution to (208) and (209) is not unique: if $\rho^* \equiv [\rho_1^*,...,\rho_T^*]$ and $\beta^* \equiv [\beta_1^*,...,\beta_N^*]$ solve (208) and (209), then so do $[\rho_1^*+\lambda,...,\rho_T^*+\lambda]$ and $[\beta_1^*-\lambda,...,\beta_N^*-\lambda]$ for all λ. Thus we can set $\rho_1^* = 0$ in equations (208) and drop the first equation in (208) and use linear algebra to find a unique solution for the resulting equations.

Define the estimated *price levels* π_t^* and *quality adjustment factors* α_n^* by definitions (203). The Weighted Time Product Dummy price level for period t is defined as $p_{WTPD}^t \equiv \pi_t^*$ for t = 1,...,T. Substitute these definitions into equations (208) and (209). After some rearrangement, equations (208) and (209) become the following ones:

(210) $\pi_t^* = \exp[\Sigma_{n\in S(t)} s_{tn}\ln(p_{tn}/\alpha_n^*)] \equiv p_{WTPD}^t$; t = 1,...,T;

(211) $\alpha_n^* = \exp[\Sigma_{t\in S^*(n)} s_{tn}\ln(p_{tn}/\pi_t^*)/\Sigma_{t\in S^*(n)} s_{tn}]$; n = 1,...,N.

Once the estimates for the π_t and α_n have been computed, we have the usual two methods for constructing period by period price and quantity levels, P^t and Q^t for t = 1,...,T; see (205) and (206) above.[178]

The new price levels π_t^* defined by (210) are functions of the T price vectors, $p^1,...,p^T$ and the T quantity vectors $q^1,...,q^T$. If there are missing products, the corresponding prices and quantities, p_{tn} and q_{tn}, are temporarily set equal to 0. The new price level functions defined by (210) have the same axiomatic properties (i)-(iv) which were noted earlier in this section.[179] The present price level functions take the economic importance of the products into account and thus are a clear improvement over their unweighted counterparts which were discussed in section 14. If the estimated errors $e_{tn}^* \equiv \ln p_{tn} - \rho_t^* - \beta_n^*$ that implicitly appear in the weighted least squares minimization problem (207) turn out to be small, then the underlying exact model, $p_{tn} = \pi_t\alpha_n$ for t = 1,...,T, $n\in S(t)$, provides a good approximation to reality and thus this weighted time product dummy regression model can be used with some confidence.

The solution to the weighted least squares minimization problem defined by (207), π_t^* for t = 1,...,T and α_n^* for n = 1,..,N can be used to define (neutral) *reservation prices* for missing observations. For any missing price for product n in period t, define p_{tn}^* as follows:

(212) $p_{tn}^* \equiv \pi_t^*\alpha_n^*$; $n\notin S^*(t)$.

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178 The counterparts to definitions (205) are now: $P^{t*} \equiv \pi_t^* = \Pi_{n\in S(t)} \exp[s_{tn}\ln(p_{tn}/\alpha_n^*)]$, a share weighted geometric mean of the quality adjusted prices present in period t, and $Q^{t*} \equiv \Sigma_{n\in S(t)}p_{tn}q_{tn}/P^{t*}$ for t = 1,...,T. The counterparts to equations (206) are now: $Q^{t**} \equiv \Sigma_{n\in S(t)} \alpha_n^*q_{tn}$ and $P^{t**} \equiv \Sigma_{n\in S(t)}p_{tn}q_{tn}/Q^{t**} = \Sigma_{n\in S(t)}p_{tn}q_{tn}/\Sigma_{n\in S(t)} \alpha_n^*q_{tn} = \Sigma_{n\in S(t)}p_{tn}q_{tn}/\Sigma_{n\in S(t)} \alpha_n^* (p_{tn})^{-1}p_{tn}q_{tn} = [\Sigma_{n\in S(t)} s_{tn}(p_{tn}/\alpha_n^*)^{-1}]^{-1}$, a share weighted harmonic mean of the quality adjusted prices present in period t. Thus using Schlömilch's inequality (see Hardy, Littlewood and Polyá (1934; 26)), we see that $P^{t**} \leq P^{t*}$ which in turn implies that $Q^{t**} \geq Q^{t*}$ for t = 1,...,T. This algebra is due to de Haan (2004b) (2010) and de Haan and Krsinich (2018; 763). If the variance of prices increases over time, it is likely that P^{t**}/P^{1**} will be less that P^{t*}/P^{1*} and vice versa if the variance of prices decreases; see de Haan and Krsinich (2018; 771) and Diewert (2018;10) on this last point. Note that the work of de Haan and Krsinich provides us with a concrete formula for the difference between P^{t*} and P^{t**}. The model used by de Haan and Krsinich is a more general hedonic regression model which includes the time dummy model used in the present section as a special case.

179 However, we would like the P^{t*} to satisfy a strong identity test as noted above; i.e., we would like P^{t*} to equal P^{r*} if the prices in periods t and r are identical. The P^{t*} equal to the π_t^* where the π_t^* are defined by (210) do not satisfy this strong identity test for price levels. However, the Q^{t**} defined as $\Sigma_{n\in S(t)} \alpha_n^*q_{tn}$ do satisfy the strong identity test for quantities and this suggests that the P^{t**}, Q^{t**} decomposition of period t sales may be a better choice than the P^{t*}, Q^{t*} decomposition.

In what follows, we will use the prices defined by (212) to replace the 0 prices in the vectors p^t for t = 1,...,T so with the use of these imputed prices, all price vectors p^t have positive components. Of course, the quantities q_{tn} and the shares s_{tn} that correspond to the imputed prices defined by (212) are still equal to 0.

The weighted time product dummy price level functions p_{WTPD}^t defined by (210) have the same unsatisfactory property that their unweighted counterparts had in previous sections: a product that is available only in one period out of the T periods has no influence on the aggregate price levels $p_{WTPD}^t \equiv \pi_t^*$.[180] This means that the price of a new product that appears in period T has no influence on the price levels and thus the benefits of an expanding consumption set are not measured by this multilateral method. This is a significant shortcoming of this method. However, on the positive side of the ledger, this method does satisfy the strong identity test for the companion quantity index, a property that it shares with the GK multilateral method.[181]

Once the WTPD price levels p_{WTPD}^t have been defined[182], the *weighted time product dummy price index* for period t (relative to period 1) is defined as $P_{WTPD}^t \equiv p_{WTPD}^t/p_{WTPD}^1$ and the logarithm of P_{WTPD}^t is equal to the following expression:

$$(213)\ \ln P_{WTPD}^t = \Sigma_{n=1}^N s_{tn}(\ln p_{tn} - \beta_n^*) - \Sigma_{n=1}^N s_{1n}(\ln p_{1n} - \beta_n^*)\ ; \qquad t = 1,\ldots,T.$$

With the above expression for $\ln P_{WTPD}^t$ in hand, we can compare $\ln P_{WTPD}^t$ to $\ln P_T^t$. Using (213) and definition (40),[183] we can derive the following expressions for t = 1,2,...,T:

$$(214)\ \ln P_{WTPD}^t - \ln P_T^t = \tfrac{1}{2}\Sigma_{n=1}^N (s_{tn} - s_{1n})(\ln p_{tn} - \beta_n^*) + \tfrac{1}{2}\Sigma_{n=1}^N (s_{tn} - s_{1n})(\ln p_{1n} - \beta_n^*).$$

Since $\Sigma_{n=1}^N (s_{tn} - s_{1n}) = 0$ for each t, the two sets of terms on the right hand side of equation t in (214) can be interpreted as normalizations of the covariances between the vectors $s^t - s^1$ and $\ln p^t - \beta^*$ for the first set of terms and between $s^t - s^1$ and $\ln p^1 - \beta^*$ for the second set of terms. If the products are highly substitutable with each other, then a low p_{tn} will usually imply that $\ln p_{tn}$ is less than the average log price for product n, β_n^*, and it is also likely that s_{tn} is greater than s_{1n} so that $(s_{tn} - s_{1n})(\ln p_{tn} - \beta_n^*)$ is likely to be negative. Hence the covariance between $s^t - s^1$ and $\ln p^t - \beta^*$ will tend to be negative. On the other hand, if p_{1n} is unusually low, then $\ln p_{1n}$ will be less than the average log price β_n^* and it is likely that s_{1n} is greater than s_{tn} so that $(s_{tn} - s_{1n})(\ln p_{1n} - \beta_n^*)$ is likely to be positive. Hence the covariance between $s^t - s^1$ and $\ln p^1 - \beta^*$ will tend to be positive. Thus the first set of terms on the right hand side of (214) will tend to be negative while the second set will tend to be positive. If there are no divergent trends in log prices and sales shares, then it is

[180] See Diewert (2004) for a proof or modify the proof in section 16 above.

[181] Both methods are basically quality adjusted unit value methods. Thus if the products under consideration are highly substitutable, then both methods may give satisfactory results. From the viewpont of the economic approach to index number theory, the GK method is consistent with utility maximizing behavior if purchasers have either Leontief (no substitution) preferences or linear preferences (perfect substitution preferences after quality adjustment). The weighted time product dummy method is consistent with utility maximizing behavior if purchasers have either Cobb Douglas preferences or linear preferences. Note that Cobb Douglas preferences are not consistent with situations where there are new and disappearing products.

[182] See (210) above.

[183] If product n in period t is missing, we use the imputed price p_{tn}^* defined by (212) as the positive reservation price for this observation in the definitions for both P_{WTPD}^t and P_T^t which appear in equations (213) and (214). Thus the summations in (213) and (214) are over all N products.

likely that these two terms will largely offset each other and under these conditions, P_{WTPD}^t is likely to approximate P_T^t reasonably well. However, with divergent trends and highly substitutable products, it is likely that the first set of negative terms will be larger in magnitude than the second set of terms and thus P_{WTPD}^t is likely to be below P_T^t under these conditions.[184] But if some product n is not available in period 1 so that $s_{1n} = 0$ and if the logarithm of the imputed price for this product p_{1n}^* defined by (212) is greater than β_n^*, then it can happen that the second covariance term on the right hand side of (214) becomes very large and positive so that it overwhelms the first negative covariance term and thus P_{WTPD}^t ends up above P_T^t rather than below it.

To sum up, the weighted time product indexes can be problematic in the elementary index context when price and quantity data are available as compared to a fixed base superlative index (that uses reservation prices):

- If there are no missing products and the products are strong substitutes, the WTPD indexes will tend to have a downward bias.
- If there are no missing products and the products are weak substitutes, the WTPD indexes will tend to have an upward bias.
- If there are missing products in period 1, the relationship between the WTPD indexes and the corresponding Törnqvist Theil indexes is uncertain.
- If there are missing products, the weighted time product dummy price levels and price indexes do not depend on reservation prices (which could be regarded as an advantage of the WTPD indexes for price statisticians who want to avoid making imputations).

18. Linking Based on Relative Price Similarity

The GEKS multilateral method treats each set of price indexes using the prices of one period as the base period as being equally valid and hence an averaging of the resulting parities seems to be appropriate under this hypothesis. Thus the method is "democratic" in that each bilateral index number comparison between any two periods gets the same weight in the overall method. However, it is not the case that all bilateral comparisons of price between two periods are equally accurate: if the relative prices in periods r and t are very similar, then the Laspeyres and Paasche price indexes will be very close to each other and hence it is likely that the "true" price comparison between these two periods (using the economic approach to index number theory) will be very close to the bilateral Fisher index that compares prices between the two periods under consideration. In particular, if the two price vectors are exactly proportional, then we want the price index between these two periods to be equal to the factor of proportionality and the direct Fisher index between these two periods satisfies this proportionality test. On the other hand, the GEKS index comparison between the two periods would not in general satisfy this proportionality test.[185] Also if prices are identical between two periods but the quantity vectors

[184] If the products are not highly substitutable so that when a price goes up, the quantity purchased goes down but the expenditure share also goes up, then the inequalities are reversed; i.e., if there are no missing products and long term trends in prices and quantities, then P_{WTPD}^t is likely to be above P_T^t. If preferences of purchasers are Cobb Douglas, then expenditure shares will remain constant over time and P_{WTPD}^t will equal P_T^t for $t = 1,...,T$.

[185] If both prices *and* quantities are proportional to each other for the two periods being compared, then the GEKS price index between the two periods will satisfy this (weak) proportionality test. However, we would like the GEKS price index between the two periods to satisfy the strong proportionality test; i.e., if the two price vectors are proportional (and the two quantity vectors are not necessarily proportional to each other), then we would like the GEKS price index between the two periods to equal the factor of proportionality.

are different, then GEKS price index between the two periods would not equal unity in general.[186] The above considerations suggest that a more accurate set of price indexes could be constructed if initially a bilateral comparison was made between the two periods that have the most *similar relative price structures*. At the next stage of the comparison, look for a third period that had the most similar relative price structure to the first two periods and link in this third country to the comparisons of volume between the first two countries and so on. At the end of this procedure, a pathway through the periods in the window would be constructed, that minimized the sum of the relative price dissimilarity measures. In the context of making comparisons of prices across countries, this method of linking countries with the most similar structure of relative prices has been pursued by Hill (1997) (1999a) (1999b) (2009), Hill and Timmer (2006), Diewert (2009) (2013) (2018) and Hill, Rao, Shankar and Hajargasht (2017). Hill (2001) (2004) also pursued this similarity of relative prices approach in the time series context. Our conclusion is that similarity linking using Fisher ideal price indexes as the bilateral links is an attractive alternative to GEKS.

A key aspect of this methodology is the choice of the measure of similarity (or dissimilarity) of the relative price structures of two countries. Various measures of the similarity or dissimilarity of relative price structures have been proposed by Allen and Diewert (1981), Kravis, Heston and Summers (1982; 104-106), Hill (1997) (2009), Sergeev (2001) (2009), Hill and Timmer (2006), Aten and Heston (2009) and Diewert (2009) (2021a).

In this section, we will discuss the following *weighted asymptotic linear index of relative price dissimilarity*, Δ_{AL}, suggested by Diewert (2009):[187]

$$(215)\ \Delta_{AL}(p^r,p^t,q^r,q^t) \equiv \sum_{n=1}^{N} ½(s_{rn}+s_{tn})\{(p_{tn}/P_F(p^r,p^t,q^r,q^t)p_{rn}) + (P_F(p^r,p^t,q^r,q^t)p_{rn}/p_{tn}) - 2\}$$

where $P_F(p^r,p^t,q^r,q^t) \equiv [p^t{\cdot}q^r p^t{\cdot}q^t/p^r{\cdot}q^r\ p^r{\cdot}q^t]^{1/2}$ is the bilateral Fisher price index linking period t to period r and p^r, q^r, s^r and p^t, q^t, s^t are the price, quantity and share vectors for periods r and t respectively. This measure turns out to be nonnegative and the bigger $\Delta_{AL}(p^r,p^t,q^r,q^t)$ is, the more dissimilar are the relative prices for periods r and t. Note that if $p^t = \lambda p^r$ for some positive scalar so that if prices are proportional for the two periods, then $\Delta_{AL}(p^r,p^t,q^r,q^t) = 0$. Note also that *all prices need to be positive* in order for $\Delta_{AL}(p^r,p^t,q^r,q^t)$ to be well defined. Thus if there are missing products in one of the two periods being compared, reservation prices need to be estimated for the missing product prices in each period.[188] Alternatively, inflation adjusted carry forward or carry backward prices can be used to fill in the missing prices.[189]

The method for constructing *Similarity Linked Fisher* price indexes in real time using the above measure of relative price similarity proceeds as follows. Set the similarity linked price index for period 1, $P_{AL}^1 \equiv 1$. The period 2 index is set equal to $P_F(p^1,p^2,q^1,q^2)$, the Fisher index linking the period 2 prices to the period 1 prices. Thus $P_{AL}^2 \equiv P_F(p^1,p^2,q^1,q^2)P_{AL}^1$. For period 3, evaluate the dissimilarity indexes $\Delta_{AL}(p^1,p^3,q^1,q^3)$ and $\Delta_{AL}(p^2,p^3,q^2,q^3)$ defined by (215). If $\Delta_{AL}(p^1,p^3,q^1,q^3)$ is the minimum of the two numbers, $\Delta_{AL}(p^1,p^3,q^1,q^3)$ and $\Delta_{AL}(p^2,p^3,q^2,q^3)$, define $P_{AL}^3 \equiv P_F(p^1,p^3,q^1,q^3)P_{AL}^1$. If $\Delta_{AL}(p^2,p^3,q^2,q^3)$ is the minimum of these two numbers, define $P_{AL}^3 \equiv P_F(p^2,p^3,q^2,q^3)P_{AL}^2$. For period 4, evaluate the dissimilarity indexes $\Delta_{AL}(p^r,p^4,q^r,q^4)$ for r = 1,2,3.

[186] See Zhang, Johansen and Nygaard (2019; 689) on this point.
[187] The discussion paper version of Diewert (2009) appeared in (2002).
[188] See section 14 of Diewert (2021a) for additional information on reservation prices.
[189] See the discussion in the following section. Section A6 of the Appendix compares P_{AL}^t computed using reservation prices and P_{ALC}^t which uses inflation adjusted carry forward/backward prices for missing products. For our particular empirical example, there were small differences in the resulting indexes.

Let r^* be such that $\Delta_{AL}(p^{r*},p^4,q^{r*},q^4) = \min_r \{\Delta_{AL}(p^r,p^4,q^r,q^4);\ r = 1,2,3\}$.[190] Then define $P_{AL}^4 \equiv P_F(p^{r*},p^4,q^{r*},q^4)P_{AL}^{r*}$. Continue this process in the same manner; i.e., for period t, let r^* be such that $\Delta_{AL}(p^{r*},p^t,q^{r*},q^t) = \min_r \{\Delta_{AL}(p^r,p^t,q^r,q^t);\ r = 1,2,...,t-1\}$ and define $P_{AL}^t \equiv P_F(p^{r*},p^t,q^{r*},q^t)P_{AL}^{r*}$. This procedure allows for the construction of similarity linked indexes in real time.

Diewert (2018) implemented the above procedure with a retail outlet scanner data set and compared the resulting similarity linked index, P_{AL}^t, to other indexes that are based on the use of superlative indexes and the economic approach to index number theory. The data set he used is listed in section 1 of the Appendix and his results are listed in the Appendix along with some additional results. The comparison indexes in his study were the fixed base Fisher and Törnqvist indexes, P_F^t and P_T^t, and the multilateral indexes, P_{GEKS}^t and P_{CCDI}^t. The sample means for these five indexes, P_{AL}^t, P_F^t, P_T^t, P_{GEKS}^t and P_{CCDI}^t, were 0.97069, 0.97434, 0.97607, 0.97417 and 0.97602. Thus on average, P_{AL}^t was about 0.5 percentage points below P_T^t and P_{CCDI}^t and about 0.35 percentage points below P_F^t and P_{GEKS}^t. These are fairly significant differences.[191]

What are some of the advantages and disadvantages of using either P_{AL}^t, P_F^t, P_T^t, P_{GEKS}^t or P_{CCDI}^t as target indexes for an elementary index in a CPI? All of these indexes are equally consistent with the economic approach to index number theory. The problem with the fixed base Fisher and Törnqvist indexes is that they depend too heavily on the base period. Moreover, sample attrition means that the base must be changed fairly frequently, leading to a potential chain drift problem. The GEKS and CCDI indexes also suffer from problems associated with the existence of seasonal products: it makes little sense to include bilateral indexes between all possible periods in a window of periods in the context of seasonal commodities. The similarity linked indexes address both the problem of sample attrition and the problem of seasonal commodities. Moreover, Walsh's multiperiod identity test is always satisfied using this methodology. Finally, there is no need to choose a window length and use a rolling window approach to construct the time series of indexes if the price similarity linking method is used: the window length simply grows by one period as the data for an additional period becomes available.[192]

The procedure for constructing the time series of similarity linked Fisher price indexes, P_{AL}^t, is a *real time procedure*; i.e., there is no preliminary time period that is required in order to produce the final time series of aggregate price levels. However, the resulting pattern of bilateral links may not be "optimal" in the sense that the most similar sets of relative prices are linked to one another in the first year or so. This is apparent when the price level P_{AL}^2 is constructed: it is simply equal to the Fisher index linking period 2 to 1; there are no other choices for a linking partner. A "better" set of bilateral links could potentially be obtained if a final set of bilateral links for the index could be obtained by forming a *spanning tree of comparisons* say for the first year of data.[193] Thus a year of data on prices and quantities is used to form a set of bilateral links that minimizes the sum of the associated dissimilarity measures that link the observations for the first year. This leads to a *modified* set of price levels for the first year, say P_{ALM}^t for t in the first year. For months t that follow after the first "training" year, the bilateral links are the same as indicated earlier but because the levels in the first year may have changed, the modified price

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[190] If the minimum occurs at more than one r, choose r^* to be the earliest of these minimizing periods.

[191] The final values for the five indexes (P_{AL}^t, P_F^t, P_T^t, P_{GEKS}^t and P_{CCDI}^t) were as follows: 0.92575, 0.95071, 0.95482, 0.94591 and 0.94834. Thus P_{AL}^t ended up significantly below the other indexes. P_T^t is listed in Table A.4 and the remaining indexes are listed in Table A.6 of the Appendix.

