International Trade and Macroeconomic Dynamics with Sanctions*

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Abstract

We develop a model of the international trade and macroeconomic dynamics triggered by the imposition of financial and/or trade sanctions. We begin with a tractable two-country model in which one of the countries (Foreign) has an advantage in production of a commodity, interpreted as gas. Both countries use gas as input in production of differentiated consumption goods, but Home supplements its domestic production of gas with imports from Foreign to meet domestic and Foreign demand of final goods. There is endogenous producer entry in each country's consumption sector, and fixed trade costs imply that only a subset of producers export. Countries trade non-contingent bonds with each other. We assume that Home is the country that imposes the sanctions. When financial sanctions are imposed, a fraction of Foreign agents is excluded from participation in the international bond market. When all Foreign agents are excluded, financial sanctions imply financial autarky. Trade sanctions can take different forms: a ban on international gas trade, a cap on the quantity traded or its price, and/or the exclusion of a fraction of Foreign exporters (the largest, most productive ones—in the limit, all of them) from international trade. We show that, for financial sanctions to have a significant impact, it is important to exclude all Foreign agents from the bond market. All types of sanctions imply costs for Home agents, but they are always more costly for Foreign ones. Our analysis sheds light on how sanctions affect the dynamics of key macroeconomic variables–such as real exchange rates, consumption, and international balances—and the underlying trade patterns.

JEL codes: F1, F4, F51.
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1 Introduction

Understanding the mechanisms of international economic interdependence is at the forefront of policy during geopolitical tensions such as those the world economy is currently experiencing. When a country (or group of countries) imposes sanctions on another country (or group of countries), and the countries involved are large enough that the use of sanctions does not impact only the targeted economy, it becomes challenging to understand the underlying mechanisms that lead to the outcome after the introduction of sanctions. Our goal in this study is to contribute to our understanding of international interdependence and clarify the transmission mechanisms of sanctions in a microfounded model of international trade and macroeconomics.

Our starting point is a suitable extension of the two-country model of international trade and macroeconomic dynamics in Ghironi and Melitz (2005) (henceforth, GM). We modify the GM model by assuming that both countries, Home and Foreign, are endowed with natural gas. In each country, an upstream, perfectly competitive production sector combines sector-specific labor and natural gas to produce usable gas. A downstream, monopolistically competitive sector uses gas and sector-specific labor to produce differentiated consumption goods. There is endogenous firm entry into this sector subject to an initial sunk cost, and firms produce with heterogeneous productivities that are drawn upon entry. Fixed trade costs imply that only the relatively more productive firms export. In the absence of sanctions, Home and Foreign gas are perfect substitutes, with price determined by equalization of world demand and supply. We assume that Foreign has a larger endowment of natural gas, but it characterized by higher sunk costs of firm entry. These assumptions imply that, in the absence of sanctions, Home imports gas from Foreign, and there is a larger mass of producers of differentiated goods in Home than in Foreign. Households in the two countries hold non-contingent bonds and shares. As in GM, we assume that only bonds are traded internationally. Each household consists of gas-sector and consumption-sector workers. Households pool their incomes so that, in the absence of sanctions, there is a representative household in each country.

We assume that sanctions are imposed by Home. There are two types of sanctions: financial and trade sanctions, and they take the form of exclusion from the international market. When financial market sanctions are imposed, a fraction of Foreign households is excluded from international bond trading (in the limit, all Foreign households are excluded). This implies that there are now two types of Foreign households: the representative sanctioned household and the representative non-sanctioned one. The latter can still trade bonds with the Home household, but the former is restricted to trading bonds only with the non-sanctioned Foreign household.

Trade sanctions can apply to gas trade and/or differentiated consumption-good trade. In the gas market, we consider the scenario of a constraint on the quantity of traded gas. In the market for consumption goods, we assume that the Home country prohibits trade in the outputs of Home firms with productivity above a certain threshold. The idea is that sanctions are imposed by the larger Home exporters. In the absence of sanctions, all Home firms with productivity above the cutoff implied by the fixed cost of trade export to Foreign. When sanctions are introduced, there is a second, higher productivity cutoff, such that it is only the Home firms with productivity between the two cutoffs export to Foreign. In the limit, the sanction-determined cutoff coincides with the trade-cost determined one, and no Home firm exports to Foreign. A similar market exclusion sanction can be imposed on Foreign firms, prohibiting them from exporting to Home.

In this environment, we study the impact of sanctions in the short, medium, and long term. We are interested in their effects on international relative prices, balances, standard macroeconomic aggregates, and ultimately how the Home and Foreign economies respond in terms of welfare.

The overall intuition comes from the product variety effects. The fluctuations in real exchange rate do not only depend on the ratio of effective wages across the border, but also on the number of entrants, exporters, and their average productivity. Sanctions generate a shift of resources in Foreign economy and factor prices play an allocative role by moving the extensive margins of production.

The simulations that we have conducted in this preliminary version of our paper indicate that Foreign economy suffers more from sanctions in terms of the initial response of consumption. In a model with more than two economies, this would translate into a less pronounced response of consumption in Home economy.