[192] In practice, as the number of periods grow and the structure of the economy evolves, it will become increasingly unlikely that a current observation will be linked to a distant observation. Thus eventually, it becomes practical to move to a rolling window framework with a large window length.

[193] See Hill (2001) (2004) for explanations of how this can be done.

levels $P_{ALM}{}^t$ for months t that follow after the first year may differ from the real time price levels $P_{AL}{}^t$ described earlier. However, the trends in the two series will be similar. In section 5 of the Appendix, we calculate both $P_{AL}{}^t$ and $P_{ALM}{}^t$ for the data set listed in section 1 of the Appendix. There is little difference in these two series for our example data set and in fact, both series end up at the same point.[194]. Normally, we do not expect much difference between the original real time method and the modified method but the modified method is useful in the context of constructing price indexes for strongly seasonal commodities because it will tend to reduce the magnitude of seasonal fluctuations.

Similarity linked price indexes suffer from at least two problems:

- A measure of relative price dissimilarity must be chosen and there may be many "reasonable" choices for the measure of dissimilarity. These different choices can lead to different indexes, which in turn can lead users to question the usefulness of the method.
- The measures of weighted price dissimilarity suggested by Diewert (2009) require that all prices in the comparison of prices between two periods be positive.

These problems will be addressed in section 20 below where an alternative measure of price dissimilarity that does not require strictly positive prices will be defined. Using the scanner data set listed in section 1 of the Appendix, this new measure of price (and quantity) dissimilarity generates indexes $P_{SP}{}^t$ that are very similar to the $P_{AL}{}^t$ indexes discussed in the present section.

It is a difficult econometric exercise to estimate reservation prices and so a simpler method may be required in order to construct imputed prices for missing products in a scanner data set. In the following section, a standard method used by price statisticians is explained.

19. Inflation Adjusted Carry Forward and Backward Imputed Prices

When constructing elementary indexes, statistical agencies often encounter situations where a product in an elementary index disappears. At the time of disappearance, it is unknown whether the product is temporarily unavailable so the missing price could be set equal to the last available price; i.e., the missing price could be replaced by a carry *forward price*. Thus carry forward prices could be used in place of reservation prices, which are much more difficult to construct. This procedure is, in general, not a recommended one. A much better alternative to the use of a carry forward price is an *inflation adjusted carry forward price*; i.e., the last available price is escalated using the maximum overlap index between the period when the product was last available and the current period where an appropriate index number formula is used.[195] In this section, we use inflation adjusted carry forward and carry backward prices in place of the reservation prices for our scanner data set and compare the resulting indexes with our earlier indexes that used the econometrically estimated reservation prices that were constructed by Diewert and Feenstra (2017) for the scanner data set listed in Appendix 1.

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[194] See Table A.7 and Chart 9 in the Appendix. Although $P_{AL}{}^t$ and $P_{ALM}{}^t$ end up at the same level, the mean of the $P_{AL}{}^t$ was 0.97069 and the mean of the $P_{ALM}{}^t$ was 0.96437. The fluctuations in the $P_{ALM}{}^t$ series were somewhat smaller. This tendency for the modified series to be a bit smoother than the corresponding real time series becomes important in the context of constructing indexes for strongly seasonal commodities. In this context, the use of the modified similarity linking method is recommended in order to reduce seasonal fluctuations.

[195] Triplett (2004; 21-29) calls these two methods for replacing missing prices the *link to show no change method* and the *deletion method*. See section 14 in Diewert (2021a) and Diewert, Fox and Schreyer (2017) for a more extensive discussion on the problems associated with finding replacements for missing prices.

Suppose we have price and quantity data for N products for T periods as usual. Let $p^t \equiv [p_{t1},...,p_{tN}]$ and $q^t \equiv [q_{t1},...,q_{tN}]$ denote the period t price and quantity vectors. If product n is not present in period t, define (for now) the corresponding p_{tn} and q_{tn} to be 0. Define S(t) to be the set of products that are present in period t; i.e., $S(t) \equiv \{n: p_{tn} > 0\}$.[196] Suppose that we want to make a Fisher index number comparison between periods r and t where r < t. The *maximum overlap set of products* that are present in periods r and t is the intersection set, $S(r)\cap S(t)$. We assume that this set is nonempty. Define the vectors p^{r*}, p^{t*}, q^{r*}, q^{t*} as the vectors that have only the products that are present in periods r and t. Define the *maximum overlap Fisher price index* for period t relative to period r as $P_{FM}(p^{r*},p^{t*},q^{r*},q^{t*})$. If there are products present in period r that are not present in period t, define the *inflation adjusted carry forward price* for such products as follows:

(216) $p_{tn} \equiv p_{rn}P_{FM}(p^{r*},p^{t*},q^{r*},q^{t*})$; $n\in S(r)$; $n\notin S(t)$.

The corresponding quantities q_{tn} remains at their initially defined 0 levels. If there are products present in period t that are not present in period r, define the *inflation adjusted carry backward price* for such products as follows:

(217) $p_{rn} \equiv p_{tn}/P_{FM}(p^{r*},p^{t*},q^{r*},q^{t*})$; $n\in S(t)$; $n\notin S(r)$.

The corresponding quantities q_{rn} remain at their initial 0 levels.

Using the above definitions, we will have new price and quantity vectors that have well defined price and quantity vectors p^{r**}, p^{t**}, q^{r**}, q^{t**} that have positive prices for products that belong to the union set of products that are present in both periods r and t, $S(r)\cup S(t)$. Denote the Fisher index for period t relative to period r over this union set of products as $P_F^*(p^{r**},p^{t**},q^{r**},q^{t**})$. This index can be used as the Fisher index linking periods r and t. Thus the carry forward and carry backward prices defined by (216) and (217) can replace econometrically estimated reservation prices and the similarity linked price indexes defined in the previous section can be calculated using the Fisher linking indexes $P_F^*(p^{r**},p^{t**},q^{r**},q^{t**})$ in place of the $P_F(p^r,p^t,q^r,q^t)$ used in the previous section. Note that the components of the period t price vector p^{t**} will be equal to the components of the original period t price vector p^t except for components that correspond to missing products.

It should be emphasized that, usually, it is important to make the index number adjustments to the carry forward and backward prices defined by (216) and (217) instead of simply carrying existing prices from one period to another period. Failure to make these index number adjustments could lead to substantial biases if substantial general inflation (or deflation) is present. From the perspective of the economic approach to index number theory, it is likely that the use of inflation adjusted carry backward prices in place of estimated reservation prices will in general lead to an upward bias in the linking index since the "true" reservation prices are likely to be higher than the adjusted prices in order to induce consumers to purchase zero units of the unavailable products in the prior period. Of course the bias in using carry forward prices for disappearing products works in the opposite direction.

In section A6 of the Appendix, we used our scanner data to compute the GEKS, Fisher, Chained Fisher and the real time similarity linked index explained in the previous section which used the Δ_{AL} dissimilarity measure defined by (215). We also calculated the real time Predicted Share

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[196] Recall that this notation was used in previous sections.

similarity linked indexes that use the Δ_{SP} dissimilarity measure that will be defined by (218) in the following section. Denote the resulting period t index by P_{SP}^t. There were missing products in our scanner data set. As noted above, the missing prices were initially set equal to reservation prices calculated using econometrics. Denote these indexes for period t (which used reservation prices) by P_{GEKS}^t, P_F^t, P_{FCH}^t, P_{AL}^t and P_{SP}^t. The same five indexes were recomputed using inflation adusted carry forward and carry backward prices for the missing product prices.[197] Denote the resulting period t indexes by P_{GEKSC}^t, P_{FC}^t, P_{FCHC}^t, P_{ALC}^t and P_{SPC}^t. As noted earlier,it turns out that Geary Khamis index (P_{GK}^t) and Weighted Time Product Dummy index (P_{WTPD}^t) do not depend on the values of the missing prices and so these indexes do not have to be recomputed using carry forward prices in place of reservation prices. P_{GK}^t and P_{WTPD}^t are listed in Table A.6 in section A5 of the Appendix. The series P_{AL}^t, P_{ALC}^t, P_{SP}^t, P_{SPC}^t, P_{GEKS}^t, P_{GEKSC}^t are listed in Table A.8 in section A6 of the Appendix A6 along with the Fisher and Chained Fisher indexes using reservation prices, denoted by P_F^t and P_{FCH}^t, and using carry forward prices, denoted by P_{FC}^t and P_{FCHC}^t.

A summary of the results using econometrically estimated reservation prices versus using carry forward and backward prices for the missing products is as follows: for our example, there was very little difference between the resulting index pairs using reservation prices versus using inflation adjusted carry forward prices. This is likely due to the fact that only 20 out of 741 prices were missing; i.e., only 2.7% of the sample had missing products. (0.97542) and $P_{FCH}^A = 1.0589$ (1.0589). Our tentative conclusion here is that *for products that are highly substitutable, the use of inflation adjusted forward and backward prices for missing products will probably generate weighted indexes that are comparable to their counterparts that use econometrically estimated reservation prices*. For products which are not highly substitutable, it is likely that reservation prices will be higher than their inflation adjusted carry forward prices and thus it is likely that the indexes will differ in a more substantial manner. This conclusion is only tentative and further research on the use of reservation prices is required.

20. Linking Based on Relative Price and Quantity Similarity

A problem with the measure of relative price dissimilarity $\Delta_{AL}(p^r,p^t,q^r,q^t)$ defined by (215) is that it requires that all prices in the two periods being compared must be positive. Thus if there are missing prices for some products present in one of the two periods but not in the other period, then the Δ_{AL} dissimilarity measure is not well defined.[198]

The following *Predicted Share measure of relative price dissimilarity*, $\Delta_{SP}(p^r,p^t,q^r,q^t)$, is well defined even if some product prices in the two periods being compared are equal to zero:

(218) $\Delta_{SP}(p^r,p^t,q^r,q^t) \equiv \Sigma_{n=1}^N [s_{tn} - (p_{rn}q_{tn}/p^r{\cdot}q^t)]^2 + \Sigma_{n=1}^N [s_{rn} - (p_{tn}q_{rn}/p^t{\cdot}q^r)]^2$

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[197] Inflation adjusted carry forward prices were used to compute prices for missing products except when a product was missing in period 1. In the latter case, inflation adjusted carry backward prices were computed for the missing products.

[198] Diewert (2009; 205-206) recommended two other measures of price dissimilarity but they also have the problem that they are also not well defined if some product prices are equal to 0. These alternative measures are the *weighted log quadratic measure of relative price dissimilarity*, $\Delta_{PLQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n^1 + s_n^2)[\ln(p_n^2/p_n^1P(p^1,p^2,q^1,q^2))]^2$, and the *weighted asymptotically quadratic measure of relative price dissimilarity*, $\sum_{n=1}^N (1/2)(s_n^1 + s_n^2)\{[(p_n^2/p_n^1P(p^1,p^2,q^1,q^2) - 1]^2 + [(P(p^1,p^2,q^1,q^2)p_n^1/p_n^2) - 1]^2\} \equiv \Delta_{WAQ}(p^1,p^2,q^1,q^2)$, where $P(p^1,p^2,q^1,q^2)$ is any superlative bilateral price index formula. It can be shown that $\Delta_{PLQ}(p^1,p^2,q^1,q^2)$ approximates $\Delta_{AL}(p^r,p^t,q^r,q^t)$ to the second order around any point where $p^1 = p^2 >> 0_N$ and $q^1 = q^2 >> 0_N$.

where $s_{tn} \equiv p_{tn}q_{tn}/p^t\cdot q^t$ is the share of product n in period t expenditures on the N products for t = 1,...,T and n = 1,...,N. We require that $p^r\cdot q^t > 0$ for r = 1,...,T and t = 1,...,T in order for $\Delta_{SP}(p^r,p^t,q^r,q^t)$ to be well defined for any pair of periods, r and t. Since the two summations on the right hand side of (218) are sums of squared terms, we see that $\Delta_{SP}(p^r,p^t,q^r,q^t) \geq 0$.

The first set of N terms on the right hand side of (218) is $\Sigma_{n=1}^N [s_{tn} - (p_{rn}q_{tn}/p^r\cdot q^t)]^2$. Note that the terms $p_{rn}q_{tn}/p^r\cdot q^t$ for n = 1,...,N are (hybrid) *shares*; i.e., these terms are nonnegative and they sum to unity so that $\Sigma_{n=1}^N (p_{rn}q_{tn}/p^r\cdot q^t) = 1$. These shares use the prices of period r and the quantities of period t. They can be regarded as *predictions* for the actual period t shares, s_{tn}, using the prices of period r but using the quantities of period t. A similar interpretation applies to the second set of N terms on the right hand side of (218); the hybrid shares that use the prices of period t and the quantities of period r, $p_{tn}q_{rn}/p^t\cdot q^r$, can be regarded as predictors for the actual period r shares, s_{rn}. Since each share s_{tn} in the first set of terms is already weighted by its economic importance, there is no need for any further weighting of the first set of N squared terms in the summation to account for economic importance. The same analysis applies to the second set of N sum of squared terms; each term in the summation is already weighted by its economic importance.

If prices in period t are proportional to prices in period r (so that $p^t = \lambda_t p^r$ for some scalar $\lambda_t > 0$ or $p^r = \lambda_r p^t$ for some $\lambda_r > 0$), then it is easy to verify that $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) is equal to 0.

Now consider the implications on p^t and p^r if $\Delta_{SP}(p^r,p^t,q^r,q^t) = 0$. We need to consider a number of cases, depending on assumptions about the positivity of the prices and quantities in periods r and t. In all cases listed below, it is assumed that $p^r\cdot q^t > 0$ for r = 1,...,T and t = 1,...,T.

Case (i): $\Delta_{SP}(p^r,p^t,q^r,q^t) = 0$ and $q^t >> 0_N$ *or* $q^r >> 0_N$; i.e., assume that all components of the period t *or* period r quantity vectors are positive. If $q^t >> 0_N$ and $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) is 0, then the first sum of squared terms, $\Sigma_{n=1}^N [s_{tn} - (p_{rn}q_{tn}/p^r\cdot q^t)]^2 = 0$, which implies that $p_{tn}q_{tn} = (p^t\cdot q^t/p^r\cdot q^t)p_{rn}q_{tn}$ which in turn implies that $p_{tn} = (p^t\cdot q^t/p^r\cdot q^t)p_{rn}$ since $q_{tn} > 0$ for n = 1,...,N. Thus $p^t = \lambda_{tr}p^r$ where $\lambda_{tr} \equiv p^t\cdot q^t/p^r\cdot q^t > 0$ which implies that the period t price vector is proportional to the period r price vector. If $q^r >> 0_N$ and $\Delta_{SP}(p^r,p^t,q^r,q^t)$ is 0, then the second set of terms on the right hand side of (218) is equal to zero. Thus we must have $p_{rn} = (p^r\cdot q^r/p^t\cdot q^r)p_{tn}$ for n = 1,...,N. Thus $p^r = \lambda_{rt}p^t$ where $\lambda_{rt} \equiv p^r\cdot q^r/p^t\cdot q^r > 0$ which in turn implies that the period r price vector is proportional to the period t price vector.

Case (ii): $\Delta_{SP}(p^r,p^t,q^r,q^t) = 0$ and $q^r + q^t >> 0_N$ so that each product is present in at least one of the two periods, periods r and t, whose prices are being compared. We further assume that there is at least one product n^* that is present in both periods being compared; i.e., there exists an n^* such that $q_{rn^*} > 0$ and $q_{tn^*} > 0$. Following the same type of argument that was pursued for Case (i) above, we find that our assumptions imply that $p_{tn} = \lambda_{tr}p_{rn}$ for n such that $q_{tn} > 0$ and $p_{rn} = \lambda_{rt}p_{tn}$ for n such that $q_{rn} > 0$. For products n^* that are present in both periods r and t, we have $p_{tn^*} = \lambda_{tr}p_{rn^*}$ and $p_{rn^*} = \lambda_{rt}p_{tn^*}$ and thus $\lambda_{tr} = 1/\lambda_{rt}$. These equalities imply that the period t price vector must be proportional to the period r price vector under our present assumptions.

Case (iii): Some products are not present in both periods r and t. This case can be reduced down to one of the previous cases for a new N^* that just includes the products that are present in in at least one of periods r and t.

Using the above analysis, it can be seen that $\Delta_{SP}(p^r,p^t,q^r,q^t)$ equals 0 if and only if the period r and t price vectors are proportional. If the price vectors are not proportional, then $\Delta_{SP}(p^r,p^t,q^r,q^t)$ will be positive. A larger value for $\Delta_{SP}(p^r,p^t,q^r,q^t)$ indicates a bigger deviation from price proportionality. Thus $\Delta_{SP}(p^r,p^t,q^r,q^t)$ is a "reasonable" measure of bilateral relative price dissimilarity.

There are some aspects of the predicted price measure of relative price dissimilarity that require further discussion. When comparing the prices of periods r and t, suppose product 1 is present in period t but not present in period r. More precisely, suppose $q_{t1} > 0$ (and $p_{t1} > 0$) but $q_{r1} = 0$. What is the corresponding price for the missing product in period r; i.e., what exactly is p_{r1}? Suppose we set $p_{r1} = 0$. For simplicity, suppose further that prices and quantities for products 2 to N are the same in periods r and t, so that $p_{rn} = p_{tn}$ and $q_{rn} = q_{tn}$ for n = 2,3,...,N. Under these conditions, we find that $\Delta_{SP}(p^r,p^t,q^r,q^t)$ is equal to the following sum of squared terms:

$$\begin{aligned}(219)\ \Delta_{SP}(p^r,p^t,q^r,q^t) &\equiv \Sigma_{n=1}^N [s_{tn} - (p_{rn}q_{tn}/p^r\cdot q^t)]^2 + \Sigma_{n=1}^N [s_{rn} - (p_{tn}q_{rn}/p^t\cdot q^r)]^2 \\ &= [s_{t1} - 0]^2 + \Sigma_{n=2}^N [s_{tn} - s_{rn}]^2 + \Sigma_{n=1}^N [s_{rn} - s_{rn}]^2 \\ &= s_{t1}^2 + \Sigma_{n=2}^N [s_{tn} - s_{rn}]^2 \\ &> 0\end{aligned}$$

where the inequality follows since under our assumptions, $s_{t1} > 0$. Thus even if all prices and quantities are the same for products that are present in both periods r and t, the dissimilarity measure defined by (218) will be positive as long as there are some products that are present in only one of the two periods being compared. Thus if we set the prices for missing products equal to 0, then the predicted share measure of relative price dissimilarity will automatically register a positive measure; i.e., the measure will *penalize* a lack of matching of prices if we set the prices for missing products equal to 0.

Hill and Timmer were the first to point out the importance of having a measure of relative price dissimilarity that would penalize a lack of matching of the prices in the two periods being compared:

"In a survey of the literature on reliability measures, Rao and Timmer (2003) concluded that the main problem of existing measures, such as Hill's (1999) Paasche-Laspeyres spread and Diewert's (2002) class of relative price dissimilarity measures, is that they fail to make adjustments for gaps in the data. Rao and Timmer drew a distinction between statistical and index theoretic measures of reliability. The former take a sampling perspective; bilateral comparisons based on a small number of matched product headings or a low coverage of total expenditure or production (averaged across the two countries) are deemed less reliable. In addition to the standard statistical arguments regarding small samples and a low coverage not being representative, little overlap in the product headings priced by the two countries implies that they are very different and, by implication, inherently difficult to compare. Index theoretic measures, in contrast, focus on the sensitivity of a bilateral comparison to the choice of price index formula. Most of the reliability measures proposed in the literature, including Hill's (1999) Paasche-Laspeyres spread and Diewert's (2002) class of relative price dissimilarity measures, are of this type. Although these measures perform well when there are few gaps in the data, they can generate highly misleading results when there are many gaps. This is because they fail to penalize bilateral comparisons made over a small number of matched headings."

Robert J. Hill and Marcel P. Timmer (2006; 366).

The above considerations suggest that the predicted share measure of relative price dissimilarity could be used under two different sets of circumstances when there are missing prices:

- Use carry forward (or backward) prices or reservation prices for the missing prices and use the measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) to link the observations. With a complete set of prices for each period in hand, the usual bilateral Fisher index could be used as the linking index. This approach is consistent with the economic approach to index number theory.

- Do not estimate carry forward or reservation prices for the missing price observations (and set the prices of the missing products equal to 0) but still use $\Delta_{SP}(p^r,p^t,q^r,q^t)$ to link the observations. In this case, the *maximum overlap* bilateral Fisher index is used as the linking index for each pair of links chosen by the similarity linking method. This approach is more consistent with the stochastic approach to index number theory used by Hill and Timmer (2006).

Both strategies are illustrated for our empirical example in the Appendix.

Some additional properties of $\Delta_{SP}(p^r,p^t,q^r,q^t)$ are the following ones:

- *Symmetry*; i.e., $\Delta_{SP}(p^r,p^t,q^r,q^t) = \Delta_{SP}(p^t,p^r,q^t,q^r)$.
- *Invariance to changes in the units of measurement*.
- *Homogeneity of degree 0 in the components of q^r and q^t*; i.e., $\Delta_{SP}(p^r,p^t,\lambda_r q^r,\lambda_t q^t) = \Delta_{SP}(p^r,p^t,q^r,q^t)$ for all $\lambda_r > 0$ and $\lambda_t > 0$.
- *Homogeneity of degree 0 in the components of p^r and p^t*; i.e., $\Delta_{SP}(\lambda_r p^r,\lambda_t p^t,q^r,q^t) = \Delta_{SP}(p^r,p^t,q^r,q^t)$ for all $\lambda_r > 0$ and $\lambda_t > 0$.

The relative price dissimilarity indexes $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) can be used in place of the dissimilarity indexes $\Delta_{AL}(p^r,p^t,q^r,q^t)$ defined by (215) in section 18 above in order to link together bilateral Fisher indexes. Thus set the new relative price similarity linked Fisher price index for period 1 equal to unity; i.e., set $P_{SP}^1 \equiv 1$. The period 2 index is set equal to $P_F(p^1,p^2,q^1,q^2)$, the Fisher index linking the period 2 prices to the period 1 prices.[199] Thus $P_{SP}^2 \equiv P_F(p^1,p^2,q^1,q^2)P_{SP}^1$. For period 3, evaluate the dissimilarity indexes $\Delta_{SP}(p^1,p^3,q^1,q^3)$ and $\Delta_{SP}(p^2,p^3,q^2,q^3)$ defined by (218). If $\Delta_{SP}(p^1,p^3,q^1,q^3)$ is the minimum of these two numbers, define $P_{SP}^3 \equiv P_F(p^1,p^3,q^1,q^3)P_{PS}^1$. If $\Delta_{SP}(p^2,p^3,q^2,q^3)$ is the minimum of these two numbers, define $P_{SP}^3 \equiv P_F(p^2,p^3,q^2,q^3)P_{SP}^2$. For period 4, evaluate the dissimilarity indexes $\Delta_{SP}(p^r,p^4,q^r,q^4)$ for $r = 1,2,3$. Let r^* be such that $\Delta_{SP}(p^{r*},p^4,q^{r*},q^4) = \min_r \{\Delta_{SP}(p^r,p^4,q^r,q^4);\ r = 1,2,3\}$.[200] Then define $P_{SP}^4 \equiv P_F(p^{r*},p^4,q^{r*},q^4)P_{SP}^{r*}$. Continue this process in the same manner; i.e., for period t, let r^* be such that $\Delta_{SP}(p^{r*},p^t,q^{r*},q^t) = \min_r \{\Delta_{SP}(p^r,p^t,q^r,q^t);\ r = 1,2,...,t-1\}$ and define $P_{SP}^t \equiv P_F(p^{r*},p^t,q^{r*},q^t)P_{SP}^{r*}$. Again, as in section 18, this procedure allows for the construction of similarity linked indexes in real time.

Using the scanner data listed in Appendix 1 which included reservation prices for missing products, the new similarity linked price indexes P_{SP}^t were calculated and compared to the price similarity linked price indexes P_{AL}^t that were defined in section 18 above. The new measure of relative price dissimilarity led to a different pattern of bilateral links: 7 of the 38 bilateral links changed when the dissimilarity measure was changed from $\Delta_{AL}(p^r,p^t,q^r,q^t)$ to $\Delta_{SP}(p^r,p^t,q^r,q^t)$. However, the price indexes generated by these alternative methods for linking observations were very similar: the sample averages for P_{AL}^t and P_{SP}^t were 0.97069 and 0.97109 respectively and the correlation coefficient between the two indexes was 0.99681. Both indexes ended up at 0.9275. Thus even though the two measures of price dissimilarity generated a different pattern of bilateral links, the underlying indexes P_{Al}^t and P_{SP}^t approximated each other very closely.

Both of the similarity linked price indexes P_{AL}^t and P_{SP}^t satisfy *a strong identity test*; i.e., if $p^r = p^t$, then $P_{AL}^r = P_{AL}^t$ and $P_{SP}^r = P_{SP}^t$. It is not necessary for q^r to equal q^t for this strong identity test to

[199] In the present context, it is not necessary to have all prices positive in computing the Fisher indexes. However, if the economic approach to index number theory is applied, then it is preferable to impute the missing prices. Missing quantities should be left at their 0 values using the economic approach.
[200] If the minimum occurs at more than one r, choose r^* to be the earliest of these minimizing periods.

be satisfied. Thus these similarity linked indexes have an advantage over the corresponding GEKS and CCDI multilateral indexes in that in order to ensure that $P_{GEKS}{}^r = P_{GEKS}{}^t$ and $P_{CCDI}{}^r = P_{CCDI}{}^t$, we require that $p^r = p^t$ *and* $q^r = q^t$; i.e., we require that quantities be equal for the two periods as well as prices.

The above material can be adapted to measuring the *relative similarity of quantities* in place of prices. The incentive to use similarity of relative quantities is as follows: if the period r and t quantity vectors are proportional, then both the Laspeyres, Paasche and Fisher quantity indexes will be equal to this factor of quantity proportionality. In particular, if $q^r = q^t$, then the Laspeyres, Paasche, Fisher and any superlative quantity index will be equal to unity, without requiring p^t and p^r to be equal. Thus when the quantity vectors are proportional, it makes sense to define the price indexes residually using the Product Test. Thus define the following measure of *relative quantity similarity* between the quantity vectors for periods r and t as follows:[201]

(220) $\Delta_{SQ}(p^r,p^t,q^r,q^t) \equiv \Sigma_{n=1}^N [s_{tn} - (p_{tn}q_{rn}/p^t\cdot q^r)]^2 + \Sigma_{n=1}^N [s_{rn} - (p_{rn}q_{tn}/p^r\cdot q^t)]^2$.

If the quantity vectors q^r and q^t are proportional to each other, then it is straightforward to verify that $\Delta_{SQ}(p^r,p^t,q^r,q^t) = 0$. On the other hand, if $\Delta_{SQ}(p^r,p^t,q^r,q^t) = 0$, then one can repeat Cases (i)-(iii) above, with prices and quantities interchanged, to show that q^r and q^t must be proportional to each other. Thus $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ equals 0 if and only if the period r and t quantity vectors are proportional. If the quantity vectors are not proportional, then $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ will be positive. A larger value for $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ indicates a bigger deviation from quantity proportionality. An advantage of the measure of dissimilarity defined by (220) is that it can deal with q_{tn} that are equal to 0.[202]

The new dissimilarity measure $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ can be used in place of $\Delta_{SP}(p^r,p^t,q^r,q^t)$ in order to construct a new pattern of bilateral Fisher price index links,[203] leading to a new series of price indexes, say $P_{SQ}{}^t$ for t = 1,...,T. The advantage in computing this sequence of price indexes is that they will satisfy the following *fixed basket test*: if $q^r = q^t \equiv q$ for $r < t$, then $P_{SQ}{}^t/P_{SQ}{}^r = p^t\cdot q/p^r\cdot q$. Note that this test does not require that $p^t = p^r$. Once the sequence of price indexes $P_{SQ}{}^t$ has been constructed, corresponding quantity levels can be defined as $Q_{SQ}{}^t \equiv p^t\cdot q^t/P_{SQ}{}^t$ for t = 1,...,T. The fixed basket test for price indexes translates into the following *strong identity test* for quantity indexes: if $q^r = q^t \equiv q$ for $r < t$, then $Q_{SQ}{}^t/Q_{SQ}{}^r = 1$. Note that this test does not require that $p^r = p^t$. It can be seen that this is the advantage in using $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ as the dissimilarity measure in place of $\Delta_{SP}(p^r,p^t,q^r,q^t)$: if $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ is used, then the strong identity test for quantities will be satisfied by the resulting quantity indexes, $Q_{SQ}{}^t$. On the other hand if $\Delta_{SP}(p^r,p^t,q^r,q^t)$ is used as the measure of relative price dissimilarity, then the resulting price indexes $P_{SP}{}^t$ will satisfy the strong identity test for prices.

[201] It can be seen that $\Delta_{SQ}(p^r,p^t,q^r,q^t) = \Delta_{SP}(q^r,q^t,p^r,p^t)$; i.e., the role of prices and quantities is interchanged in the above measure of price dissimilarity $\Delta_{SP}(p^r,p^t,q^r,q^t)$.

[202] If one takes the economic approach to index number theory and adopts the reservation price methodology due to Hicks (1940), then 0 prices can be avoided by using reservation prices or approximations to them such as inflation adjusted carry forward or backward prices. However, 0 quantities cannot be avoided so we need measures of price and quantity dissimilarity that can accomodate 0 prices and quantities in a sensible way.

[203] The implicit Fisher price index that is defined residually using the Product Test turns out to be equal to the usual Fisher price index that is defined directly as the geometric mean of the Laspeyres and Paasche price indexes.

It is possible to design a measure that combines relative price dissimilarity with relative quantity dissimilarity such that the resulting dissimilarity measure when used with Fisher price index bilateral links in the usual manner gives rise to a sequence of price indexes (relative to period 1) $P_{SPQ}{}^t$ that will satisfy both the fixed basket test and the strong identity test for prices. Define the following index for *relative price and quantity dissimilarity* between periods r and t, $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$, as follows:[204]

(221) $\Delta_{SPQ}(p^r,p^t,q^r,q^t) \equiv \min \{\Delta_{SP}(p^r,p^t,q^r,q^t), \Delta_{SQ}(p^r,p^t,q^r,q^t)\}$.

Thus if prices are equal to each other for periods r and t, then $\Delta_{SP}(p^r,p^t,q^r,q^t)$ and $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ will both equal 0 and our linking procedure will lead to equal price levels for periods r and t. On the other hand, if quantities are equal to each other for periods r and t, then $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ and $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ will both equal 0 and our linking procedure will lead to equal quantity levels for periods r and t.[205] Denote the price indexes relative to period 1 generated using $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ as the measure of dissimilarity by $P_{SPQ}{}^t$ for t = 1,...,T. Call this method the *SPQ multilateral method*. Thus the similarity linked indexes that are generated using the dissimilarity measure defined by (221) will lead to index levels that satisfy both a strong identity test for prices and a strong identity test for quantities. Thus if prices are identical in the two periods being compared ($p^r = p^t$), then the similarity linked price levels for periods r and t are equal *and* if quantities are identical in the two periods being compared ($q^r = q^t$), then the similarity linked quantity levels for periods r and t are equal. No of the other multilateral methods studied in this chapter have this very strong property. *This property rules out chain drift both in the price and quantity levels*.

Using the scanner data listed in Appendix 1, the new similarity linked price indexes that combine price and quantity similarity linking, $P_{SPQ}{}^t$, were calculated and compared to the price similarity linked price indexes $P_{SP}{}^t$ that were defined in the beginning of this section. For our sample data set, it turned out that predicted share quantity dissimilarity was always greater than the corresponding measure of predicted share price dissimilarity for each pair of observations in our sample. Under these conditions, it can be seen that $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ will equal $\Delta_{SP}(p^r,p^t,q^r,q^t)$ for all periods r and t. Thus the same set of bilateral Fisher index links that were generated using $\Delta_{SP}(p^r,p^t,q^r,q^t)$ were also generated using $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ defined by (221) as the measure of dissimilarity. It turns out that it was always the case that $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ was much bigger than the corresponding $\Delta_{SP}(p^r,p^t,q^r,q^t)$; i.e., in all cases, *relative quantity dissimilarity was much bigger than the corresponding relative price dissimilarity*.[206]

In section A5 of the Appendix, some variations on the multilateral indexes $P_{AL}{}^t$ and $P_{SP}{}^t$ are considered and evaluated using the price and quantity data for our empirical example. The indexes $P_{ALM}{}^t$ and $P_{SPM}{}^t$ use the same tables of dissimilarity measures that were used to define the

[204] This approach that combines measures of relative price dissimilarity with measures of relative quantity dissimilarity is due to Allen and Diewert (1981), Hill (2004) and Hill and Timmer (2006; 277). Hill and Timmer also noted that, usually, the relative price dissimilarity measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$ will be smaller than the relative quantity dissimilarity measure $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ in which case the combined measure $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ reduces to the price measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$. Diewert and Allen (1981) found this to be the case with their empirical example and we find the same to be true for our empirical example in the Appendix.

[205] Thus a strong version of Walsh's multiperiod identity test will hold using this procedure; i.e., if $p^r = p^t$, then the period r and t price levels will coincide and if $q^r = q^t$, then the period r and t quantity levels will coincide. Note that these tests will hold no matter how large the number of observations T is.

[206] Allen and Diewert (1981) and Hill and Timmer (2006) found the same pattern for their empirical examples using their measures of price and quantity dissimilarity.

bilateral links for the indexes P_{AL}^t and P_{SP}^t but instead of generating real time indexes, the new *modified* indexes P_{ALM}^t and P_{SPM}^t use the observations for the first year of data in the sample to construct a *spanning tree* of comparisons; i.e., the Robert Hill (2001) methodology is used to construct the set of bilateral comparisons for all months in the first year such that the resulting set of bilateral comparisons minimizes the sum of the dissimilarity measures for the chosen bilateral links. Once the set of bilateral links for the first year has been determined, subsequent months are linked to previous months in real time. Thus the bilateral links for P_{AL}^t and P_{ALM}^t to the index levels of previous months are the same for all months t beyond the first year. Similar comments apply to P_{SP}^t and P_{SPM}^t. It follows that the longer term trends in P_{AL}^t and P_{ALM}^t will be the same as will the trends in P_{SP}^t and P_{SPM}^t. [207]

The indexes P_{AL}^t, P_{SP}^t, P_{ALM}^t and P_{SPM}^t all use reservation prices for the prices of missing products. These reservation prices were estimated econometrically in an earlier study by Diewert and Feenstra (2017). It is not easy to estimate reservation prices. Moreover, reservation prices rely on the applicability of the economic approach to index number theory and many assumptions are required in order to implement this approach. Thus many statistical agencies will want to avoid the use of estimated reservation prices when constructing their consumer price indexes. As was indicated in the discussion below equation (219), the predicted share measure of relative price dissimilarity $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) is well defined even if the prices for missing products are set equal to zero.[208] As was mentioned earlier in this section, it is possible to use $\Delta_{SP}(p^r,p^t,q^r,q^t)$ as a guide to linking the observations even if the prices of missing products are set equal to 0. We explain how alternative versions of P_{SP}^t and P_{SPM}^t can be produced when the price vectors p^t have 0 components for missing products in period t in the following paragraph.

In order to explain how the alternative version of P_{SP}^t (call it P_{SP}^{t*}), it is first necessary to calculate all possible *maximum overlap bilateral Fisher indexes* for every pair of observations in the sample. Denote the maximum overlap Fisher price index for period t relative to the base period r as $P_{FMO}(p^r,p^t,q^r,q^t)$ for all observations r and t. When calculating $P_{FMO}(p^r,p^t,q^r,q^t)$, the usual inner products $p^r\cdot q^t = \Sigma_{n=1}^N p_{rn}q_{tn}$ that are used to construct the Fisher index between periods r and t are replaced by summations over n where n is restricted to products that are present in both periods r and t. These four restricted inner products can be constructed very efficiently using matrix operations. As noted above, the dissimilarity measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) is well defined even if the prices for missing products are set equal to zero. Set the maximum overlap similarity linked price index P_{SP}^{1*} for period 1 equal to unity; i.e., set $P_{SP}^{1*} \equiv 1$. The period 2 index P_{SP}^{2*} is set equal to $P_{FMO}(p^1,p^2,q^1,q^2)$, the maximum overlap Fisher index linking the period 2 prices to the period 1 prices. Thus $P_{SP}^{2*} \equiv P_{FMO}(p^1,p^2,q^1,q^2)P_{SP}^{1*}$. For period 3, evaluate the dissimilarity indexes $\Delta_{SP}(p^1,p^3,q^1,q^3)$ and $\Delta_{SP}(p^2,p^3,q^2,q^3)$ defined by (218). If $\Delta_{SP}(p^1,p^3,q^1,q^3)$ is the minimum of these two numbers, define $P_{SP}^{3*} \equiv P_{FMO}(p^1,p^3,q^1,q^3)P_{PS}^{1*}$. If $\Delta_{SP}(p^2,p^3,q^2,q^3)$ is the minimum of these two numbers, define $P_{SP}^{3*} \equiv P_{FMO}(p^2,p^3,q^2,q^3)P_{SP}^{2*}$. For period 4, evaluate the dissimilarity indexes $\Delta_{SP}(p^r,p^4,q^r,q^4)$ for r = 1,2,3. Let r° be such that $\Delta_{SP}(p^{r^\circ},p^4,q^{r^\circ},q^4) = \min_r \{\Delta_{SP}(p^r,p^4,q^r,q^4);\ r = 1,2,3\}$.[209] Then define $P_{SP}^{4*} \equiv P_{FMO}(p^{r^\circ},p^4,q^{r^\circ},q^4)P_{SP}^{r^\circ}$. Continue this process in the same manner; i.e., for period t, let r° be such that $\Delta_{SP}(p^{r^\circ},p^t,q^{r^\circ},q^t) = \min_r \{\Delta_{SP}(p^r,p^t,q^r,q^t);\ r = 1,2,...,t-1\}$ and define $P_{SP}^{t*} \equiv P_{FMO}(p^{r^\circ},p^t,q^{r^\circ},q^t)P_{SP}^{r^\circ}$. The procedure for constructing P_{SP}^{t*} is exactly the same as the procedure for constructing P_{SP}^t except that maximum overlap Fisher indexes are

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[207] For our empirical example, P_{AL}^t, P_{SP}^t, P_{ALM}^t and P_{SPM}^t all end up at the same level for the last month in our sample; see Table A.7 and Chart 9 in the Appendix.

[208] This is not the case for the Asymptotic Linear measure of relative price dissimilarity $\Delta_{AL}(p^r,p^t,q^r,q^t)$ defined by (215).

[209] If the minimum occurs at more than one r, choose r^* to be the earliest of these minimizing periods.

used in place of regular Fisher indexes defined over all products in order to implement the "best" set of bilateral links that are used to link all of the observations in the sample up to the current period t.[210]

Recall the definition for the modified set of price levels P_{ALM}^t using the Asypmtotic Linear measure of relative price dissimilarity, which were similar to the P_{AL}^t price levels except that a year of data on prices and quantities was used to form a set of bilateral links that minimizes the sum of the associated dissimilarity measures that link the observations for the first year. The same procedure can be used in the present context where the P_{SP}^{t*} can be replaced by the *Modified Predicted Share indexes*, P_{SPM}^{t*}.[211] For months t that follow after the first "training" year, the bilateral links are the same as the links used to calculate the Predicted Share indexes P_{SP}^{t*}. [212]

The maximum overlap fixed base Fisher indexes, $P_{FMO}(p^1,p^t.q^1.p^t) \equiv P_F^{t*}$, and the GEKS indexes P_{GEKS}^{t*} using maximum overlap Fisher indexes in place of regular Fisher indexes are listed in the Appendix and can be compared to their counterparts P_F^t and P_{GEKS}^t that used reservation prices for the missing products. See Table A.7 in section A5 of the Appendix for a listing of the following indexes: P_{AL}^t, P_{ALM}^t, P_{SP}^t, P_{SPM}^t, P_{SP}^{t*}, P_{SPM}^{t*}, P_{GEKS}^t, P_{GEKS}^{t*}, P_F^t, P_F^{t*}. The final level for these ten indexes after 3 years of data where the level in month 1 was 1.00000 was as follows: 0.92725, 0.92725, 0.92725, 0.92725, 0.92612, 0.92612, 0.94591, 0.94987, 0.95071, 0.95610. Thus the first four similarity linked indexes end up at the same price level, 0.92575, while the predicted share and modified predicted share indexes that used maximum overlap prices, P_{SP}^{t*} and P_{SPM}^{t*}, ended up at the same slightly higher price level, 0.92612. The two GEKS indexes (P_{GEKS}^t used reservation prices while P_{GEKS}^{t*} used maximum overlap Fisher links that did not depend on any imputed prices) ended up about 2 percentage points above the similarity linked indexes. Finally, the fixed base Fisher index that used reservation prices and the fixed base Fisher index that used maximum overlap bilateral links, P_F^t and P_F^{t*}, ended up about 3 percentage points above the similarity linked index levels. These results lead to two important (but tentative) conclusions:

- The similarity linked indexes considered in this section and the previous sections all generate approximately the same results and
- The similarity linked indexes appear to generate lower rates of overall price change than the fixed base Fisher or the GEKS indexes generate.

The first dot point is important one if it is consistent with other empirical investigations. Some statistical agencies may prefer to use inflation adjusted carry forward prices to replace missing prices while other agencies may not wish to use any form of an imputed price in their indexes.

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[210] In addition to using P_{FMO} in place of P_F, the other difference in the two procedures is the use of 0 prices for unavailable products in place of reservation or carry forward prices when evaluating the dissimilarity measures $\Delta_{SP}(p^r,p^t,q^r,q^t)$. Thus the set of optimal bilateral links can change as we move from the P_{SP}^t indexes to their maximum overlap counterpart P_{SP}^{t*} indexes.

[211] Note that we cannot construct P_{AL}^{t*} or P_{ALM}^{t*} in the present context where we have 0 prices for missing products because $\Delta_{AL}(p^r,p^t,q^r,q^t)$ is not well defined when some prices are equal to zero.