Sanctions on Foreign gas exports force the Foreign economy to shift resources towards the consumption good sector. Number of entrants and producers in Foreign increases. Sanctions that reduce the Home consumption good exports force Foreign economy to expand domestic demand. Similarly, this sanction also generates a higher number of entrants and producers in the consumption good sector of Foreign economy.

Financial sanctions generate a more pronounced drop in Foreign consumption only if a greater share of Foreign households are sanctioned. With greater share of Foreign households are sanctioned, entry in Foreign gets dampened. Number of exporters in Foreign increase in order to improve Foreign trade balance. Increase in the number of exporters translates into lower average exporter productivity. Therefore, the average Foreign exporter price increases and Home exchange rate appreciates.

The real exchange rate behavior is qualitatively different under different types of sanctions. The real exchange rate depreciates from the Foreign economy perspective in response to gas sanctions, but appreciates (from the Foreign perspective again) in response to consumption good trade sanctions. The behavior of exchange rate crucially depends on the number of exporters in Foreign. In response to a gas sanction, Foreign expands exports in consumption goods to compensate for the loss of gas exports. In response to a consumption good sanction, domestic demand in Foreign gets stronger and the number of exporters decrease. The former contributes to the depreciation whereas the latter contributes to the appreciation of the exchange rate from Foreign perspective. In response to financial sanctions–by disconnecting a larger share of Foreign households from international bond trade–increase in the number of exporters translates into lower average exporter productivity. Therefore, the average Foreign exporter price increases and Home exchange rate appreciates.

We intentionally plan to keep our setup simple relative to quantitative versions of the Ghironi-Melitz framework that have appeared in the subsequent literature and/or relative to analyses of gas trade sanctions such as that recently produced by Bachmann et al. Our goal is to provide a set of benchmark results on international trade and macroeconomic dynamics in tense times that future literature can build on.

2 The Model

The world is composed of two asymmetric regions, Home and Foreign. Both Home and Foreign are populated by a unit mass of atomistic households. The representative household in each country consists of two groups of workers who supply labor to the two sectors of the economy, consumption goods producers and gas producers. Labor is assumed immobile across the two sectors in each country and across countries. Home is an importer of gas, whereas Foreign is an exporter of gas. We use Melitz (2003) monopolistic competition and heterogenous producers framework for the microeconomic underpinning of the consumption good producing sector as in GM. Prices are flexible. Figure 1 exhibits the model architecture.

2.1 Household Preferences

The representative household obtains utility from consumption of a basket of goods, C_t, and disutility from supplying labor, L_t, to the sector that produces consumption goods and $L_{G,t}$ to the sector that produces gas. The expected intertemporal utility function that the household maximizes is:

$$\mathbb{E}_t \left[\sum_{s=t}^{\infty} \beta^{s-t} \left(logC_s - \frac{\kappa}{2} L_s^2 - \frac{\kappa_G}{2} L_{G,s}^2 \right) \right]$$

with $\beta \in (0,1)$ and $\kappa, \kappa_G > 0$. The consumption basket is defined over a continuum of goods Ω: $C_t = \left(\int_{\omega \in \Omega} c_t(\omega)^{\frac{\theta-1}{\theta}} d\omega \right)^{\frac{\theta}{\theta-1}}$ where $\theta > 1$ is the symmetric elasticity of substitution across goods. At any time t, only a subset of goods $\Omega_t \subset \Omega$ is available. Demand for individual goods is $c_t(\omega) = \left(\frac{p_t(\omega)}{P_t} \right)^{-\theta} C_t$ where $p_t(\omega)$ is the home currency price of a good $\omega \in \Omega_t$ and $P_t = \left(\int_{\omega \in \Omega_t} p_t(\omega)^{1-\theta} d\omega \right)^{\frac{1}{1-\theta}}$. Letting $\rho_t(\omega)$ be the price of good ω relative to the price of the basket, demand for good ω is $c_t(\omega) = (\rho_t(\omega))^{-\theta} C_t$. Everything is similar in Foreign unless otherwise noted. Foreign variables are denoted with a star, and the location of gas use or good consumption below is denoted with a subscript H or F.

2.2 Gas Production

Home and Foreign are endowed with amounts of natural gas G_N and G_N^*, respectively, and we assume that Foreign has a larger endowment, *i.e.*, $G_N^* > G_N$. A perfectly competitive, upstream sector in each country produces usable gas by combining labor and natural gas. Production of usable gas by Home is

$$G_t = G_N L_{G,t}.$$

This gas can be used domestically ($G_{H,t}$) or exported ($G_{F,t}$). Hence, in equilibrium, it will be $G_N L_{G,t} = G_{H,t} + G_{F,t}$. Similarly, Foreign production of usable gas is

$$G_t^* = G_N^* L_{G,t}^*,$$

and equilibrium will imply $G_N^* L_{G,t}^* = G_{H,t}^* + G_{F,t}^*$. First-order conditions for optimal labor demand in gas production in Home and Foreign imply, respectively, $w_{G,t} = \rho_{G,t} G_N$ and $w_{G,t}^* = \rho_{G,t}^* G_N^*$, where $w_{G,t}$ and $w_{G,t}^*$ are the real wages paid to workers in this sector in Home and Foreign, and $\rho_{G,t}$ and $\rho_{G,t}^*$ are the real prices of usable gas in the two countries (both wages and prices are in units of the relevant country's consumption basket). Foreign exports gas to Home. Home and foreign produced gas is perfectly substitutable, and thus home gas market price determination ensures $\rho_{G,t} = \tau_{G,t} Q_t \rho_{G,t}^*$, where $\tau_{G,t}$ is iceberg gas trade costs, and Q_t is the consumption-based real exchange rate (units of Home consumption per unit of Foreign).