[212] It is straightforward to apply the predicted share methodology when we have zero prices and quantities for missing products to quantity indexes. Apply definition (221); i.e., define $\Delta_{SPQ}(p^r,p^t,q^r,q^t) \equiv \min \{\Delta_{SP}(p^r,p^t,q^r,q^t), \Delta_{SQ}(p^r,p^t,q^r,q^t)\}$as our new measure of relative price and quantity dissimilarity where 0 prices and quantities are allowed to appear in the price and quanity vectors. Using this measure of dissimilarity and maximum overlap Fisher price and quantity indexes leads to the price levels P_{SPQ}^{t*}. For our empirical example, it was the case that $\Delta_{SP}(p^r,p^t,q^r,q^t)$ was always less than $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ so the P_{SPQ}^{t*} ended up being equal to the P_{SP}^{t*} for all t.

The results for our empirical example suggest that it may not matter very much which strategy is chosen, provided similarity linking of observations is used.

21. The Axiomatic Approach to Multilateral Price Levels

In this section, we will look at the axiomatic or test properties of the five major multilateral methods studied in previous sections. The multilateral methods are the GEKS, CCDI, GK, WTPD and SPQ (Price and Quantity Similarity Linking) methods. The price levels for period t for the five methods are defined by definitions (69) for $p_{GEKS}{}^t$, (76) for $p_{CCDI}{}^t$, (137) for $p_{GK}{}^t$, (210) for $p_{WTPD}{}^t$ and by (221) for $P_{SPQ}{}^t$.[213] We will look at the properties of these price level functions rather than at the corresponding price indexes.[214] Denote the period t price level function for generic multilateral method M as $p_M{}^t(p^1,...,p^T;q^1,...,q^T)$ for t = 1,...,T. We will follow the example of Dalén (2001) (2017) and Zhang, Johansen and Nygaard (2019) in considering a *dynamic product universe*; i.e., we will allow for new products and disappearing products in the tests that follow. N is the total number of products that are in the aggregate over all T periods. If a product n is not available in period t, we set q_{tn} equal to 0. We will assume that the corresponding price p_{tn} is a positive Hicksian reservation price or a positive inflation adjusted carry forward or backward price. Thus for each period t, the price vector $p^t >> 0_N$ but the corresponding period t quantity vector satisfies only $q^t > 0_N$; i.e., the missing products in period t are assigned 0 values for the corresponding quantities.[215] It proves convenient to define the N by T matrices of prices and quantities as $P \equiv [p^1,...,p^T]$ and $Q \equiv [q^1,...,q^T]$. Thus the p^t and q^t are to be interpreted as column vectors of dimension N in the definitions of the matrices P and Q.

Consider the following nine tests for a system of generic multilateral price levels, $p_M{}^t(P,Q)$:

Test 1: *The strong identity test for prices*. If $p^r = p^t$, then $p_M{}^r(P,Q) = p_M{}^t(P,Q)$. Thus if prices are equal in periods r and t, then the corresponding price levels are equal even if the corresponding quantity vectors q^r and q^t are not equal.

Test 2: *The fixed basket test for prices* or *the strong identity test for quantities*.[216] If $q^r = q^t \equiv q$, then the price index for period t relative to period r is $p_M{}^t(P,Q)/p_M{}^r(P,Q)$ which is equal to $p^t{\cdot}q/p^r{\cdot}q$.[217]

Test 3: *Linear homogeneity test for prices*. Let $r \neq t$ and $\lambda > 0$. Then $p_M{}^t(p^1,...,p^{t-1},\lambda p^t,p^{t+1},...,p^T,Q)/$

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[213] The price and quantity similarity linked price levels $P_{SPQ}{}^t$ have been normalized to equal 1 in period 1. The other four sets of price levels have not been normalized.

[214] For earlier work on the axiomatic properties of multilateral price and quantity indexes, see Diewert (1988) (1999b) and Balk (2008). These earlier studies did not look at the properties of stand alone price level functions.

[215] It is necessary to have strictly positive prices in order to calculate the CCDI price levels. The remaining multilateral methods do not require strictly positive prices for all products and all periods to be well defined but our last test involves imputed prices for missing products. Thus we need to introduce these imputed prices at the outset of our axiomatic framework.

[216] The period t quantity level that matches up with the period t price level is $q_M{}^t(P,Q) \equiv p^t{\cdot}q^t/p_M{}^t(P,Q)$ for t = 1,...,T. Test 2 translates into the *strong identity test for quantity levels*; i.e., if $q^r = q^t$, then $q_M{}^r(P,Q) = q_M{}^t(P,Q)$ even if the price vectors for the two periods p^r and p^t are not equal.

[217] Tests 1 and 2 are essentially versions of Tests 1 and 2 suggested by Zhang, Johansen and Nygaard (2019).

$p_M{}^r(p^1,...,p^{t-1},\lambda p^t,p^{t+1},...,p^T,Q) = \lambda p_M{}^t(P,Q)/p_M{}^r(P,Q)$. Thus if all prices in period t are multiplied by a common scalar factor λ, then the price level of period t relative to the price level of any other period r will increase by the multiplicative factor λ.

Test 4: *Homogeneity test for quantities*. Let $\lambda > 0$. Then $p_M{}^r(P,q^1,...,q^{t-1},\lambda q^t,q^{t+1},...,q^T) = p_M{}^r(P,Q)$ for r = 1,...,T. Thus if all quantities in period t are multiplied by a common scalar factor λ, then the price level of any period r remains unchanged. This property holds for all t = 1,...,T.

Test 5: *Invariance to changes in the units of measurement*. The price level functions $p_M{}^t(P,Q)$ for t = 1,...,T remain unchanged if the N commodities are measured in different units of measurement.

Test 6: *Invariance to changes in the ordering of the commodities*. The price level functions $p_M{}^t(P,Q)$ for t = 1,...,T remain unchanged if the ordering of the N commodities is changed.

Test 7: *Invariance to changes in the ordering of the time periods*. If the T time periods are reordered by some permutation of the first T integers, then the new price level functions are equal to the same permutation of the initial price level functions. This test is considered to be an important one in the context of making cross sectional comparisons of price levels across countries. In the country context, if this test is satisfied, then all countries are treated in a symmetric manner. It is not so clear whether this test is important in the time series context.

Test 8: *Responsiveness to Isolated Products Test*: If a product is available in only one period in the window of T periods, this test asks that the price level functions $p_M{}^t(P,Q)$ *respond to changes* in the prices of these isolated products; i.e., the test asks that the price level functions *are not constant* as the prices for isolated products change. This test is a variation of Test 5 suggested by Zhang, Johansen and Nygaard (2019), which was a bilateral version of this test.[218]

Test 9: *Responsiveness to Changes in Imputed Prices for Missing Products Test*: If there are missing products in one or more periods, then there will be imputed prices for these missing products according to our methodological framework. This test asks that the price level functions $p_M{}^t(P,Q)$ *respond to changes* in these imputed prices; i.e., the test asks that the price level functions *are not constant* as the imputed prices change. This test is essentially an extension of the previous Test 8. This test allows a price level to decline if new products enter the market place during the period and for consumer utility to increase as the number of available products increases. If this test is not satisfied, then the price levels will be subject to *new products bias*.[219] This is an important source of bias in a dynamic product universe.

It can be shown that GEKS and CCDI fail Tests 1 and 2, GK fails 1,4, 8 and 9, WTPD fails 1, 2,[220] 8 and 9 and SPQ fails 7. The above five multilateral methods pass the remaining Tests. Since Test 7 may not be so important in the time series context, it appears that the price and quantity similarity method of linking, the SPQ method, is “best” for the above tests. However, other reasonable tests could be considered in a more systematic exploration of the test approach to

[218] This test was explicitly suggested by Claude Lamboray. Some care is needed in interpreting this test since the test framework assumes that there are imputed prices for the missing products.
[219] On new goods bias, see Boskin, Dulberger, Gordon, Griliches and Jorgenson (1996), Nordhaus (1997), Diewert (1998) and the references in section 14 of Diewert (2021a).
[220] The Weighted Time Product Dummy price levels fail Test 2 if definitions (205) are used to define the period t price levels. This is the option that statistical agencies are using at present. However The WTPD price levels P^{t**} and the corresponding quantity levels Q^{t**} defined by (206) will satisfy Test 2. If all errors are equal to 0, equations (205) and (206) will generate the same estimated price and quantity levels.

multilateral comparisons so our endorsement of the SPQ method is tentative at this point. Furthermore, the method needs to be tested on alternative data sets to see if "reasonable" indexes are generated by the method.

22. Summary of Results

Some of the more important results in each section of the Chapter will be summarized here.

- If there are divergent trends in product prices, the Dutot index is likely to have an upward bias relative to the Jevons index; see section 2.
- The Carli index has an upward bias relative to the Jevons index (unless all prices move proportionally over time in which case both indexes will capture the common trend). The same result holds for the weighted Carli (or Young) index relative to the corresponding weighted Jevons index; see section 3.
- The useful relationship (41) implies that the Fisher index P_F^t will be slightly less than the corresponding fixed base Törnqvist index P_T^t, provided that the products in scope for the index are highly substitutable and there are divergent trends in prices; see section 4. Under these circumstances, the following inequalities between the Paasche, Geometric Paasche, Törnqvist, Geometric Laspeyres and Laspeyres indexes are likely to hold: $P_P^t < P_{GP}^t < P_T^t < P_{GL}^t < P_L^t$.
- The covariance identity (48) provides an exact relationship between the Jevons and Törnqvist indexes. Some conditions for equality and for divergence between these two indexes are provided at the end of section 5.
- In section 6, a geometric index that uses annual expenditure sales of a previous year as weights, $P_{J\alpha}^t$, is defined and compared to the Törnqvist index, P_T^t. Equation (62) provides an exact covariance decomposition of the difference between these two indexes. If the products are highly substitutable and there are divergent trends in prices, then it is likely that $P_T^t < P_{J\alpha}^t$.
- Section 7 derives an exact relationship (65) between the fixed base Törnqvist index, P_T^t, and its chained counterpart, P_{TCh}^t. This identity is used to show that it is likely that the chained index will "drift" below its fixed base counterpart if the products in scope are highly substitutable and prices are frequently heavily discounted. However, a numerical example shows that if quantities are slow to adjust to the lower prices, then upward chain drift can occur.
- Section 8 introduces two multilateral indexes, P_{GEKS}^t and P_{CCDI}^t. The exact identity (78) for the difference between P_{CCDI}^t and P_T^t is derived. This identity and the fact that P_F^t usually closely approximates P_T^t lead to the conclusion (79) that typically, P_F^t, P_T^t, P_{GEKS}^t and P_{CCDI}^t will approximate each other fairly closely.
- Section 9 introduces the Unit Value price index P_{UV}^t and shows that if there are divergent trends in prices and the products are highly substitutable, it is likely that $P_{UV}^t < P_F^t$. However, this conclusion does not necessarily hold if there are missing products in period 1. Section 10 derives similar results for the Quality Adjusted Unit Value index, $P_{UV\alpha}^t$.
- Section 11 looks at the relationship of the Lowe index, P_{Lo}^t, with other indexes. The Lowe index uses the quantities in a base year as weights in a fixed basket type index for months that follow the base year. In using annual weights of a previous year, this index is similar in spirit to the geometric index $P_{J\alpha}^t$ that was analyzed in section 6. The covariance type identities (128) and (131) are used to suggest that it is likely that the Lowe index lies between the fixed base Paasche and Laspeyres indexes; i.e., it is likely that $P_P^t < P_{Lo}^t < P_L^t$. The identity (134) is used to suggest that the Lowe index is likely to

have an upward bias relative to the fixed base Fisher index; i.e., it is likely that $P_F^t < P_{Lo}^t$. However, if there are missing products in the base year, then these inequalities do not necessarily hold.

- Section 12 looks at an additional multilateral index, the Geary Khamis index, P_{GK}^t and shows that P_{GK}^t can be interpreted as a quality adjusted unit value index and hence using the analysis in section 10, it is likely that the Geary Khamis price index has a downward bias relative to the Fisher index; i.e., it is likely that $P_{GK}^t < P_F^t$. However, if there are missing products in the first month of the sample, the above inequality will not necessarily hold.
- Sections 13-16 look at special cases of Weighted Time Product Dummy indexes, P_{WTPD}^t. These sections show how different forms of weighting can generate very different indexes. Section 17 finally deals with the general case where there are T periods and missing products. The exact identity (214) is used to show that it is likely that P_{WTPD}^t is less than the corresponding fixed base Törnqvist Theil index, P_T^t, provided that the products are highly substitutable and there are no missing products in period 1. However, if there are missing products in period 1, the inequality can be reversed.
- It turns out that the following price indexes are not affected by reservation prices: the unit value price indexes P_{UV}^t and $P_{UV\alpha}^t$, the Geary Khamis indexes P_{GK}^t, and the Weighted Time Product Dummy indexes P_{WTPD}^t. Thus these indexes are not consistent with the economic approach to dealing with the problems associated with new and disappearing products and services.
- The final multilateral indexes were introduced in sections 18-20. These indexes use bilateral Fisher price indexes to link the price and quantity data of the current period to a prior period. The prior period that is chosen minimizes a measure of relative price (or quantity) dissimilarity. Two main measures of relative price dissimilarity were studied: the AL or Asymptotic Linear measure $\Delta_{AL}(p^r,p^t,q^r,q^t)$ defined by (215) and the SP or Predicted Share measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218). The role of prices and quantities can be interchanged in order to define the Predicted Share measure $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ of relative quantity dissimilarity which can also be used to generate a set of bilateral Fisher price index links. Finally, the minimum of the $\Delta_{SP}(p^r,p^t,q^r,q^t)$ and $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ measures can be taken to define the $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ measure of relative price and quantity dissimilarity; see definition (221). When observations are linked using this dissimilarity measure, the resulting price indexes satisfy both the identity test for prices and the corresponding identity price for quantities. Thus the SPQ method explained in section 20 has attractive axiomatic properties as is explained in section 21. For our empirical example, relative quantity dissimilarity was always greater than relative price dissimilarity so the SP and SPQ price indexes were always identical.
- For our empirical example, the similarity linked price indexes P_{AL}^t and $P_{SP}^t = P_{SPQ}^t$ ended up about 2 percentage points below P_{GEKS}^t and P_{CCDI}^t which in turn finished about 1 percentage point below P_F^t and P_T^t and finally P_{GK}^t and P_{WTPD}^t finished about 1 percentage point above P_F^t and P_T^t; see Table A.6 and Chart 8 in the Appendix. All of these indexes captured the trend in product prices quite well. More research is required in order to determine whether these differences are significant and occur in other examples.
- It is difficult to calculate reservation prices using econometric techniques. Thus section 19 looked at methods for replacing reservation prices by inflation adjusted carry forward and backward prices which are much easier to calculate.
- For our empirical example, the replacement of the reservation prices by inflation adjusted carried forward or backward prices did not make much difference to the

multilateral indexes.[221] If the products in scope are highly substitutable for each other, then we expect that this invariance result will hold (approximately). However, if products with new characteristics are introduced, then we expect that the replacement of econometrically estimated reservation prices by carried forward and backward prices would probably lead to an index that has an upward bias.

- Finally, in section 20, we introduced some similarity linked Fisher price indexes that did not require imputations for missing prices. These indexes used the Predicted Share measure of relative price dissimilarity which is well defined even if the prices of missing products are set equal to 0. The Fisher indexes that link pairs of observations that have the lowest measures of dissimilarity are maximum overlap Fisher indexes. For our empirical example, it turned out that these indexes were very close to their counterparts that used reservation prices for the missing prices. These no imputation indexes (denoted by P_{SP}^{t*} and P_{SPM}^{t*}) were calculated for our data set and listed in Table A.7 and plotted on Chart 9 in the Appendix.

Conceptually, the Price and Quantity Similarity linked indexes P_{SPQ}^{t} seem to be the most attractive solution for solving the chain drift problem since the strong identity tests for both prices and quantities will always be satisfied using this multilateral method.

The data used for the empirically constructed indexes are listed in the Appendix so that the listed indexes can be replicated and so that alternative solutions to the chain drift problem can be tested out by other statisticians and economists.

23. Conclusion

It is evident that there is no easy solution to the chain drift problem. The previous *Consumer Price Index* Manual tended to use the economic approach to index number theory as a guide to choosing between alternative index number formula; i.e., the *Manual* tended to recommend the use of a superlative index number formula as a target index. However, the existence of deeply discounted prices and the appearance and disappearance of products often lead to a substantial chain drift problem. Some of the difficulties stem from the fact that the *economic approach to index number theory* that dates back to Konüs (1924), Konüs and Byushgens (1926) and Diewert (1976) suffers from the following problems:

- The theory assumes that all purchased goods and services are consumed in the period under consideration. But in reality, when a good goes on sale at a deeply discounted price, the quantity purchased will not necessarily be consumed in the current period. If the good can be stored, it will decrease demand for the product in the subsequent period. The traditional economic approach to index number theory does not take the storage problem into account.
- Preferences over goods and services are assumed to be complete. In reality, consumers may not be aware of many new (and old) products; i.e., knowledge about products may be subject to a diffusion process.
- Our approach to the treatment of new and disappearing products uses the reservation price methodology due to Hicks (1940), which simply assumes that latent preferences for new products exist in the period before their introduction to the marketplace. Thus the consumer is assumed to have unchanging preferences over all periods. Before a new product appears, the quantity of the product is set equal to 0 in the consumer's

[221] See Table A.8 in the Appendix.

utility function. In reality, a new product may change the consumer's utility function. This makes the estimation of reservation prices very difficult if not impossible.
- Preferences are assumed to be the same across consumers so that they can be represented by a common linearly homogeneous utility function. Moreover, the preferences do not change over time. All of these assumptions are suspect.

In view of the fact that the assumptions of the economic approach to index number theory will not be satisfied precisely in the real world, we cannot rely entirely on this approach to guide advice to statistical agencies on how to deal with the chain drift problem. Thus it would be useful to develop the test approach to multilateral index number theory in more detail.

So what exactly should statistical agencies do to deal with the chain drift problem when price and quantity are available for a stratum of the CPI? At our current state of knowledge, it seems that the following methods are acceptable:

- Rolling window GEKS and CCDI. Probably the "safest" method of linking the results of one window to the previous window is to use the mean method suggested by Ivancic, Diewert and Fox (2009) and Diewert and Fox (2017). This is the method used by the Australian Bureau of Statistics (2016). However, in the case of seasonal products that are not present in all periods of the year, rolling window GEKS and CCDI can be problematic and similarity linking is preferred.
- Bilateral linking based on Price (and Quantity) Similarity. This method seems very promising. It can be adapted to work in situations where there are imputed prices for missing products or in situations where imputed prices are not allowed. The resulting indexes are guaranteed to be free of chain drift.

If only price information is available and there are no missing prices, then the Jevons index is the best alternative to use (at least from the perspective of the test approach to index number theory).

If only price information is available and there are missing prices for some products for some periods, then the time product dummy method is probably the best index to use. This method reduces to the Jevons index if there are no missing prices.[222]

We conclude this section by noting some priorities for future research:

- We need more studies on Price Similarity Linking, particularly in the context of strongly seasonal commodities.
- What is the "optimal" length of the time period for a CPI? Should statistical agencies produce weekly or daily CPIs in addition to monthly CPIs?[223]
- There is a conceptual problem in using retail outlet prices to construct a consumer price index, since tourists and governments also purchase consumer goods. It would be preferable to use the purchase data of domestic households in order to construct a CPI for residents of the country so that the welfare of residents in the country could be calculated.

[222] However, in situations where there are many missing prices, it may be preferable to adapt the predicted share similarity linking methodology to the case where only price information is available. We will explore this possibility in another chapter which deals with strongly seasonal products.

[223] The problem with making the time period shorter is that the number of price matches will decline, leading to the need for more imputations.. Also, the shorter the period, the more variance there will be in the unit value prices and the associated quantities, leading to indexes that will also have high variances. Thus the shorter the period, the less accurate the resulting indexes will be.

However, if we focus on individual households, the matching problems are substantial due to the infrequency of purchases of storable commodities. Thus it will be necessary to aggregate over demographically and locationally similar households in order to calculate indexes that minimize the number of imputations. In the perhaps distant future, it will become possible in a cashless society to utilize the data of banks and credit card companies to track the universe of purchases of individual households and thus to construct more accurate consumer price indexes. However, this development will depend on whether credit and debit card consumer transactions are also coded for the type of purchase.

- A final problem that may require some research is how to combine elementary indexes that are constructed using scanner data with elementary indexes that use web scraped data on prices or data on prices collected by employees of the statistical agency. This does not seem to be a big conceptual problem: for strata that use scanner data, we end up with an aggregate price and quantity level for each stratum. For strata that use web scraped data or collector data, we end up with a stratum elementary price level for each period and consumer expenditure survey information will generate an estimated value of consumer expenditures for the stratum in question so the corresponding stratum quantity can be defined as expenditure divided by the elementary price level. Thus the resulting CPI will be of uneven quality (because the expenditure estimates will not be current for the web scraped categories) but it will probably be of better quality than a traditional price collector generated CPI. However, as mentioned above, another problem is that the scanner data will apply not only to expenditures of domestic households but also to tourists and governments. Thus there is a need for more research on this topic of combining methods of price collection.

Appendix: Data Listing and Index Number Tables and Charts

A1. Listing of the Data

Here is a listing of the "monthly" quantities sold of 19 varieties of frozen juice (mostly orange juice) from Dominick's Store 5 in the Greater Chicago area, where a "month" consists of sales for four consecutive weeks. These data are available from the Booth School of Business at the University of Chicago (2013).[224] The weekly unit value price and quantity sold data were converted into "monthly" unit value prices and quantities.[225] Finally, the original data came in units where the package size was not standardized. We rescaled the price and quantity data into prices per ounce. Thus the quantity data are equal to the "monthly" ounces sold for each product.

Table A1: "Monthly" Unit Value Prices for 19 Frozen Juice Products

t	p_1^t	p_2^t	p_3^t	p_4^t	p_5^t	p_6^t	p_7^t	p_8^t	p_9^t
1	**0.122500**	***0.145108***	**0.147652**	***0.148593***	**0.146818**	**0.146875**	**0.147623**	**0.080199**	**0.062944**
2	**0.118682**	***0.127820***	**0.116391**	***0.128153***	**0.117901**	**0.146875**	**0.128833**	**0.090833**	**0.069167**
3	**0.120521**	***0.128608***	**0.129345**	***0.148180***	**0.131117**	**0.143750**	**0.136775**	**0.090833**	**0.048803**
4	**0.126667**	***0.128968***	**0.114604**	***0.115604***	**0.116703**	**0.143750**	**0.114942**	**0.088523**	**0.055842**
5	**0.126667**	***0.130737***	**0.140833**	***0.141108***	**0.140833**	**0.143304**	**0.140833**	**0.090833**	**0.051730**

[224] The Office for National Statistics (2020) also used the Dominick's data in order to compare many of the same indexes that are compared in this Appendix.
[225] In practice, statistical agencies will not be able to produce indexes for 13 months in a year. There are at least two possible solutions to the problem of aggregating weekly data into monthly data: (i) aggregate the data for the first three weeks in a month or (ii) split the weekly data that spans two consecutive months into imputed data for each month.

6	0.120473	*0.113822*	0.157119	*0.151296*	0.156845	0.161844	0.156342	0.090833	0.049167
7	0.164607	*0.144385*	0.154551	*0.158485*	0.156607	0.171875	0.152769	0.084503	0.069167
8	0.142004	*0.160519*	0.174167	*0.179951*	0.174167	0.171341	0.163333	0.089813	0.069167
9	0.135828	0.165833	0.154795	0.159043	0.151628	0.171483	0.160960	0.089970	0.067406
10	0.129208	0.130126	0.153415	0.158167	0.152108	0.171875	0.158225	0.078906	0.067897
11	0.165833	0.165833	0.139690	0.136830	0.134743	0.171875	0.136685	0.079573	0.058841
12	0.165833	0.165833	0.174167	0.174167	0.174167	0.171875	0.174167	0.081902	0.079241
13	0.113739	0.116474	0.155685	0.149942	0.145633	0.171875	0.146875	0.074167	0.048880
14	0.120882	0.125608	0.141602	0.147428	0.142664	0.163750	0.144911	0.090833	0.080000
15	0.165833	0.165833	0.147067	0.143214	0.144306	0.155625	0.147546	0.088410	0.080000
16	0.122603	0.118536	0.135878	0.137359	0.137480	0.155625	0.138146	0.084489	0.080000
17	0.104991	0.104659	0.112497	0.113487	0.110532	0.141250	0.113552	0.082500	0.067104
18	0.088056	0.091133	0.118440	0.120331	0.117468	0.141250	0.124687	0.085000	0.065664
19	0.096637	0.097358	0.141667	0.141667	0.141667	0.141250	0.141667	0.082500	0.080000
20	0.085845	0.090193	0.120354	0.122168	0.113110	0.136250	0.124418	0.085874	0.051003
21	0.094009	0.100208	0.121135	0.122500	0.121497	0.125652	0.121955	0.090833	0.085282
22	0.084371	0.087263	0.120310	0.123833	0.118067	0.125492	0.124167	0.085898	0.063411
23	0.123333	0.123333	0.116412	0.118860	0.113085	0.126250	0.118237	0.085891	0.049167
24	0.078747	0.081153	0.125833	0.125833	0.125833	0.126250	0.125833	0.090833	0.049167
25	0.088284	0.092363	0.098703	0.098279	0.088839	0.126250	0.100640	0.090833	0.049167
26	0.123333	0.123333	0.092725	0.096323	0.095115	0.126250	0.095030	0.090833	0.049167
27	0.101331	0.102442	0.125833	0.125833	0.125833	0.126250	0.125833	0.090833	0.049167
28	0.101450	0.108416	0.092500	0.097740	0.091025	0.126250	0.096140	0.054115	0.049167
29	0.123333	0.123333	0.118986	0.119509	0.115603	0.126250	0.118343	0.096922	0.049167
30	0.094038	0.095444	0.109096	0.113827	0.106760	0.126250	0.113163	0.089697	0.049167
31	0.130179	0.130000	0.110257	0.115028	0.112113	0.134106	0.110579	0.093702	0.049167
32	0.103027	0.103299	0.149167	0.149167	0.149167	0.149375	0.149167	0.098333	0.049167
33	0.148333	0.148333	0.089746	0.097110	0.091357	0.149375	0.094347	0.098333	0.049167
34	0.115247	0.114789	0.123151	0.123892	0.127177	0.149375	0.125362	0.094394	0.049167
35	0.118090	0.120981	0.121191	0.129477	0.128180	0.149375	0.132934	0.096927	0.049167
36	0.132585	0.131547	0.129430	0.128314	0.121833	0.134375	0.128874	0.070481	0.049167
37	0.114056	0.115491	0.138214	0.140090	0.139116	0.146822	0.142770	0.077785	0.053864
38	0.142500	0.142500	0.134677	0.133351	0.133216	0.148125	0.132873	0.108333	0.054167
39	0.121692	0.123274	0.095236	0.102652	0.093365	0.148125	0.101343	0.090180	0.054167

t	p_{10}^t	p_{11}^t	p_{12}^t	p_{13}^t	p_{14}^t	p_{15}^t	p_{16}^t	p_{17}^t	p_{18}^t	p_{19}^t
1	0.062944	0.075795	0.080625	0.087684	0.109375	0.113333	0.149167	0.122097	0.149167	0.124492
2	0.069167	0.082500	0.080625	0.112500	0.109375	0.113333	0.119996	0.109861	0.130311	0.117645
3	0.043997	0.082500	0.078546	0.106468	0.100703	0.110264	0.134380	0.109551	0.131890	0.114933
4	0.055705	0.082500	0.080625	0.099167	0.099375	0.111667	0.109005	0.106843	0.108611	0.118333
5	0.051687	0.071670	0.080625	0.094517	0.099375	0.111667	0.105168	0.106839	0.105055	0.076942
6	0.049167	0.078215	0.080625	0.115352	0.114909	0.130149	0.099128	0.134309	0.118647	0.088949
7	0.069167	0.069945	0.080625	0.124167	0.118125	0.131667	0.102524	0.128471	0.102073	0.160833
8	0.069167	0.082500	0.080625	0.107381	0.121513	0.138184	0.164245	0.141978	0.164162	0.136105
9	0.067401	0.082500	0.074375	0.112463	0.128125	0.141667	0.163333	0.153258	0.163333	0.118979
10	0.067688	0.082500	*0.100545*	0.132500	0.128125	0.141667	0.133711	0.152461	0.133806	0.118439
11	0.060008	0.082500	0.080625	0.120362	0.134151	0.144890	0.163333	0.151033	0.163333	0.120424
12	0.079325	0.071867	0.080625	0.093144	0.136875	0.148333	0.144032	0.148107	0.146491	0.160833
13	0.064028	0.069934	0.067280	0.118009	0.136875	0.148333	0.163333	0.143125	0.163333	0.131144
14	0.080000	0.078491	0.075211	0.131851	0.130342	0.143013	0.123414	0.152937	0.130223	0.122899
15	0.080000	0.082500	0.080625	0.093389	0.128125	0.141667	0.117955	0.147024	0.119786	0.128929
16	0.080000	0.086689	0.080625	0.100592	0.128125	0.141667	0.114940	0.143125	0.126599	0.124620
17	0.065670	0.088333	0.072941	0.115559	0.110426	0.139379	0.107709	0.143125	0.109987	0.145556
18	0.064111	0.091286	0.069866	0.088224	0.105625	0.105529	0.089141	0.130110	0.095463	0.140000
19	0.080000	0.094167	0.088125	0.080392	0.105625	0.131667	0.086086	0.118125	0.091020	0.109424
20	0.048613	0.094167	*0.096177*	0.080643	0.105625	0.131667	0.125000	0.114706	0.125000	0.110921
21	0.085114	0.080262	*0.064774*	0.080245	0.099375	0.125000	0.104513	0.114795	0.104228	0.134014
22	0.062852	0.086115	*0.083132*	0.087551	0.101493	0.127366	0.086484	0.118125	0.088325	0.126667
23	0.049167	0.095833	0.090625	0.089110	0.099375	0.125000	0.086263	0.118125	0.095750	0.100780
24	0.049167	0.095833	0.090625	0.090167	0.099375	0.125000	0.111859	0.114330	0.112296	0.118333
25	0.049167	0.095833	0.090625	0.072861	0.099375	0.125000	0.125000	0.113823	0.125000	0.084817
26	0.049167	0.095833	0.090625	0.086226	0.099375	0.125000	0.086088	0.114190	0.091864	0.118333

27	0.049167	0.077500	0.076875	0.081764	0.099375	0.125000	0.113412	0.114231	0.113241	0.110346
28	0.049167	0.077500	0.076875	0.104167	0.099375	0.125000	0.085803	0.118125	0.086154	0.084604
29	0.049167	0.077500	0.076875	0.086713	0.099375	0.125000	0.087410	0.118125	0.086196	0.085034
30	0.049167	0.077500	0.076875	0.104167	0.099375	0.125000	0.084953	0.114826	0.085156	0.083921
31	0.049167	0.077500	0.076875	0.095613	0.099375	0.125000	0.087372	0.125809	0.087775	0.088304
32	0.049167	0.077500	0.076875	0.112500	0.099375	0.067046	0.091827	0.143125	0.088937	0.103519
33	0.049167	0.077500	0.076875	0.104721	0.099375	0.125000	0.131399	0.143125	0.130253	0.127588
34	0.049167	0.077500	0.076875	0.088935	0.099375	0.125000	0.123037	0.143125	0.123573	0.132500
35	0.049167	0.077500	0.076875	0.112500	0.099375	0.125000	0.125832	0.137837	0.125681	0.112286
36	0.049167	0.077500	0.076875	0.089456	0.099375	0.125000	0.139240	0.141242	0.144390	0.127323
37	0.053865	0.084549	0.083343	0.107198	0.119368	0.151719	0.146126	0.154886	0.146332	0.120616
38	0.054167	0.085000	0.084375	0.127500	0.123125	0.156667	0.129577	0.138823	0.130850	0.114177
39	0.054167	0.085000	0.084375	0.102403	0.123125	0.156667	0.115965	0.149219	0.114947	0.136667

The actual prices p_2^t and p_4^t are not available for t =1,2,...,8 since products 2 and 4 were not sold during these months. However, in the above Table, we filled in these missing prices with the imputed reservation prices that were estimated by Diewert and Feenstra (2017). Similarly, p_{12}^t was missing for months t = 12, 20, 21 and 22 and again, we replaced these missing prices with the corresponding estimated imputed reservation prices in Table A1. The imputed prices appear in italics in the above Table.

Table A2: "Monthly" Quantities Sold for 19 Frozen Juice Products

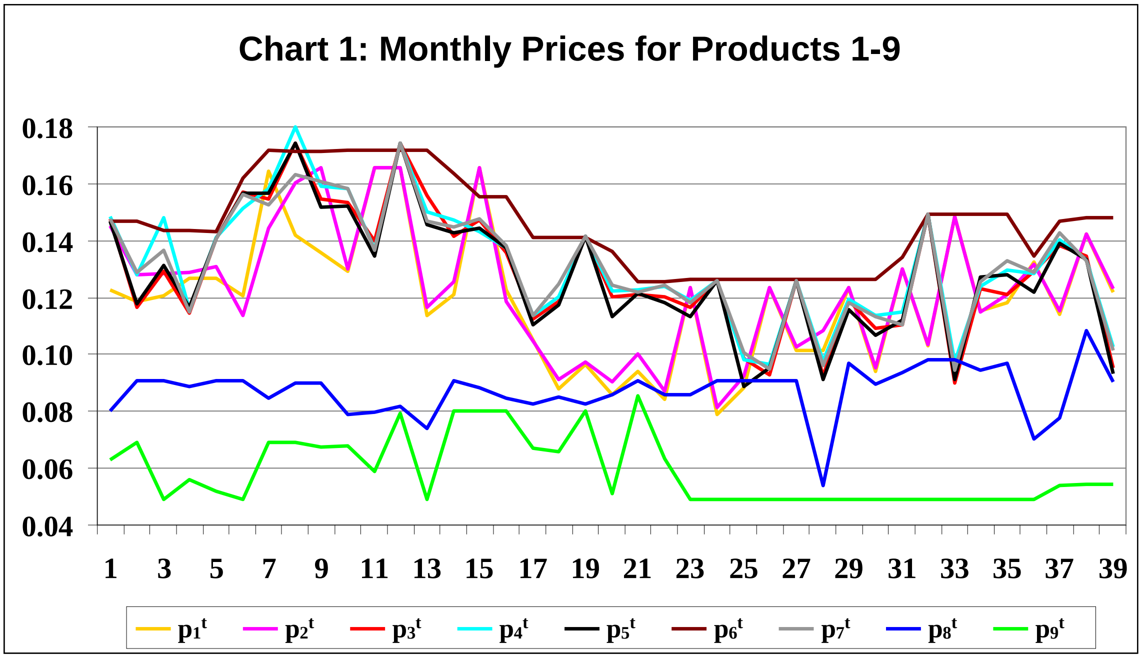
t	q_1^t	q_2^t	q_3^t	q_4^t	q_5^t	q_6^t	q_7^t	q_8^t	q_9^t
1	1704	0.000	792	0.000	4428	1360	1296	1956	1080
2	3960	0.000	3588	0.000	19344	3568	3600	2532	2052
3	5436	0.000	1680	0.000	8100	3296	2760	3000	1896
4	1584	0.000	5532	0.000	21744	3360	5160	3420	2328
5	1044	0.000	1284	0.000	5880	3360	1896	3072	1908
6	8148	0.000	1260	0.000	7860	2608	2184	3000	2040
7	636	0.000	3120	0.000	9516	2848	2784	3444	1620
8	1692	0.000	1200	0.000	4116	1872	1380	2088	1848
9	5304	1476.000	2292	1295.999	7596	2448	1740	2016	3180
10	6288	2867.993	2448	1500.000	6528	2064	2208	3840	4680
11	408	228.000	2448	2147.994	9852	2096	2700	5124	12168
12	624	384.000	948	1020.000	2916	1872	1068	2508	4032
13	6732	2964.005	1488	2064.003	8376	2224	2400	4080	8928
14	6180	3192.007	2472	2244.006	7920	1920	2256	1728	1836
15	1044	672.000	1572	1932.002	2880	1744	1728	1692	1116
16	3900	1332.002	1560	2339.997	4464	2416	2028	2112	1260
17	5328	1847.999	3528	3972.008	13524	2336	3252	2628	1524
18	7056	2100.000	2436	2748.007	6828	2544	1980	3000	1596
19	5712	3167.988	1464	1872.000	2100	2080	1572	3384	1020
20	9960	3311.996	2376	2172.003	8028	2112	1788	2460	3708
21	7368	2496.000	1992	1872.000	3708	1840	1980	1692	2232
22	9168	4835.983	2064	1980.000	10476	1504	2880	2472	7020
23	7068	660.000	1728	1955.999	6972	1888	2172	2448	12120
24	11856	5604.017	972	1464.000	2136	1296	1536	3780	7584
25	7116	2831.994	2760	2207.996	12468	1776	2580	2880	11220
26	660	504.000	3552	3755.995	17808	1296	5580	4956	7428
27	4824	3276.011	1356	1452.000	2388	1824	1524	1548	10188
28	3684	971.998	4680	2832.003	11712	1712	4308	4284	1140
29	684	1152.000	1884	2015.996	9252	1680	3144	1020	1392
30	5112	3467.996	2256	2291.994	9060	1936	2172	1452	2532
31	672	840.000	4788	2951.990	9396	1856	4644	1764	1260
32	7344	5843.997	1320	1128.000	2664	1744	1560	1548	1416
33	480	504.000	6624	5639.996	13368	1824	6888	1800	1440
34	4104	3036.001	2124	3180.009	5088	1568	2820	1668	1884
35	2688	1583.997	2220	2760.008	5244	1344	2532	1920	4956

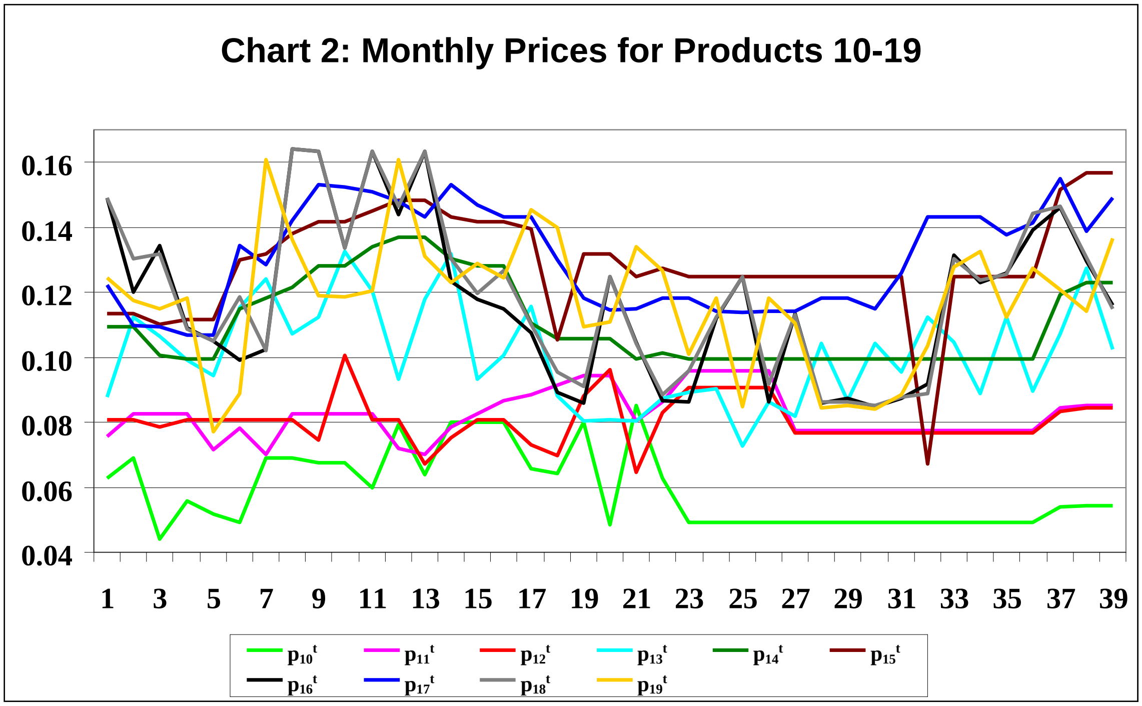
36	936	612.001	1824	2567.994	6684	1552	2772	4740	7644
37	4140	2268.001	1932	1559.997	4740	1520	2076	1752	6336
38	912	264.000	1860	2844.002	4260	1808	2064	1452	2952
39	1068	960.001	4356	2903.996	11052	1776	4356	2220	2772

t	q_{10}^t	q_{11}^t	q_{12}^t	q_{13}^t	q_{14}^t	q_{15}^t	q_{16}^t	q_{17}^t	q_{18}^t	q_{19}^t
1	540	2088	1744.000	30972	3728	792	1512	1712	600	2460
2	1308	4212	3824.000	11796	6480	2712	12720	3312	2376	1788
3	1416	3900	4848.010	18708	10064	2652	4116	3184	1476	3756
4	1716	3156	5152.000	19656	10352	2472	15420	3120	3888	900
5	1452	6168	3360.000	42624	7360	1590	9228	2800	5652	13560
6	1068	5088	3296.000	10380	7712	1884	12012	1808	3348	7824
7	1116	6372	3712.000	11772	7920	1680	29592	3296	11712	708
8	1296	3684	3216.000	21024	5856	1206	11184	1744	4344	6036
9	2220	4512	3024.000	24420	5856	1398	2040	1648	1176	7896
10	4152	4572	0.000	8328	6384	1740	9168	1296	2832	9120
11	9732	3432	3360.000	18372	5808	1638	2412	1568	972	7176
12	3024	6132	1792.000	48648	4672	1770	7512	2208	2052	3564
13	2160	6828	6271.998	15960	4736	1662	1740	2896	1176	3216
14	1356	5088	2991.997	9432	5872	1902	4968	1488	2064	6420
15	1188	4656	2976.000	33936	3872	1452	9060	1744	2712	3876
16	816	3108	4784.000	23772	6272	1578	8496	2832	1488	4128
17	696	3252	4879.997	10656	7648	1836	9000	2704	2292	648
18	720	2940	4848.021	26604	6448	4086	14592	1552	3108	732
19	624	4320	2480.000	27192	4944	1140	19056	1808	5088	5676
20	3288	2784	0.000	23796	5120	1284	2196	2896	1260	3876
21	1848	12324	0.000	25824	5248	1140	8640	1952	2940	588
22	4824	6468	0.000	18168	3872	930	15360	1520	4728	276
23	10092	3708	1744.000	14592	4336	870	14232	1504	2040	1128
24	6372	3264	2016.000	16548	4608	858	6696	1792	2496	792
25	7284	3480	2032.000	38880	4064	750	1836	1232	636	7608
26	6588	3768	2208.000	14724	3760	768	9096	1296	4248	480
27	2832	4692	2592.000	31512	5344	930	5796	2080	5244	1416
28	900	3180	2624.000	8172	5776	810	13896	1328	7536	6744
29	1128	3948	2608.000	19440	5792	954	12360	1552	5796	7296
30	1284	5232	2960.000	6552	6320	924	13932	2304	8064	14520
31	864	5928	3280.000	16896	5888	852	14340	2064	8412	3768
32	948	5784	2496.000	5880	5088	15132	14496	1600	10440	4044
33	708	5232	2704.000	15180	4800	618	4812	976	3204	1812
34	1152	4692	2736.000	25344	5648	600	6552	1360	3876	1344
35	4248	4668	2800.000	8580	5488	498	28104	1872	11292	4152
36	6492	4872	2256.000	30276	5504	510	4080	1328	3768	1860
37	5976	3396	1743.995	8208	2832	384	1092	528	1284	2028
38	1812	3660	2416.000	4392	4144	534	4752	1504	2436	4980
39	2844	3852	1888.000	16704	3488	708	6180	1600	4236	804