2.3 Consumption Good Production

Consumption Goods Producer Differentiated consumption goods are produced by monopolistically competitive firms using gas and labor as inputs. Home and Foreign gas are perfect substitutes in production of consumption goods. Home firm ω produces output $y_t(\omega)$ of good ω with production function:

$$y_t(\omega) = z Z_t \left(G_{H,t}(\omega) + \frac{G_{H,t}^*(\omega)}{\tau_{G,t}} \right)^{\alpha} L_t(\omega)^{1-\alpha},$$

where z is exogenous, heterogeneous productivity determined upon firm entry, Z_t is an exogenous sector-wide productivity shock, $G_{H,t}(\omega) + G_{H,t}^*(\omega)/\tau_{G,t}$ is the firm's total use of gas (domestic and imported, with gas import subject to an iceberg trade cost $\tau_{G,t} \geq 1$), $L_t(\omega)$ is the firm's use of labor, and $0 \leq \alpha < 1$. We set Foreign not to import gas from Home. Foreign firms use only domestic gas, $G_{F,t} = 0$.

Using w_t to denote the real wage paid to consumption-sector workers (in units of consumption), the firm's marginal cost is $\rho_{G,t}^{\alpha} w_t^{1-\alpha}/(z Z_t)$. Given Dixit-Siglitz preferences, the real price charged by the firm for sales in the Home market is

$$\rho_{H,t}(z) = \left(\frac{\theta}{\theta - 1}\right) \frac{\rho_{G,t}^{\alpha} w_t^{1-\alpha}}{z Z_t},$$

where we dropped the identifier ω and replaced it with the heterogeneous productivity z. Exporting is costly, and producers are subject to an iceberg trade cost, $\tau_t \geq 1$, and a per-period fixed export cost, $f_{X,t}$. The fixed export cost requires use of consumption-sector labor with effectiveness determined by the aggregate shock Z_t. We assume that $f_{X,t}$ is in units of effective labor. Hence, the fixed export cost in units of consumption is $w_t f_{X,t}/Z_t$. The fixed export cost implies that only firms with sufficiently high productivity z will export. The iceberg cost implies that, if a firm exports, the price it charges in the Foreign market (in units of the Foreign consumption basket) is

$$\rho_{F,t}(z) = \left(\frac{\theta}{\theta - 1}\right) \frac{\tau_t \rho_{G,t}^{\alpha} w_t^{1-\alpha}}{Q_t z Z_t}.$$

Total firm profits $d_t(z)$, distributed to households as dividends, are equal to $\rho_{H,t}(z) c_{H,t}(z) + Q_t \rho_{F,t}(z) c_{F,t}(z) - \rho_{G,t}\left(G_{H,t}(z) + G^*_{H,t}(z)\right) - w_t L_t(z) - I_t(z) \frac{w_t}{Z_t} f_{X,t}$, where $c_{H,t}(z) = (\rho_{H,t}(z))^{-\theta} C_t$, $c_{F,t}(z) = (\rho_{F,t}(z))^{-\theta} C^*_t$, and $I_t(z)$ is an indicator function that takes the value 1 if the firm exports and 0 otherwise. Firm z exports if and only if the profit from exporting is positive, i.e., if and only if $\left[Q_t \rho_{F,t}(z) - \rho_{G,t}^{\alpha} w_t^{1-\alpha}/(z Z_t)\right] c_{F,t}(z) > w_t f_{X,t}/Z_t$. This condition implies a cutoff productivity level $z_{X,t}$ such that $\left[Q_t \rho_{F,t}(z) - \rho_{G,t}^{\alpha} w_t^{1-\alpha}/(z Z_t)\right] c_{F,t}(z_{X,t}) = w_t f_{X,t}/Z_t$. Only firms with $z > z_{X,t}$ export to Foreign.

First-order conditions for optimal demand of Home gas, Foreign gas, and labor by Home firm z lead to

$$\alpha w_t L_t(z) = (1-\alpha)\left(\rho_{G,t} G_{H,t}(z) + Q_t \rho^*_{G,t} G^*_{H,t}(z)\right)$$

and

$$\rho_{G,t} = Q_t \rho^*_{G,t} \tau_{G,t}.$$

Number of Firms, Exporters, and Their Averages Following Melitz (2003), define the market-share weighted productivity average $\tilde{z}_D$ for all producing firms in each country as:

$$\tilde{z}_D \equiv \left(\int_{zmin}^{\infty} z^{\theta-1} dF(z)\right)^{\frac{1}{\theta-1}}, \tag{1}$$

and the market-share weighted productivity averages for Home and Foreign exporters as, respectively:

$$\tilde{z}_{X,t} \equiv \left(\frac{1}{1-F(z_{X,t})}\int_{z_{X,t}}^{\infty} z^{\theta-1} dF(z)\right)^{\frac{1}{\theta-1}}, \tag{2}$$

and

$$\tilde{z}^*_{X,t} \equiv \left(\frac{1}{1-F\left(z^*_{X,t}\right)}\int_{z^*_{X,t}}^{\infty} z^{\theta-1} dF(z)\right)^{\frac{1}{\theta-1}}. \tag{3}$$