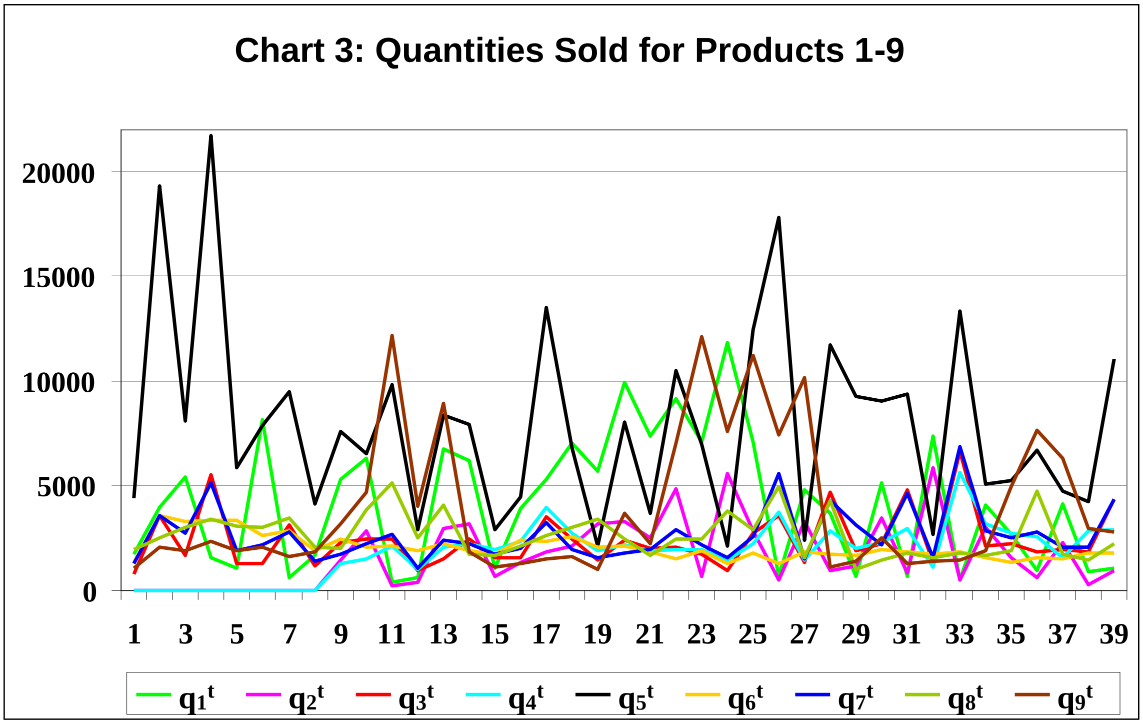
It can be seen that there were no sales of Products 2 and 4 for months 1-8 and there were no sales of Product 12 in month 10 and in months 20-22.

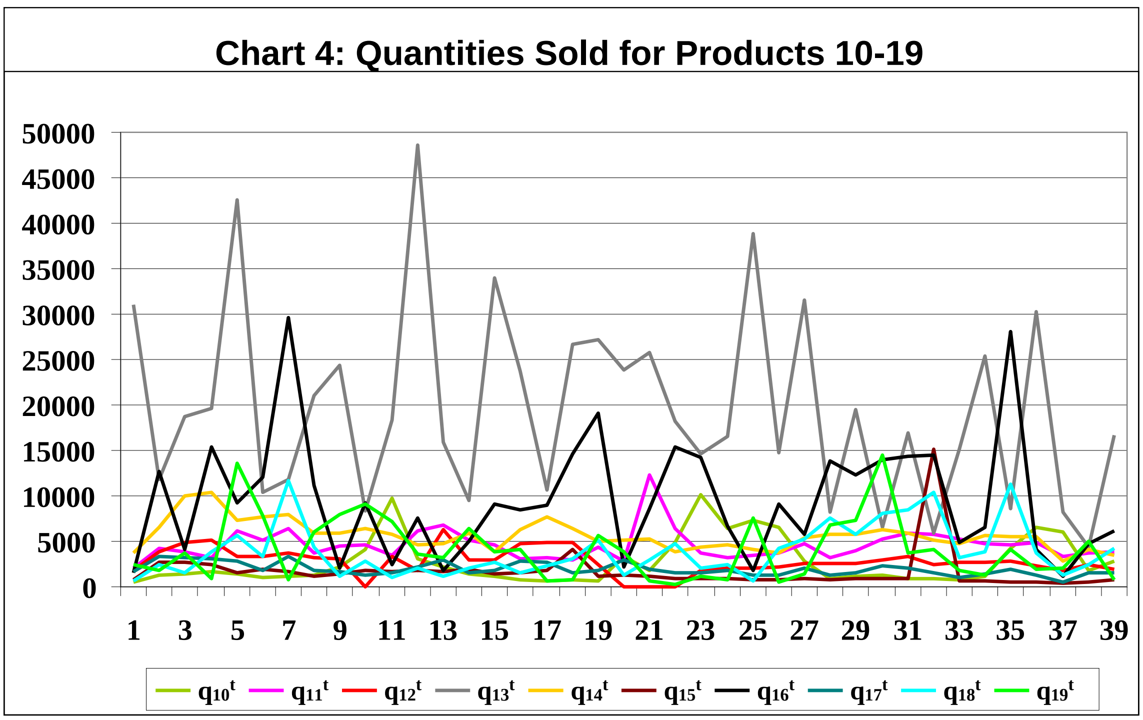
Charts that plot the data in the above tables follow below.





It can be seen that there is a considerable amount of variability in these per ounce monthly unit value prices for frozen juice products. There are also differences in the average level of the prices of these 19 products. These differences can be interpreted as quality differences.





It can be seen that the quantity volatility of the products is much bigger than the volatility in prices.

A2: Unweighted Price Indexes

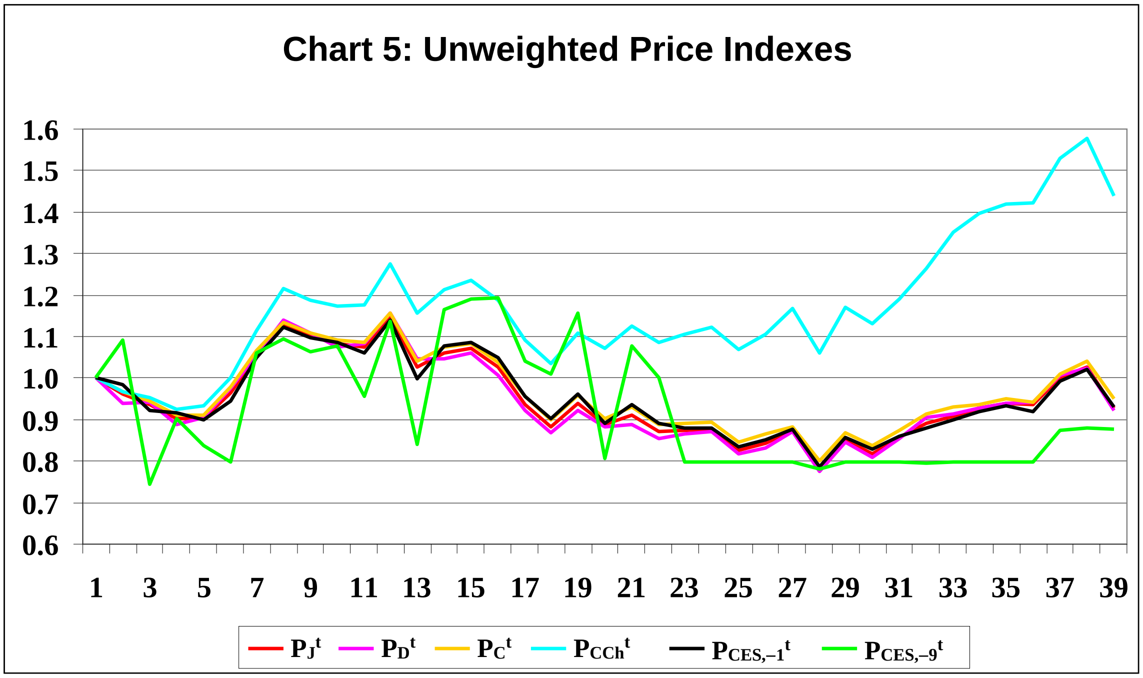
In this section, we list the unweighted indexes[226] that were defined in sections 2 and 3 in the main text. We used the data that is listed in Appendix 1 above in order to construct the indexes. We list the Jevons, Dutot, Carli, Chained Carli, CES with r = –1 and r = –9 which we denote by P_J^t, P_D^t, P_C^t, P_{CCh}^t, $P_{CES,-1}^t$ and $P_{CES,-9}^t$ respectively for month t.[227]

Table A.3 Jevons, Dutot, Fixed Base and Chained Carli and CES Price Indexes

t	P_J^t	P_D^t	P_C^t	P_{CCh}^t	$P_{CES,-1}$	$P_{CES,-9}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.96040	0.94016	0.96846	0.96846	0.98439	1.09067
3	0.93661	0.94070	0.94340	0.95336	0.92229	0.74456
4	0.90240	0.88954	0.91068	0.92488	0.91540	0.90219
5	0.90207	0.90438	0.91172	0.93347	0.89823	0.83603
6	0.96315	0.97490	0.97869	1.00142	0.94497	0.79833
7	1.05097	1.05468	1.06692	1.11301	1.04802	1.06093
8	1.13202	1.13825	1.13337	1.21622	1.12388	1.09382
9	1.10373	1.10769	1.10739	1.18820	1.09706	1.06198
10	1.08176	1.07574	1.09119	1.17299	1.08685	1.07704
11	1.07545	1.08438	1.08516	1.17758	1.06038	0.95660
12	1.14864	1.15654	1.15517	1.27479	1.13881	1.13589
13	1.02772	1.04754	1.03943	1.15786	0.99848	0.84113
14	1.06109	1.04636	1.07433	1.21248	1.07853	1.16587
15	1.07130	1.06066	1.08164	1.23459	1.08565	1.19072
16	1.02572	1.00635	1.03655	1.18788	1.04979	1.19448
17	0.93668	0.92185	0.95548	1.09129	0.95529	1.04194
18	0.88243	0.86882	0.89940	1.03405	0.90087	1.01058
19	0.93855	0.92175	0.96016	1.10908	0.96244	1.15748
20	0.88855	0.88248	0.90225	1.07164	0.89126	0.80633
21	0.91044	0.88862	0.92930	1.12593	0.93740	1.07775
22	0.87080	0.85512	0.88891	1.08595	0.89107	1.00076
23	0.87476	0.86577	0.89065	1.10508	0.88042	0.79864
24	0.87714	0.87111	0.89384	1.12219	0.87980	0.79810
25	0.82562	0.81640	0.84467	1.06793	0.83434	0.79708
26	0.84210	0.83168	0.86532	1.10572	0.85123	0.79827
27	0.87538	0.87012	0.88197	1.16687	0.87714	0.79760
28	0.78149	0.77534	0.80014	1.05919	0.78770	0.78068
29	0.85227	0.84699	0.86721	1.17131	0.85568	0.79718
30	0.81870	0.80899	0.83656	1.13006	0.82799	0.79688
31	0.85842	0.85377	0.87514	1.19113	0.86118	0.79741
32	0.89203	0.90407	0.91440	1.26420	0.87884	0.79524
33	0.90818	0.91368	0.93127	1.35047	0.89955	0.79775
34	0.92659	0.92742	0.93489	1.39685	0.91949	0.79804
35	0.93981	0.93944	0.94941	1.42023	0.93256	0.79825
36	0.93542	0.94295	0.94087	1.42210	0.92057	0.79654
37	1.00182	1.00595	1.01060	1.52914	0.99139	0.87270
38	1.02591	1.02295	1.04068	1.57788	1.02072	0.87939
39	0.92689	0.92334	0.95090	1.44006	0.93017	0.87789

[226] It would be more accurate to call these indexes equally weighted indexes.
[227] All of these indexes were defined in section 2 except that the Carli and Chained Carli index were defined in section 3.

The above price indexes are plotted on Chart 5.



The Chained Carli index, P_{CCh}^t, is well above the other indexes as is expected. The fixed base Carli index P_C^t is slightly above the corresponding Jevons index P_J^t which in turn is slightly above the corresponding Dutot index P_D^t. The CES index with r = –1 (this corresponds to σ = 2) is on average between the Jevons and fixed base Carli indexes while the CES index with r = –9 (this corresponds to σ = 10) is well below all of the other indexes on average (and is extremely volatile).[228]

The Jevons, Dutot and fixed base Carli indexes, P_J^t, P_D^t and P_C^t, are quite close to each other. They turn out to end up about 3 percentage points below the fixed base Fisher indexes, P_F^t, at the end of the sample period. However, in the Office for National Statistics (2020) study that also compares unweighted with weighted indexes, they find larger differences between these unweighted indexes and their superlative index counterparts.[229] The problem with the unweighted indexes is that they do not weight price changes by their economic importance so if weights change dramatically along with dramatic price changes, the unweighted indexes can differ significantly from their symmetrically weighted counterpart indexes like the Fisher index. For another example of this phenomenon, see the Appendix to chapter 6, where it is shown that there are large differences between P_J^t, P_D^t, P_C^t and P_F^t.

[228] The sample means of the P_J^t, P_D^t, P_C^t, P_{CCh}^t, $P_{CES,-1}^t$ and $P_{CES,-9}^t$ are: 0.9496, 0.9458, 0.9628, 1.1732, 0.9520 and 0.9237 respectively.

[229] The ONS makes the following important point about differences between their GEKS-J unweighted index (essentially our P_J^t index) and an appropriately weighted index: "Two main observations can be made from the observations of these case studies. ... Secondly, there is an apparent upward bias from the GEKS-J methods in comparison to the weighted methods; this is likely because consumers substitute towards products that are on sale and this is not accounted for when using unweighted methods. This again highlights that having information on sales values, or approximates thereof, is arguably more important than the choice between weighted index number methods themselves." ONS (2020; 43).

We turn now to a listing of standard bilateral indexes using the three years of data and the econometrically estimated reservation prices.

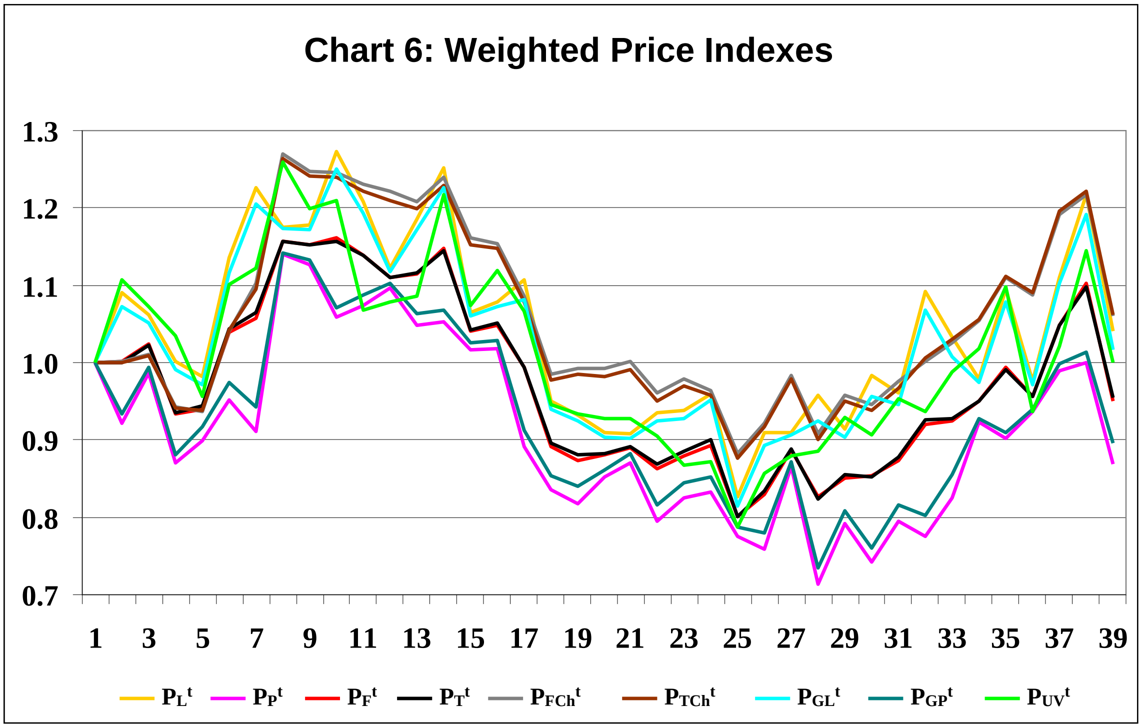
A3. Commonly Used Weighted Price Indexes

We list the fixed base and chained Laspeyres, Paasche, Fisher and Törnqvist indexes in Table A.4 below. The Geometric Laspeyres and Geometric Paasche and Unit Value indexes are also listed in this table.