As shown by Melitz (2003), the model is isomorphic to one in which $N_{D,t}$ ($N^*_{D,t}$) firms with productivity $\tilde{z}_D$ produce in the Home (Foreign) country and $N_{X,t}$ ($N^*_{X,t}$) firms with productivity $\tilde{z}_{X,t}$ ($\tilde{z}^*_{X,t}$) export to Foreign (Home). The expression of the Home price index P_t then implies $N_{D,t}\left(\tilde{\rho}_{D,t}\right)^{1-\theta} + N^*_{X,t}\left(\tilde{\rho}^*_{X,t}\right)^{1-\theta} = 1$, where $\tilde{\rho}_{D,t} \equiv \rho_{D,t}(\tilde{z}_D)$ and $\tilde{\rho}^*_{X,t} \equiv \rho^*_{X,t}\left(\tilde{z}^*_{X,t}\right)$ are the average relative prices of Home producers and Foreign exporters in the Home market. Moreover, given average profits from domestic and export sales$\tilde{d}_{D,t} \equiv d_{D,t}(\tilde{z}_D)$ and $\tilde{d}_{X,t} \equiv d_{X,t}(\tilde{z}_{X,t})$, average total profits of Home firms are $\tilde{d}_t \equiv \tilde{d}_{D,t} + (1-F(z_{X,t}))\,\tilde{d}_{X,t}$, where $1-F(z_{X,t})$ is the proportion of Home firms that export, i.e., $1-F(z_{X,t}) = N_{X,t}/N_{D,t}$.

Firm Entry and Exit There is an unbounded mass of potential entrants in each country. Entry requires use of consumption-sector labor with effectiveness determined by the aggregate shock Z_t. Prior to entry, all firms are identical and face a sunk entry cost $f_{E,t}$ in units of effective labor. Hence, the sunk entry cost in units of consumption is $w_t f_{E,t}/Z_t$. Upon entry, firms draw the firm-specific productivity level z from a cumulative distribution function $F(z)$ with support $[z_{min}, \infty)$. This productivity level remains fixed thereafter. We assume that $f^*_{E,t} \geq f_{E,t}$, allowing for the possibility that the gas-rich country features less consumption-sector firms as a consequence of inefficiencies of various type that can characterize the firm creation process.

We also assume a one-period time-to-build requirement: It takes one period between the time of entry and the time when firms start producing and generating profits. All firms in the economy, incumbent and new entrants, are subject to an exogenous shock that causes them to exit with probability $\delta \in (0,1)$ at the end of each period. Therefore, the mass $N_{D,t}$ of producing Home firms in period t is determined by $N_{D,t} = (1-\delta)\left(N_{D,t-1} + N_{E,t-1}\right)$, where $N_{E,t-1}$is the number of firms that entered in period t-1.

Given these definition, firm entry decisions are determined as follows. Prospective entrants are forward looking and compute the rational expectation of the stream of average total profits that they will generate post entry. This determines the average value of an entrant, $\tilde{v}_t$, as:

$$\tilde{v}_t \equiv E_t \left\{ \sum_{s=t+1}^{\infty} \left[\beta\left(1-\delta\right)\right]^{s-t} \left(\frac{C_s}{C_t}\right)^{-1} \tilde{d}_s \right\}. \tag{4}$$

Entry occurs until this value is equated to the sunk entry cost, implying the free-entry condition $\tilde{v}_t = w_t f_{E,t}/Z_t$. We assume that macroeconomic shocks are never large enough to cause zero entry in any period (or $\tilde{v}_t < w_t f_{E,t}/Z_t$) so that the entry condition always holds with equality (in other words, there is always a positive number of entrants). Since both new entrants and incumbent firms face the same probability δ of exit at the end of each period regardless of their firm-specific productivity, $\tilde{v}_t$ is also the average value of incumbent firms after production has occurred.

2.4 Household Budget Constraint, Asset Holding, and Labor Supply Decisions

International financial markets are incomplete as only non-contingent, riskless real bonds are traded internationally. The representative Home household's Holdings of Home bonds entering period t are denoted with $B_{H,t}$. The household receives the risk-free real interest rate r_t on these bonds during period t. The household's Holdings of Foreign real bonds entering period t are denoted with $B^*_{H,t}$, and they pay the risk-free real interest rate r^*_t (Foreign bonds and interest rate are in units of Foreign consumption). We assume that firms are fully owned domestically. Specifically, the representative household enters the period with share holdings x_t in a mutual fund of $N_{D,t}$ Home producing firms. During period t, the household receives dividends from its share holdings, $\tilde{d}_t$ per share, and the value of selling its share portfolio at the price $\tilde{v}_t$ per share. Besides its financial assets and the income they generate, the representative household's resources in period t also include the income from labor supplied in the gas production sector $(w_{G,t}L_{G,t})$ and in the consumption sector ($w_t L_t$). Finally, the household also receives a lump-sum rebate of fees that it pays to financial intermediaries in order to enter period $t+1$ (these fees serve the purpose of pinning down holdings of Home and Foreign bonds at their steady state values in the deterministic steady state of the model). During period t, the household uses its resources to buy consumption, to buy bonds with which it will enter period $t+1$ ($B_{H,t+1}$ and $B^*_{H,t+1}$), to pay fees $0.5\eta(B_{H,t+1} - B_H)^2$ and $0.5\eta Q_t(B^*_{H,t+1} - B^*_H)^2$, with . Also, the household buys share holdings x_{t+1}in a mutual fund of $N_t \equiv N_{D,t} + N_{E,t}$ firms with fees, $0.5\eta(x_{t+1} - 1)^2$. Only $1 - \delta$ of these N_t firms will be around to produce and generate profits in period $t + 1$. The household does not know which firms will be hit by the exit-inducing shock and, therefore, it finances continued operations by all currently producing firms and entry by all producers who choose to enter the market, with the risk of firm exit at the end of period t reflected in the share price that will be determined by the Euler equation for optimal share holdings. The budget constraint of the representative Home household is thus:

$$\begin{aligned} C_t + B_{H,t+1} + Q_t B^*_{H,t+1} + \tfrac{\eta}{2}(B_{H,t+1} - B_H)^2 + \tfrac{\eta}{2} Q_t (B^*_{H,t+1} - B^*_H)^2 + \tilde{v}_t N_t x_{t+1} + \tfrac{\eta}{2}(x_{t+1} - 1)^2 \\ = (1 + r_t) B_{H,t} + Q_t (1 + r^*_t) B^*_{H,t} + w_{G,t} L_{G,t} + w_t L_t + \left(\tilde{d}_t + \tilde{v}_t\right) N_{D,t} x_t + T^f_t . \end{aligned} \tag{5}$$

where $T^f_t = 0.5\eta(B_{H,t+1} - B_H)^2 + 0.5\eta Q_t(B^*_{H,t+1} - B^*_H)^2 + 0.5\eta(x_{t+1} - 1)^2$.

The Euler equations for optimal holdings of Home and Foreign bonds are, respectively:

$$C_t^{-1} \left(1 + \eta(B_{H,t+1} - B_H)\right) = \beta \left(1 + r_{t+1}\right) E_t \left(C_{t+1}^{-1}\right), \tag{6}$$

and

$$C_t^{-1} \left(1 + \eta(B^*_{H,t+1} - B^*_H)\right) = \beta \left(1 + r^*_{t+1}\right) E_t \left(\frac{Q_{t+1}}{Q_t} C_{t+1}^{-1}\right). \tag{7}$$

The Euler equation for optimal share holdings implies:

$$\tilde{v}_t = \beta \left(1 - \delta\right) E_t \left[\left(\frac{C_{t+1}}{C_t}\right)^{-1} \left(\tilde{v}_{t+1} + \tilde{d}_{t+1}\right)\right]. \tag{8}$$

Forward iteration of this equation and the relevant transversality condition imply the expression for $\tilde{v}_t$ in the free-entry condition above, thus establishing the general equilibrium link between firm entry decisions and household decisions regarding the financing of entry.

Finally, the first-order conditions for optimal supply of labor to the gas and consumption sectors are $\kappa_G L_{G,t} = w_{G,t}/C_t$ and $\kappa L_t = w_t/C_t$.

2.5 Market Clearing and Aggregate Accounting

The price of usable gas, $\rho_{G,t}$, is determined by gas market clearing conditions:

$$G_N L_{G,t} = G_{H,t},$$

$$G_N^* L_{G,t}^* = G_{H,t}^* + G_{F,t}^*,$$

where $G_{H,t} = N_{D,t} G_{H,t}(\tilde{z}_D) + N_{X,t} G_{H,t}(\tilde{z}_{X,t})$, $G_{H,t}^* = N_{D,t} G_{H,t}^*(\tilde{z}_D) + N_{X,t} G_{H,t}^*(\tilde{z}_{X,t})$, and $G_{F,t}^* = N_{D,t}^* G_{F,t}^*(\tilde{z}_D) + N_{X,t}^* G_{F,t}^*\left(\tilde{z}_{X,t}^*\right)$.

Market clearing for individual goods requires $y_t(z) = c_{H,t}(z) + c_{F,t}(z)$ for the product of a Home firm with productivity z and $y_t^*(z) = c_{H,t}^*(z) + c_{F,t}^*(z)$ for the product of a Foreign firm with the same productivity.

Labor market clearing in gas production in Home and Foreign requires $L_{G,t} = w_{G,t}/(\kappa_G C_t)$ and $L_{G,t}^* = w_{G,t}^*/(\kappa_G C_t^*)$, respectively. Since $w_{G,t} = \rho_{G,t} G_N$ and $w_{G,t}^* = \rho_{G,t}^* G_N^*$, it follows that $L_{G,t} = \rho_{G,t} G_N/(\kappa_G C_t)$ and $L_{G,t}^* = \rho_{G,t}^* G_N^*/(\kappa_G C_t^*) = \rho_{G,t} G_N^*/(\kappa_G Q_t C_t^*)$, where the last equality uses the fact that $\rho_{G,t} = \tau_{G,t} Q_t \rho_{G,t}^*$. Ceteris paribus, the amount of labor employed in gas production in each country is larger the larger the country's endowment of natural gas and the higher the price of gas; instead, labor in the gas sector is smaller the higher the country's consumption and, intuitively, the higher the weight of the disutility of labor. Since a real depreciation of the Home currency (an increase in Q_t) causes a higher real price of usable gas in Home, it causes a decrease in gas-sector employment in Foreign, as there is an incentive to shift production to Home.