Table A.4: Fixed Base and Chained Laspeyres, Paasche, Fisher and Törnqvist, Geometric Laspeyres, Geometric Paasche and Unit Value Indexes

t	P_L^t	P_P^t	P_F^t	P_T^t	P_{LCh}^t	P_{PCh}^t	P_{FCh}^t	P_{TCh}^t	P_{GL}^t	P_{GP}^t	P_{UV}^t
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.08991	0.92151	1.00218	1.00036	1.08991	0.92151	1.00218	1.00036	0.98194	1.07214	1.10724
3	1.06187	0.98637	1.02342	1.02220	1.12136	0.91193	1.01124	1.00905	0.97979	1.05116	1.07205
4	1.00174	0.87061	0.93388	0.93445	1.06798	0.83203	0.94265	0.94077	0.91520	0.99062	1.03463
5	0.98198	0.89913	0.93964	0.94387	1.11998	0.78417	0.93715	0.93753	0.91048	0.97176	0.95620
6	1.13639	0.95159	1.03989	1.04311	1.27664	0.84845	1.04075	1.04165	0.99679	1.11657	1.10159
7	1.22555	0.91097	1.05662	1.06555	1.42086	0.85482	1.10208	1.09531	1.07355	1.20485	1.12167
8	1.17447	1.14057	1.15740	1.15743	1.75897	0.91677	1.26987	1.26340	1.14865	1.17300	1.25911
9	1.17750	1.12636	1.15164	1.15169	1.73986	0.89414	1.24727	1.24135	1.12700	1.17162	1.19939
10	1.27247	1.05895	1.16081	1.15735	1.80210	0.86050	1.24528	1.23902	1.12514	1.25074	1.20900
11	1.20770	1.07376	1.13876	1.13875	1.86610	0.81117	1.23034	1.22114	1.12189	1.19276	1.06812
12	1.12229	1.09688	1.10951	1.10976	2.01810	0.73863	1.22091	1.20993	1.12209	1.11767	1.07795
13	1.18583	1.04861	1.11511	1.11677	2.17862	0.66995	1.20813	1.19943	1.09272	1.17231	1.08595
14	1.25239	1.05236	1.14803	1.14485	2.30844	0.66552	1.23948	1.22942	1.09463	1.22682	1.21698
15	1.06527	1.01701	1.04086	1.04292	2.32124	0.58025	1.16056	1.15215	1.03397	1.06020	1.07438
16	1.07893	1.01866	1.04836	1.05073	2.34342	0.56876	1.15449	1.14720	1.01310	1.07256	1.11895
17	1.10767	0.89217	0.99410	0.99352	2.28924	0.51559	1.08642	1.07832	0.95895	1.08127	1.06696
18	0.95021	0.83559	0.89105	0.89584	2.14196	0.45252	0.98452	0.97741	0.86911	0.94010	0.94589
19	0.93250	0.81744	0.87308	0.88137	2.21416	0.44435	0.99189	0.98454	0.88768	0.92447	0.93364
20	0.91010	0.85188	0.88051	0.88230	2.37598	0.41411	0.99193	0.98133	0.88109	0.90423	0.92812
21	0.90831	0.87050	0.88920	0.89209	2.48204	0.40411	1.00150	0.99069	0.87548	0.90221	0.92800
22	0.93448	0.79545	0.86217	0.86876	2.44050	0.37816	0.96068	0.95081	0.85191	0.92460	0.90448
23	0.93852	0.82477	0.87981	0.88494	2.54428	0.37672	0.97902	0.96923	0.85916	0.92722	0.86752
24	0.95955	0.83212	0.89357	0.90008	2.61768	0.35461	0.96347	0.95725	0.88900	0.95127	0.87176
25	0.82659	0.77523	0.80050	0.80120	2.54432	0.30555	0.88172	0.87662	0.80638	0.81529	0.78713
26	0.90933	0.75806	0.83026	0.83456	2.84192	0.29847	0.92100	0.91714	0.82419	0.89313	0.85607
27	0.90913	0.86638	0.88749	0.88866	3.22816	0.29960	0.98344	0.97818	0.87350	0.90653	0.87957
28	0.95748	0.71369	0.82665	0.82378	3.27769	0.25120	0.90739	0.90090	0.80609	0.92446	0.88558
29	0.91434	0.79178	0.85086	0.85489	3.58621	0.25612	0.95839	0.95091	0.83824	0.90372	0.92881
30	0.98306	0.74159	0.85383	0.85285	3.63285	0.24640	0.94612	0.93848	0.83636	0.95691	0.90674
31	0.96148	0.79467	0.87411	0.87827	3.82999	0.24849	0.97557	0.96637	0.85604	0.94519	0.95259
32	1.09219	0.77559	0.92038	0.92577	4.36079	0.23020	1.00192	1.00563	0.93404	1.06859	0.93739
33	1.03387	0.82587	0.92403	0.92835	5.45066	0.19325	1.02632	1.03039	0.92860	1.00783	0.98847
34	0.97819	0.92286	0.95012	0.95072	5.95659	0.18655	1.05412	1.05647	0.93004	0.97390	1.01750
35	1.09532	0.90246	0.99422	0.99086	6.44252	0.19130	1.11015	1.11105	0.97904	1.07872	1.09820
36	0.97574	0.93603	0.95568	0.95607	6.69005	0.17668	1.08720	1.08989	0.94745	0.97198	0.93645
37	1.10952	0.99004	1.04808	1.04846	7.50373	0.18937	1.19204	1.19665	1.03628	1.10176	1.02142
38	1.21684	0.99944	1.10280	1.09863	7.90093	0.18768	1.21774	1.22145	1.06914	1.19166	1.14490
39	1.04027	0.86886	0.95071	0.95482	7.16398	0.15715	1.06105	1.06219	0.93030	1.01682	0.99999

Note that the chained Laspeyres index ends up at 7.164 while the chained Paasche index ends up at 0.157. The corresponding fixed base indexes end up at 1.040 and 0.869 so it is clear that these chained indexes are subject to tremendous chain drift. The chain drift carries over to the Fisher and Törnqvist indexes; i.e., the fixed base Fisher index ends up at 0.9548 while its chained counterpart ends up at 1.061. Chart 6 plots the above indexes with the exceptions of the chained Laspeyres and Paasche indexes (these indexes exhibit too much chain drift to be considered further).



It can be seen that all nine of the weighted indexes which appear on Chart 6 capture an underlying general trend in prices. However, there is a considerable dispersion between the indexes. Our preferred indexes for this group of indexes are the fixed base Fisher and Törnqvist indexes, P_F^t and P_T^t. These two indexes approximate each other very closely and can barely be distinguished in the Chart. The Paasche and Geometric Paasche indexes, P_P^t and P_{GP}^t, lie below our preferred indexes while the remaining indexes generally lie above our preferred indexes. The chained Fisher and Törnqvist indexes, P_{FCh}^t and P_{TCh}^t, approximate each other very closely but both indexes lie well above their fixed base counterparts; i.e., they exhibit a considerable amount of chain drift. Thus chained superlative indexes are not recommended for use with scanner data where the products are subject to large fluctuations in prices and quantities. The fixed base Laspeyres and Geometric Laspeyres indexes, P_L^t and P_{GL}^t, are fairly close to each other and are well above P_F^t and P_T^t. The unit value price index, P_{UV}^t, is subject to large fluctuations and generally lies above our preferred indexes.

We turn now to weighted indexes that use annual weights from a base year.

A4. Indexes which Use Annual Weights

The Weighted Jevons or Geometric Young index, $P_{J\alpha}^t$ or P_{GY}^t, was defined by (54) in section 6. This index uses the arithmetic average of the monthly shares in year 1 as weights in a weighted geometric index for subsequent months in the sample. The Lowe index, P_{Lo}^t, was defined by

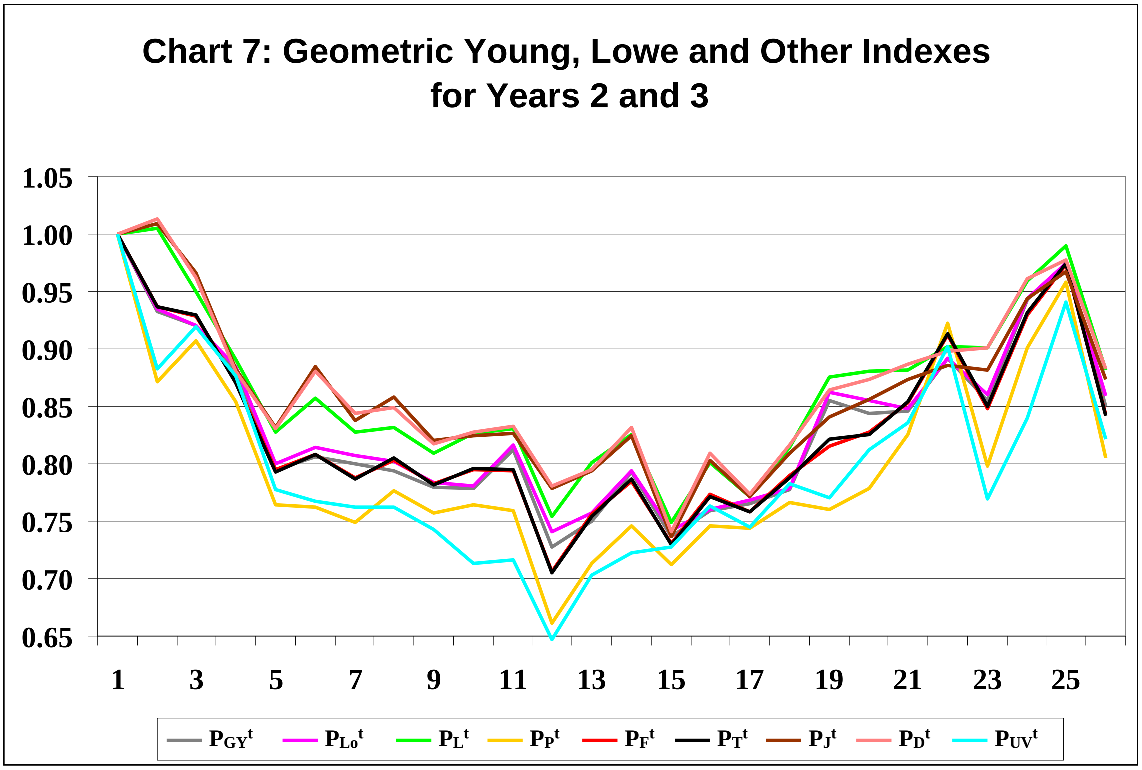
(124) in section 11. This index is a fixed basket index that uses the average quantities in the base year as the vector of quantity weights. We calculated both of these indexes for the months in years 2 and 3 for our sample using the weights from year 1 of our sample. For comparison purposes, we also list the fixed base Laspeyres, Paasche, Fisher and Törnqvist indexes, $P_L{}^t$, $P_P{}^t$, $P_F{}^t$ and $P_T{}^t$ for the "months" in years 2 and 3 of our sample. It is also of interest to list the Jevons, Dutot and Unit Value indexes, $P_J{}^t$, $P_D{}^t$ and $P_{UV}{}^t$ for years 2 and 3 in order to see how unweighted indexes compare to the weighted indexes. The sample averages for these indexes are listed in the last row of the table.

Table A.5: Geometric Young, Lowe, Laspeyres, Paasche, Fisher, Törnqvist, Jevons, Dutot and Unit Value Indexes for Years 2 and 3

t	$P_{GY}{}^t$	$P_{Lo}{}^t$	$P_L{}^t$	$P_P{}^t$	$P_F{}^t$	$P_T{}^t$	$P_J{}^t$	$P_D{}^t$	$P_{UV}{}^t$
14	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
15	0.93263	0.93494	1.00555	0.87189	0.93634	0.93715	1.00962	1.01366	0.88282
16	0.92082	0.92031	0.95041	0.90755	0.92873	0.92986	0.96667	0.96175	0.91945
17	0.87997	0.88507	0.89085	0.85384	0.87215	0.87032	0.88275	0.88100	0.87673
18	0.79503	0.80022	0.82733	0.76386	0.79496	0.79283	0.83163	0.83032	0.77724
19	0.80573	0.81468	0.85664	0.76193	0.80790	0.80865	0.88451	0.88091	0.76718
20	0.79981	0.80700	0.82757	0.74904	0.78733	0.78635	0.83739	0.84337	0.76264
21	0.79437	0.80164	0.83126	0.77659	0.80346	0.80489	0.85802	0.84924	0.76254
22	0.77921	0.78355	0.80911	0.75762	0.78294	0.78149	0.82067	0.81723	0.74322
23	0.77876	0.78087	0.82688	0.76468	0.79517	0.79606	0.82440	0.82741	0.71285
24	0.81228	0.81680	0.83070	0.75908	0.79408	0.79472	0.82664	0.83251	0.71633
25	0.72801	0.74112	0.75452	0.66120	0.70632	0.70498	0.77809	0.78023	0.64679
26	0.75011	0.75684	0.80141	0.71350	0.75618	0.75377	0.79362	0.79483	0.70344
27	0.79254	0.79375	0.82527	0.74559	0.78442	0.78661	0.82498	0.83156	0.72275
28	0.73664	0.74226	0.74893	0.71223	0.73035	0.72970	0.73650	0.74098	0.72769
29	0.75964	0.76031	0.80135	0.74576	0.77306	0.77165	0.80321	0.80946	0.76321
30	0.76531	0.76828	0.77149	0.74410	0.75767	0.75781	0.77157	0.77315	0.74507
31	0.77786	0.77867	0.81448	0.76635	0.79005	0.78811	0.80900	0.81594	0.78275
32	0.85506	0.86201	0.87512	0.76018	0.81563	0.82138	0.84067	0.86401	0.77026
33	0.84365	0.85499	0.88099	0.77811	0.82795	0.82554	0.85589	0.87320	0.81223
34	0.84601	0.84804	0.88159	0.82588	0.85328	0.85422	0.87325	0.88632	0.83608
35	0.89199	0.89177	0.90170	0.92254	0.91206	0.91320	0.88570	0.89782	0.90240
36	0.85506	0.85983	0.90132	0.79811	0.84815	0.84966	0.88156	0.90117	0.76948
37	0.94264	0.94402	0.95898	0.90084	0.92946	0.93135	0.94414	0.96137	0.83931
38	0.97419	0.97462	0.99009	0.95811	0.97397	0.97413	0.96684	0.97762	0.94077
39	0.85043	0.85908	0.88213	0.80516	0.84277	0.84144	0.87353	0.88242	0.82170
Mean	0.83338	0.83772	0.86329	0.80014	0.83094	0.83100	0.86080	0.86644	0.79634

As usual, $P_F{}^t$ and $P_T{}^t$ approximate each other very closely. Indexes with substantial upward biases relative to these two indexes are the Laspeyres, Jevons and Dutot indexes, $P_L{}^t$, $P_J{}^t$ and $P_D{}^t$. The Geometric Young index and the Lowe index, $P_{GY}{}^t$ and $P_{Lo}{}^t$, were about 0.25 and .67 percentage points above the superlative indexes on average. The Paasche and Unit Value indexes, $P_P{}^t$ and $P_{UV}{}^t$, had substantial downward biases relative to the superlative indexes. These inequalities agree with our a priori expectations about biases. The nine indexes are plotted in Chart 7.

It can be seen that all nine indexes capture the trend in the product prices with $P_F{}^t$ and $P_T{}^t$ in the middle of the indexes (and barely distinguishable from each other in the chart). The unit value index $P_{UV}{}^t$ is the lowest index followed by the Paasche index $P_P{}^t$. The Geometric Young and Lowe indexes, $P_{GY}{}^t$ and $P_{Lo}{}^t$, are quite close to each other and close to the superlative indexes in the first part of the sample but then they drift above the superlative indexes in the latter half of the sample. We expect the Lowe index to have some upward substitution bias and with highly substitutable products, we expect the Geometric Young index to also have an upward substitution bias. Finally, the Laspeyres, Jevons and Dutot indexes are all substantially above the superlative indexes with P_J^t and P_D^t approximating each other quite closely.



We turn our attention to multilateral indexes.

A5. Multilateral Indexes

We considered seven main multilateral indexes in the main text:[230]

- P_{GEKS}^t (see definition (70) in section 8);
- P_{CCDI}^t (see definition (77) in section 8);
- P_{GK}^t (see definition (137) in section 12);
- P_{WTPD}^t (see definition (149) in section 13);
- P_{AL}^t, the price similarity linked indexes defined below definition (215) which defined the *asymptotic linear measures of relative price dissimilarity* $\Delta_{AL}(p^r,p^t,q^r,q^t)$;
- P_{SP}^t, the price similarity linked indexes defined below definitions (218) which defined the *predicted share measures of relative price dissimilarity* $\Delta_{SP}(p^r,p^t,q^r,q^t)$ and

[230] We also defined the quantity similarity linked price indexes P_{SQ}^t below definitions (219) which were constructed using the predicted share measures of relative quantity dissimilarity $\Delta_{SQ}(p^r,p^t,q^r,q^t)$. However, the indexes P_{SQ}^t are absorbed into the definition of the superior indexes P_{SPQ}^t and so we did not list the P_{SQ}^t here. We also considered some variants of P_{AL}^t and P_{SP}^t which will be considered later in this section and in section A6.

- P_{SPQ}^t, the price and quantity similarity linked indexes defined below definition (221) which defined the *predicted share measures of relative price and quantity dissimilarity* $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$.

It turned out that the similarity linked price indexes P_{SP}^t were equal to their counterparts P_{SPQ}^t for each time period t so we list only the P_{SP}^t indexes in Table A.6 below.[231] The above six multilateral indexes are listed in Table A.6 along with the fixed base Fisher and Törnqvist indexes P_F^t and P_T^t. All of these indexes were evaluated using estimated reservation prices for the missing products. The sample mean for each index is listed in the last row of Table A.6.

Table A.6: Six Multilateral Indexes and the Fixed Base Fisher and Törnqvist Indexes

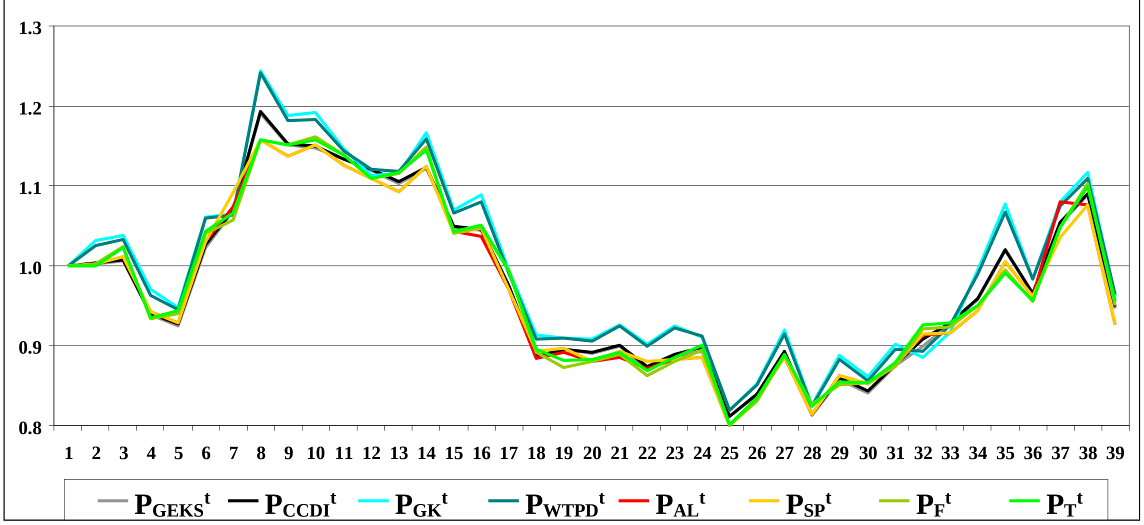
t	P_{GEKS}^t	P_{CCDI}^t	P_{GK}^t	P_{WTPD}^t	P_{AL}^t	P_{SP}^t	P_F^t	P_T^t
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.00233	1.00395	1.03138	1.02468	1.00218	1.00218	1.00218	1.00036
3	1.00575	1.00681	1.03801	1.03322	1.01124	1.01124	1.02342	1.02220
4	0.93922	0.94020	0.97021	0.96241	0.94262	0.94262	0.93388	0.93445
5	0.92448	0.92712	0.94754	0.94505	0.92812	0.92812	0.93964	0.94387
6	1.02249	1.02595	1.06097	1.05893	1.03073	1.03073	1.03989	1.04311
7	1.06833	1.06995	1.06459	1.06390	1.07314	1.09146	1.05662	1.06555
8	1.19023	1.19269	1.24385	1.24192	1.15740	1.15740	1.15740	1.15743
9	1.15115	1.15206	1.18818	1.18231	1.13680	1.13680	1.15164	1.15169
10	1.14730	1.15007	1.19184	1.18333	1.15156	1.15156	1.16081	1.15735
11	1.13270	1.13301	1.14662	1.14308	1.12574	1.12574	1.13876	1.13875
12	1.11903	1.12079	1.11332	1.12082	1.10951	1.10951	1.10951	1.10976
13	1.10247	1.10487	1.11561	1.11838	1.09229	1.09229	1.11511	1.11677
14	1.12136	1.12345	1.16579	1.15912	1.12489	1.12489	1.14803	1.14485
15	1.04827	1.04883	1.06958	1.06608	1.04237	1.04086	1.04086	1.04292
16	1.04385	1.04539	1.08842	1.08044	1.03692	1.04704	1.04836	1.05073
17	0.97470	0.97550	0.99512	0.99145	0.97013	0.97013	0.99410	0.99352
18	0.88586	0.88695	0.91319	0.90765	0.88455	0.89319	0.89105	0.89584
19	0.89497	0.89597	0.90990	0.90923	0.89118	0.89702	0.87308	0.88137
20	0.88973	0.89126	0.90822	0.90578	0.88051	0.88051	0.88051	0.88230
21	0.89904	0.89990	0.92641	0.92503	0.88482	0.89346	0.88920	0.89209
22	0.87061	0.87363	0.90145	0.89880	0.87151	0.88001	0.86217	0.86876
23	0.88592	0.88868	0.92421	0.92158	0.88280	0.88280	0.87981	0.88494
24	0.89282	0.89799	0.91127	0.91198	0.88502	0.88502	0.89357	0.90008
25	0.81132	0.81115	0.81875	0.81913	0.79966	0.79966	0.80050	0.80120
26	0.83799	0.83914	0.85168	0.85089	0.83378	0.83378	0.83026	0.83456
27	0.89063	0.89246	0.91906	0.91398	0.88481	0.88481	0.88749	0.88866
28	0.81304	0.81411	0.82600	0.82419	0.81336	0.81336	0.82665	0.82378
29	0.85763	0.85934	0.88821	0.88248	0.86271	0.86271	0.85086	0.85489
30	0.84103	0.84305	0.86121	0.85556	0.85166	0.85230	0.85383	0.85285
31	0.87495	0.87639	0.90123	0.89600	0.87568	0.87568	0.87411	0.87827
32	0.89936	0.90831	0.88553	0.89332	0.91368	0.91398	0.92038	0.92577
33	0.92670	0.92878	0.91672	0.92625	0.91517	0.91517	0.92403	0.92835
34	0.95721	0.95846	0.99507	0.98974	0.94435	0.94435	0.95012	0.95072
35	1.01848	1.02026	1.07728	1.06779	1.00422	1.00422	0.99422	0.99086
36	0.96507	0.96601	0.98339	0.98282	0.96122	0.96122	0.95568	0.95607
37	1.05250	1.05448	1.08019	1.07514	1.07953	1.03556	1.04808	1.04846

[231] For every pair of observations, the measure of predicted share relative price dissimilarity was always smaller than the corresponding measure of predicted share relative quantity dissimilarity.

38	**1.08819**	**1.08961**	**1.11648**	**1.10963**	**1.07546**	**1.07546**	**1.10280**	**1.09863**
39	**0.94591**	**0.94834**	**0.96156**	**0.96453**	**0.92575**	**0.92575**	**0.95071**	**0.95482**
Mean	**0.97417**	**0.97602**	**0.99764**	**0.99504**	**0.97069**	**0.97109**	**0.97434**	**0.97607**

If the eight indexes are evaluated according to their sample means, the two Similarity Linked indexes $P_{AL}{}^t$ and $P_{SP}{}^t = P_{SPQ}{}^t$ generated the lowest indexes on average. The $P_{GEKS}{}^t$, $P_{CCDI}{}^t$, $P_F{}^t$ and $P_T{}^t$ indexes are tightly clustered in the middle and the $P_{GK}{}^t$ and $P_{WTPD}{}^t$ are about 2 percentage points above the middle indexes on average. Looking at the index levels at the final sample observation, the two indexes that use similarity linking end up at 0.9275 which is about 2 percentage points below where the GEKS, CCDI, fixed base Fisher and Törnqvist Theil indexes ended up. The Geary Khamis and Weighted Time Product Dummy indexes ended up approximately 4 percentage points above the two similarity linked indexes. These differences are substantial. Chart 8 plots the eight indexes.

Chart 8: Six Multilateral and Two Superlative Indexes



All eight indexes capture the trend in product prices reasonably well. It is clear that the Geary-Khamis and Weighted Time Product Dummy indexes have some upward bias relative to the remaining six indexes. The two similarity linked indexes, $P_{AL}{}^t$ and $P_{SP}{}^t$, both end up at the same index level and in general, they are very close.

The following table lists the real time $P_{AL}{}^t$ and $P_{SP}{}^t$ again and compares them with their modified counterparts, $P_{ALM}{}^t$ and $P_{SPM}{}^t$. These latter indexes use the first 13 “months” as a “training” year where a spanning tree of observations is linked similtaneously. Here is the spanning tree or path of bilateral links that minimizes the sum of the dissimilarity measures associated with the links for $P_{ALM}{}^t$:

```
12     13
|      |
1 - 8 - 9 - 11
       |
       3 - 4 - 2 - 10 - 6 - 5
```

|
7

Here is the corresponding set of optimal links for P_{SPM}^t for "months" 1-13:

13
|
12 - 1 - 8 - 9 - 11
|
10 - 2 - 4 - 3 - 5 - 6 – 7.

The above spanning trees are similar but are not identical. Nevertheless, the index levels generated by the two alternative measures of price dissimilarity end up being the same.

At the end of section 20, the fixed base maximum overlap Fisher indexes P_F^{t*} were defined along with the GEKS index that uses the geometric mean of the maximum overlap Fisher indexes for each choice of a base, P_{GEKS}^{t*}. We also defined the counterparts to the predicted share multilateral indexes P_{SP}^t and P_{SPM}^t using maximum overlap Fisher indexes to do the linking of observations in place of regular Fisher indexes. These maximum overlap indexes (which do not use imputations) were denoted by P_{SP}^{t*} and P_{SPM}^{t*}. All of these indexes are listed in Table A.8. The set of optimal links for P_{SPM}^{t*} for "months" 1-13 are as follows:

13
|
12 - 1 - 8 - 9 - 11 - 10
|
2 - 4 - 3 - 5 - 6 - 7

All ten of the above indexes are listed in Table A.7 and plotted on Chart 9.

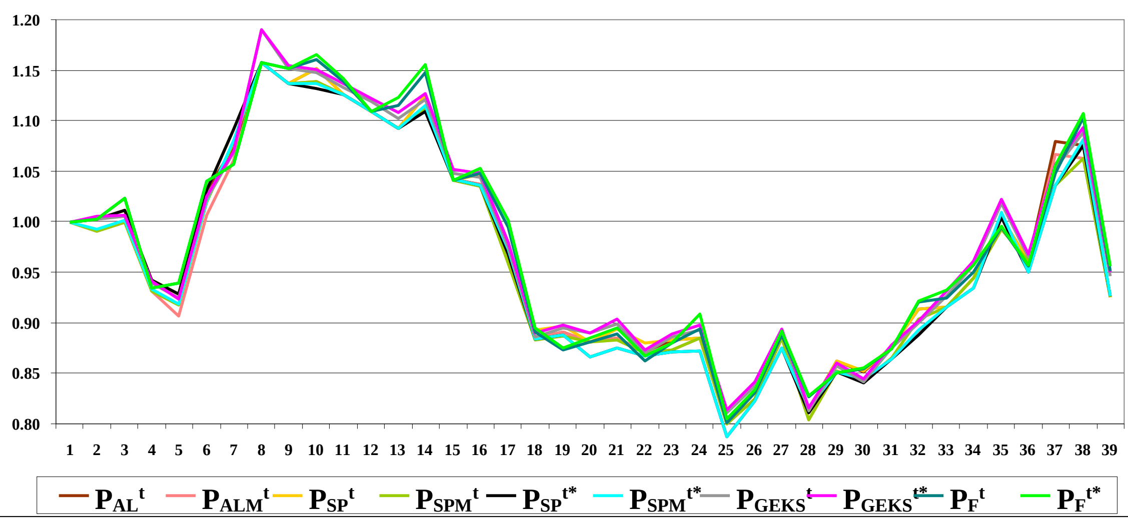
Table A.7: Six Similarity Linked Multilateral, Two GEKS and Two Fisher Indexes

t	P_{AL}^t	P_{ALM}^t	P_{SP}^t	P_{SPM}^t	P_{SP}^{t*}	P_{SPM}^{t*}	P_{GEKS}^t	P_{GEKS}^{t*}	P_F^t	P_F^{t*}
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.00218	0.99067	1.00218	0.99067	1.00218	0.99213	1.00233	1.00546	1.00218	1.00218
3	1.01124	0.99960	1.01124	0.99960	1.01124	1.00108	1.00575	1.00673	1.02342	1.02342
4	0.94262	0.93180	0.94262	0.93180	0.94262	0.93317	0.93922	0.94156	0.93388	0.93388
5	0.92812	0.90620	0.92812	0.91744	0.92812	0.91879	0.92448	0.92384	0.93964	0.93964
6	1.03073	1.00638	1.03073	1.01886	1.03073	1.02037	1.02249	1.02505	1.03989	1.03989
7	1.07314	1.06081	1.09146	1.07890	1.09146	1.08049	1.06833	1.06965	1.05662	1.05662
8	1.15740	1.15740	1.15740	1.15740	1.15740	1.15740	1.19023	1.19015	1.15740	1.15740
9	1.13680	1.13680	1.13680	1.13680	1.13726	1.13726	1.15115	1.15502	1.15164	1.15209
10	1.15156	1.13833	1.15156	1.13833	1.13142	1.13707	1.14730	1.15094	1.16081	1.16529
11	1.12574	1.12574	1.12574	1.12574	1.12620	1.12620	1.13270	1.13707	1.13876	1.14153
12	1.10951	1.10951	1.10951	1.10951	1.10876	1.10876	1.11903	1.12242	1.10951	1.10876
13	1.09229	1.09229	1.09229	1.09229	1.09273	1.09273	1.10247	1.10798	1.11511	1.12264
14	1.12489	1.11196	1.12489	1.11196	1.10948	1.11502	1.12136	1.12651	1.14803	1.15567
15	1.04237	1.04237	1.04086	1.04086	1.04167	1.04167	1.04827	1.05159	1.04086	1.04105
16	1.03692	1.03692	1.04704	1.03502	1.03622	1.03622	1.04385	1.04814	1.04836	1.05283
17	0.97013	0.95899	0.97013	0.95899	0.96764	0.97246	0.97470	0.97951	0.99410	1.00156
18	0.88455	0.88455	0.89319	0.88293	0.88396	0.88396	0.88586	0.88943	0.89105	0.89486
19	0.89118	0.89118	0.89702	0.88672	0.88775	0.88775	0.89497	0.89780	0.87308	0.87462
20	0.88051	0.88051	0.88051	0.88051	0.86666	0.86666	0.88973	0.89037	0.88051	0.88462
21	0.88482	0.88482	0.89346	0.88319	0.87503	0.87503	0.89904	0.90403	0.88920	0.89505

22	**0.87151**	**0.87151**	**0.88001**	**0.86991**	**0.86764**	**0.86764**	**0.87061**	**0.87296**	**0.86217**	**0.86759**
23	**0.88280**	**0.87265**	**0.88280**	**0.87265**	**0.87100**	**0.87100**	**0.88592**	**0.88869**	**0.87981**	**0.88008**
24	**0.88502**	**0.88502**	**0.88502**	**0.88502**	**0.87164**	**0.87164**	**0.89282**	**0.89785**	**0.89357**	**0.90877**
25	**0.79966**	**0.79966**	**0.79966**	**0.79966**	**0.78672**	**0.78672**	**0.81132**	**0.81419**	**0.80050**	**0.80492**
26	**0.83378**	**0.82421**	**0.83378**	**0.82421**	**0.82264**	**0.82264**	**0.83799**	**0.84106**	**0.83026**	**0.83325**
27	**0.88481**	**0.88481**	**0.88481**	**0.88481**	**0.87500**	**0.87500**	**0.89063**	**0.89395**	**0.88749**	**0.89223**
28	**0.81336**	**0.80401**	**0.81336**	**0.80401**	**0.81126**	**0.81531**	**0.81304**	**0.81584**	**0.82665**	**0.82771**
29	**0.86271**	**0.85280**	**0.86271**	**0.85280**	**0.85118**	**0.85118**	**0.85763**	**0.86015**	**0.85086**	**0.85009**
30	**0.85166**	**0.84188**	**0.85230**	**0.84250**	**0.84063**	**0.84482**	**0.84103**	**0.84407**	**0.85383**	**0.85566**
31	**0.87568**	**0.86562**	**0.87568**	**0.86562**	**0.86398**	**0.86398**	**0.87495**	**0.87775**	**0.87411**	**0.87393**
32	**0.91368**	**0.89210**	**0.91398**	**0.90346**	**0.88825**	**0.89268**	**0.89936**	**0.90222**	**0.92038**	**0.92131**
33	**0.91517**	**0.91517**	**0.91517**	**0.91517**	**0.91554**	**0.91554**	**0.92670**	**0.93126**	**0.92403**	**0.93241**
34	**0.94435**	**0.94435**	**0.94435**	**0.94435**	**0.93388**	**0.93388**	**0.95721**	**0.96113**	**0.95012**	**0.95662**
35	**1.00422**	**0.99266**	**1.00422**	**0.99266**	**1.00461**	**1.00963**	**1.01848**	**1.02253**	**0.99422**	**0.99561**
36	**0.96122**	**0.96122**	**0.96122**	**0.96122**	**0.95057**	**0.95057**	**0.96507**	**0.96835**	**0.95568**	**0.95746**
37	**1.07953**	**1.06710**	**1.03556**	**1.03556**	**1.03597**	**1.03597**	**1.05250**	**1.05702**	**1.04808**	**1.05585**
38	**1.07546**	**1.06308**	**1.07546**	**1.06308**	**1.07588**	**1.08124**	**1.08819**	**1.09293**	**1.10280**	**1.10739**
39	**0.92575**	**0.92575**	**0.92575**	**0.92575**	**0.92612**	**0.92612**	**0.94591**	**0.94987**	**0.95071**	**0.95610**
Mean	**0.97069**	**0.96437**	**0.97109**	**0.96461**	**0.96464**	**0.96410**	**0.97417**	**0.97731**	**0.97434**	**0.97745**

The four similarity linked indexes that used reservation prices, P_{AL}^t, P_{ALM}^t, P_{SP}^t and P_{SMP}^t ended up at the same level for the last observation, 0.92575. The predicted share similarity linked indexes that did not use imputations for the prices of missing products, P_{SP}^{t*} and P_{SMP}^{t*}, ended up at the slightly higher level, 0.92612. Thus all of the similarity linked indexes behaved in a similar manner for our particular data set.

Chart 9: Similarity Linked, GEKS and Fisher Price Indexes



A6. Multilateral and Fisher Indexes Using Reservation Prices versus Carry Forward Prices

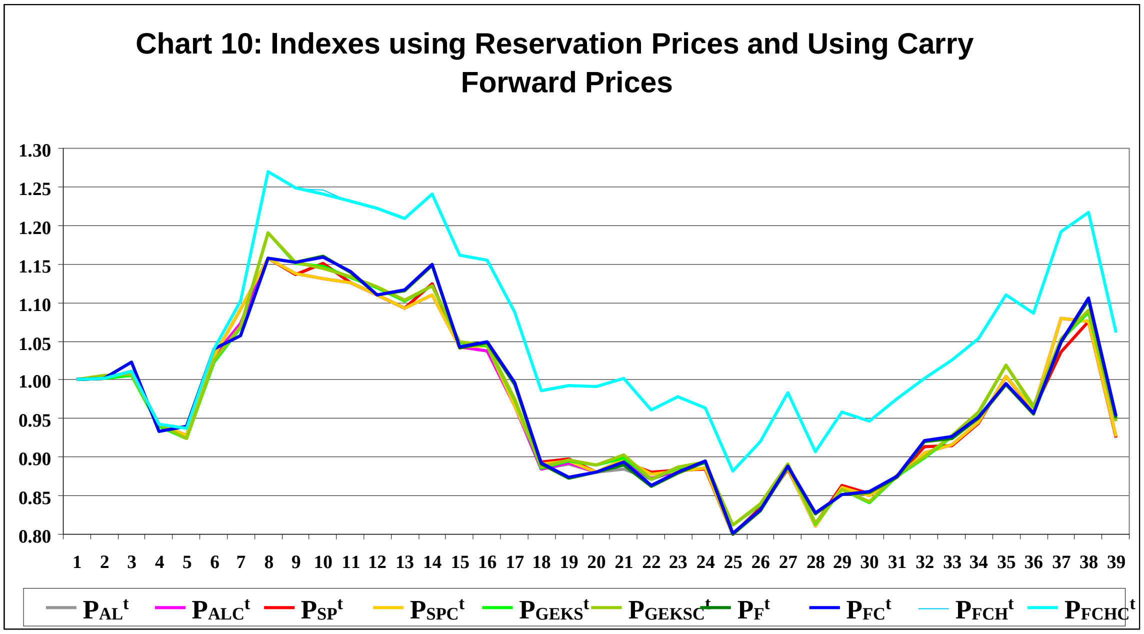
Finally, we compare P_{AL}^t (Asymptotic Linear), P_{SP}^t (Predicted Share), P_{GEKS}^t (GEKS), P_F^t (Fixed Base Fisher) and P_{FCH}^t (Chained Fisher) using reservation prices with their counterparts using inflation adjusted Carry Forward or Carry Backward prices, P_{ALC}^t, P_{SPC}^t, P_{GEKSC}^t, P_{FC}^t and P_{FCHC}^t, in Table A.8. The ten indexes are plotted on Chart 10.

Table A.8: Six Multilateral Indexes and Four Fisher Indexes Using Reservation Prices and Using Inflation Adjusted Carry Forward or Backward Prices

t	P_{AL}^t	P_{ALC}^t	P_{SP}^t	P_{SPC}^t	P_{GEKS}^t	P_{GEKSC}^t	P_F^t	P_{FC}^t	P_{FCH}^t	P_{FCHC}^t
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.00218	1.00218	1.00218	1.00218	1.00233	1.00600	1.00218	1.00218	1.00218	1.00218
3	1.01124	1.01124	1.01124	1.01124	1.00575	1.00765	1.02342	1.02342	1.01124	1.01124
4	0.94262	0.94262	0.94262	0.94262	0.93922	0.94238	0.93388	0.93388	0.94265	0.94265
5	0.92812	0.92812	0.92812	0.92812	0.92448	0.92458	0.93964	0.93964	0.93715	0.93715
6	1.03073	1.03073	1.03073	1.03073	1.02249	1.02595	1.03989	1.03989	1.04075	1.04075
7	1.07314	1.07314	1.09146	1.09146	1.06833	1.06926	1.05662	1.05662	1.10208	1.10208
8	1.15740	1.15740	1.15740	1.15740	1.19023	1.19049	1.15740	1.15740	1.26987	1.26987
9	1.13680	1.13743	1.13680	1.13743	1.15115	1.15235	1.15164	1.15265	1.24727	1.24796
10	1.15156	1.13117	1.15156	1.13117	1.14730	1.14432	1.16081	1.15847	1.24528	1.24110
11	1.12574	1.12637	1.12574	1.12637	1.13270	1.13413	1.13876	1.14017	1.23034	1.23142
12	1.10951	1.11015	1.10951	1.11015	1.11903	1.12033	1.10951	1.11015	1.22091	1.22199
13	1.09229	1.09290	1.09229	1.09290	1.10247	1.10348	1.11511	1.11667	1.20813	1.20919
14	1.12489	1.10982	1.12489	1.10982	1.12136	1.12230	1.14803	1.14991	1.23948	1.24057
15	1.04237	1.04298	1.04086	1.04215	1.04827	1.04951	1.04086	1.04215	1.16056	1.16159
16	1.03692	1.03752	1.04704	1.04435	1.04385	1.04502	1.04836	1.04993	1.15449	1.15551
17	0.97013	0.96643	0.97013	0.96643	0.97470	0.97582	0.99410	0.99631	1.08642	1.08738
18	0.88455	0.88507	0.89319	0.89089	0.88586	0.88680	0.89105	0.89233	0.98452	0.98539
19	0.89118	0.89169	0.89702	0.89471	0.89497	0.89577	0.87308	0.87401	0.99189	0.99277
20	0.88051	0.88066	0.88051	0.88066	0.88973	0.88931	0.88051	0.88066	0.99193	0.99178
21	0.88482	0.89189	0.89346	0.89776	0.89904	0.90338	0.88920	0.89369	1.00150	1.00135
22	0.87151	0.87235	0.88001	0.87809	0.87061	0.87144	0.86217	0.86337	0.96068	0.96053
23	0.88280	0.88115	0.88280	0.88115	0.88592	0.88697	0.87981	0.88078	0.97902	0.97871
24	0.88502	0.88616	0.88502	0.88616	0.89282	0.89324	0.89357	0.89470	0.96347	0.96316
25	0.79966	0.80045	0.79966	0.80045	0.81132	0.81211	0.80050	0.80141	0.88172	0.88144
26	0.83378	0.83223	0.83378	0.83223	0.83799	0.83906	0.83026	0.83184	0.92100	0.92071
27	0.88481	0.88608	0.88481	0.88608	0.89063	0.89137	0.88749	0.88840	0.98344	0.98313
28	0.81336	0.81025	0.81336	0.81025	0.81304	0.81400	0.82665	0.82783	0.90739	0.90710
29	0.86271	0.86110	0.86271	0.86110	0.85763	0.85859	0.85086	0.85183	0.95839	0.95809
30	0.85166	0.85007	0.85230	0.85007	0.84103	0.84177	0.85383	0.85488	0.94612	0.94582
31	0.87568	0.87405	0.87568	0.87405	0.87495	0.87600	0.87411	0.87539	0.97557	0.97526
32	0.91368	0.90516	0.91398	0.90516	0.89936	0.89984	0.92038	0.92116	1.00192	1.00161
33	0.91517	0.91568	0.91517	0.91568	0.92670	0.92799	0.92403	0.92695	1.02632	1.02600
34	0.94435	0.94571	0.94435	0.94571	0.95721	0.95811	0.95012	0.95213	1.05412	1.05379
35	1.00422	1.00400	1.00422	1.00400	1.01848	1.01961	0.99422	0.99551	1.11015	1.10980
36	0.96122	0.96261	0.96122	0.96261	0.96507	0.96626	0.95568	0.95713	1.08720	1.08686
37	1.07953	1.07929	1.03556	1.07929	1.05250	1.05337	1.04808	1.04986	1.19204	1.19167
38	1.07546	1.07522	1.07546	1.07522	1.08819	1.08970	1.10280	1.10574	1.21774	1.21735
39	0.92575	0.92626	0.92575	0.92626	0.94591	0.94704	0.95071	0.95246	1.06105	1.06071
Mean	0.97069	0.96968	0.97109	0.97082	0.97417	0.97526	0.97434	0.97542	1.0589	1.0589

Basically, each index that uses reservation prices is close to its counterpart index that uses inflation adjusted carry forward or backward prices. This is to be expected since there are only 20 missing product prices out of a sample of 19x39 = 741 price and quantity observations.

The two Fisher chained indexes, P_{FCH}^t (uses reservation prices) and P_{FCHC}^t (uses inflation adjusted carry forward or backward prices) cannot be distinguished from each other in Chart 10. These indexes are subject to substantial upward chain drift. The remaining indexes (which are not subject to chain drift) are quite close to each other.



A7. Conclusion

Conceptually, the price and quantity similarity linked indexes P_{SPQ}^t based on the combined price and quantity dissimilarity measure $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ seem to be the most attractive solution for solving the chain drift problem.[232] In practice, $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ will typically equal the predicted share price dissimilarity measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$ so that P_{SPQ}^t will typically equal P_{SP}^t. The indexes P_{SPQ}^t and P_{SP}^t can be implemented using either reservation prices or some form of carry forward prices or if the statistical agency does not want to use explicit imputations for missing product prices, these indexes can be calculated without using imputations.

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[232] They can deal with seasonal products more adequately than the other indexes that are considered in this paper. They also satisfy the strong identity test (and thus are not subject to chain drift) as well as the fixed basket test.

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