Labor market clearing in the consumption sectors of the two countries requires

$$N_{D,t} L_t(\tilde{z}_D) + N_{X,t} L_t(\tilde{z}_{X,t}) + N_{E,t} \frac{f_{E,t}}{Z_t} + N_{X,t} \frac{f_{X,t}}{Z_t} = \frac{w_t}{\kappa C_t}$$

and

$$N_{D,t}^* L_t^*(\tilde{z}_D) + N_{X,t}^* L_t^*\left(\tilde{z}_{X,t}^*\right) + N_{E,t}^* \frac{f_{E,t}^*}{Z_t^*} + N_{X,t}^* \frac{f_{X,t}^*}{Z_t^*} = \frac{w_t^*}{\kappa C_t^*}$$

.

Market clearing for bonds issued by Home requires $B_{H,t+1} + B_{F,t+1} = B_{H,t} + B_{F,t} = 0$ in every period, and for bonds issued by Foreign: $B^*_{H,t+1} + B^*_{F,t+1} = B^*_{H,t} + B^*_{F,t} = 0$ in every period. Stock market clearing in each country requires $x_{t+1} = x_t = 1$ and $x^*_{t+1} = x^*_t = 1$ in every period. Since costs of adjusting bond holdings away from zero are rebated back to households in equilibrium, imposing equilibrium conditions on the household budget constraint yields:

$$C_t + \tilde{v}_t N_{E,t} + B_{H,t+1} + Q_t B^*_{H,t+1} = (1 + r_t) B_{H,t} + Q_t (1 + r^*_t) B^*_{H,t} + w_{G,t} L_{G,t} + w_t L_t + N_{D,t} \tilde{d}_t, \quad (9)$$

in Home and:

$$C^*_t + \tilde{v}^*_t N^*_{E,t} + \frac{B_{F,t+1}}{Q_t} + B^*_{F,t+1} = \frac{(1 + r_t)}{Q_t} B_{F,t} + (1 + r^*_t) B^*_{F,t} + w^*_{G,t} L^*_{G,t} + w^*_t L^*_t + N^*_{D,t} \tilde{d}^*_t. \quad (10)$$

These two equations together, and bond market equilibrium, imply that Home net foreign assets obey the law of motion:

$$\begin{aligned} & B_{H,t+1} + Q_t B^*_{H,t+1} \\ & = (1 + r_t) B_{H,t} + Q_t (1 + r^*_t) B^*_{H,t} + \tfrac{1}{2} \left(w_{G,t} L_{G,t} - Q_t w^*_{G,t} L^*_{G,t} \right) + \tfrac{1}{2} (w_t L_t - Q_t w^*_t L^*_t) \\ & \quad + \tfrac{1}{2} \left(N_{D,t} \tilde{d}_t - Q_t N^*_{D,t} \tilde{d}^*_t \right) - \tfrac{1}{2} (C_t - Q_t C^*_t) - \tfrac{1}{2} \left(\tilde{v}_t N_{E,t} - Q_t \tilde{v}^*_t N^*_{E,t} \right), \end{aligned} \quad (11)$$

or that Home's current account is determined by:

$$CA_t \equiv B_{H,t+1} + Q_t B^*_{H,t+1} - \left(B_{H,t} + Q_t B^*_{H,t} \right) = r_t B_{H,t} + Q_t r^*_t B^*_{H,t} + TB_t, \quad (12)$$

where TB_t is the trade balance:

$$\begin{aligned} TB_t \equiv \; & \tfrac{1}{2} \left(w_{G,t} L_{G,t} - Q_t w^*_{G,t} L^*_{G,t} \right) + \tfrac{1}{2} (w_t L_t - Q_t w^*_t L^*_t) \\ & + \tfrac{1}{2} \left(N_{D,t} \tilde{d}_t - Q_t N^*_{D,t} \tilde{d}^*_t \right) - \tfrac{1}{2} (C_t - Q_t C^*_t) - \tfrac{1}{2} \left(\tilde{v}_t N_{E,t} - Q_t \tilde{v}^*_t N^*_{E,t} \right) \end{aligned} \quad (13)$$

Finally, the trade balance can be rewritten as:

$$TB_t \equiv \frac{1}{2} (Y_t - Q_t Y^*_t) - \frac{1}{2} (C_t - Q_t C^*_t) - \frac{1}{2} \left(\tilde{v}_t N_{E,t} - Q_t \tilde{v}^*_t N^*_{E,t} \right) \quad (14)$$

once we recognize that $w_{G,t}L_{G,t} + w_t L_t + N_{D,t}\tilde{d}_t$ is total Home income from labor and dividends (or Home GDP, Y_t) and $w^*_{G,t}L^*_{G,t} + w^*_t L^*_t + N^*_{D,t}\tilde{d}^*_t$ is total Foreign income from labor and dividends (or Foreign GDP, Y^*_t). Home and Foreign current accounts and trade balances are such that $CA_t + Q_t CA^*_t = TB_t + Q_t TB^*_t = 0$.

3 Analytical Insights

Like the GM model we build on, our model cannot be fully solved analytically. However, it is possible to obtain intermediate analytical results on key variables of interest. We present some of these results below, focusing on two prices: the price of gas and the real exchange rate.

3.1 Gas Price

Using gas market clearing conditions, production functions, optimal prices, and marginal cost expressions, it is possible to express the price of gas, $\rho_{G,t}$, as:

$$\rho_{G,t} = \frac{(1-\alpha)^{(1-\alpha)(\theta-1)}\,\kappa_G C_t}{G_N^2} \left\{ \frac{1 + \xi_t + \tau_{G,t}^{-1}\left[1 - (1-\alpha)^{(1-\alpha)(\theta-1)}\right]\xi_t}{\left[1 + \tau_{G,t}^{-1}\left[1 - (1-\alpha)^{(1-\alpha)(\theta-1)}\right]\xi_t\right](1+\xi_t)} \right\} \tag{15}$$

where $\xi_t \equiv (G^*_N/G_N)^2\left[\kappa_G/(\kappa^*_G \tau_{G,t})\right]\left[C_t/(Q_t C^*_t)\right]$.[1] For given level of gas trade cost, $\tau_{G,t}$, fluctuations in the price of gas paid by Home consumption-sector firms are driven by fluctuations in Home consumption and in the extent to which the relation between Home and Foreign consumptions deviates from the complete markets outcome (under complete markets, the ratio $C_t/(Q_t C^*_t)$ would be constant, and changes in $\tau_{G,t}$ would be the only reason for ξ_t to move).

To build intuition for the implications of equation (15), suppose that markets are indeed complete, so that, up to a constant, $C_t = Q_t C^*_t$. Suppose also that $\tau_{G,t} = 1$, $G_N = G^*_N$, and $\kappa_G = \kappa^*_G$. then, equation (15) becomes:

$$\rho_{G,t} = \frac{(1-\alpha)^{(1-\alpha)(\theta-1)}\,\kappa_G C_t}{G_N^2} \left\{ \frac{3 - (1-\alpha)^{(1-\alpha)(\theta-1)}}{2\left[2 - (1-\alpha)^{(1-\alpha)(\theta-1)}\right]} \right\}. \tag{16}$$

The expression in curly brackets is smaller than 1. It tends to 1 if the share of gas in consumption

[1] See Appendix A.1 for details.

production, α, tends to 0 or 1. Interestingly, both the cases in which there is no international trade in gas ($\alpha \to 0$) or there is the highest need for Home to import gas ($\alpha \to 1$) imply that the price of gas tends to $\kappa_G C_t / G_N^2$. We show in Appendix A.2 that there is a non-monotonicity in gas price behavior as the share of gas in consumption production varies. For given Home consumption, if α is sufficiently high, further increases in α cause a higher gas price. If instead α is sufficiently low, increases in α have the opposite effect on $\rho_{G,t}$. When α is high, the effect of rising α on gas demand prevails, resulting in a higher price. If α is low, demand does not increase enough to offset the effect of substitution toward labor, and the price of gas falls.

The effects of κ_G, C_t, and G_N on $\rho_{G,t}$ in equation (16) are also consistent with intuition: If the weight of the disutility of supplying labor to gas production increases, the price of gas increases as agents reduce gas labor supply. If consumption increases, the price of gas increases, because there is more demand for consumption goods. If efficiency in gas production (or the endowment of natural gas) increases, the price of gas decreases as its supply rises.

In the general case in which $\rho_{G,t}$ is determined by equation (15), we can build intuition by considering the version of equation (15) that is obtained by log-linearizing it around the steady state. We show in Appendix A.3 that it is:

$$\rho_{G,t} = (1 - \Gamma_1)\, \mathsf{C}_t - \mathsf{Q}_t - \mathsf{C}_t^* - (\Gamma_1 - \Gamma_2)\, \tau_{G,t} \tag{17}$$

where Sans Serif fonts denote percentage deviations from the steady state, and the coefficients Γ_1 and Γ_2 are given by, respectively:

$$\begin{aligned}
\Gamma_1 &\equiv \frac{\eta \bar{\xi} \bar{\tau}_G^{-1} \left(1 - \eta \bar{\tau}_G^{-1}\right) \left[1 + 2\bar{\xi}\left(1 + \bar{\xi}\right)\right]}{\left(1 + \bar{\xi}\right)\left(1 + \eta \bar{\xi} \bar{\tau}_G^{-1}\right)\left[1 + \bar{\xi}\left(1 + \eta \bar{\tau}_G^{-1}\right)\right]} > 0, \\
\Gamma_2 &\equiv \frac{\eta \bar{\xi}^2 \bar{\tau}_G^{-1}}{\left(1 + \eta \bar{\xi} \bar{\tau}_G^{-1}\right)\left[1 + \bar{\xi}\left(1 + \eta \bar{\tau}_G^{-1}\right)\right]} > 0.
\end{aligned}$$

In these expressions, $\eta \equiv 1 - (1-\alpha)^{(1-\alpha)(\theta-1)}$, and we denote steady-state levels of variables by dropping the time subscript and using an overbar. Since both η and $\bar{\tau}_G^{-1}$ are between 0 and 1, both Γ_1 and Γ_2 are strictly positive. Assuming $\bar{\xi} < 1$ and $2\eta\bar{\tau}_G^{-1} < 1$ is sufficient (but not necessary) to ensure $\Gamma_1 < 1$ and $\Gamma_1 > \Gamma_2$. If Γ_1 and Γ_2 fulfill these inequalities, the effects of Home consumption, the real exchange rate, foreign consumption, and the iceberg cost of gas trade on the gas price paid by Home firms are intuitive: Higher $\mathsf{C_t}$ causes higher demand of gas for production by Home firms, hence a higher price of gas. The effect of $\mathsf{Q_t}$ in equation (17) is tied to the role of the real exchange rate in international risk sharing and is best understood in conjunction with that of $\mathsf{C_t}^*$.[2] Higher $\mathsf{Q_t} + \mathsf{C_t^*}$ implies an increase in gas demand by Foreign firms relative to Home (given a share on non-traded consumption goods larger than 1/2). Gas demand shifts toward Foreign, causing the gas price in Home to decrease. Higher $\tau_{G,t}$ causes Home firms to reduce their demand of gas, hence a lower price. Any policy action (including sanctions) that causes Home consumption, the real exchange rate, Foreign consumption, and/or the iceberg cost of gas trade to change will have an effect on the price of gas facing Home consumption-sector firms that can be understood based on these results.[3]

A final observation on the gas price $\rho_{G,t}$ concerns its measurement: $\rho_{G,t}$ is measured in units of consumption, i.e., in welfare-consistent units. It can fluctuate because of pure variety effects that are not accounted for in available data on the price index P_t. This implies that, while understanding the dynamics of $\rho_{G,t}$ is important to understand the welfare-effects of sanctions through their impact on the price of gas, if we want to have a model-implied measure of real gas price that can be compared to data, we must deflate the nominal price of gas $p_{G,t}$ using a measure of the Home price index that has been purged of pure variety effects. As in Feenstra (1994) and GM, this measure of the Home price level is given by $\tilde{P}_t \equiv N_t^{\frac{1}{\theta}}$, where $N_t \equiv N_{D,t} + N_{X,t}^*$ is the total number of products available to Home consumers. Deflating $p_{G,t}$ with $\tilde{P}_t$ yields the data-consistent gas price $\tilde{\rho}_{G,t} \equiv p_{G,t}/\tilde{P}_t$. Notice that this gas price is such that $\tilde{\rho}_{G,t} = N_t^{\frac{1}{1-\theta}} \rho_{G,t}$. Hence, the log-linear equation for $\tilde{\rho}_{G,t}$ follows immediately from this relation and equation (17) as:

$$\tilde{\rho}_{G,t} = (1 - \Gamma_1)\mathsf{C_t} - \mathsf{Q_t} - \mathsf{C_t^*} - (\Gamma_1 - \Gamma_2)\tau_{G,t} - \frac{1}{\theta - 1}\mathsf{N_t} \tag{18}$$

In addition to the effects through $\rho_{G,t}$, policy actions affect the data-consistent gas price by changing the number of products available to Home consumers. Actions that reduce product variety in the Home country cause $\tilde{\rho}_{G,t}$ to depreciate. The reason follows from the effect of product variety

[2]With complete markets, we would have $\mathsf{C_t} - \mathsf{C_t^*} = \mathsf{Q_t}$, which would imply that the ceteris paribus scenario of a change in $\mathsf{Q_t}$ in equation (17) without at least one between $\mathsf{C_t}$ and $\mathsf{C_t^*}$ also moving would be impossible.

[3]If Home imposes a full embargo on Foreign gas, there no longer is any arbitrage force that ensures the condition $\rho_{G,t} = \tau_{G,t} Q_t \rho_{G,t}^*$, which is used in obtaining equation (15). In case of a full embargo, the price of gas in Home is determined solely by $\rho_{G,t} = \frac{w_{G,t}}{G_N}$.

on welfare via the price index P_t. Holding product prices constant, this price index decreases if product variety expands, implying that consumers can buy more consumption (and hence obtain more welfare) by spending a given nominal amount. The data-consistent price index $\tilde{P}_t$ removes this variety effect by augmenting with $N_t^{\frac{1}{\theta-1}}$. Since $\tilde{\rho}_{G,t}$ is obtained by deflating $p_{G,t}$ with $\tilde{P}_t$, it follows that higher N_t would cause $\tilde{\rho}_{G,t}$ to decrease, and lower N_t causes it to increase.

3.2 Real Exchange Rate

Similar to the gas price $\rho_{G,t}$, the real exchange rate Q_t is in welfare-consistent units that are not comparable to data because of unmeasured variety effects. As in GM, the data-consistent real exchange rate $\tilde{Q}_t$ is related to Q_t by the equation:

$$\tilde{Q}_t = \left(\frac{N_t^*}{N_t}\right)^{\frac{1}{\theta-1}} Q_t, \tag{19}$$

where $N_t^* \equiv N_{D,t}^* + N_{X,t}$ is the total number of products available to Foreign consumers.

Using price index equations and optimal price setting by Home and Foreign consumption-sector firms yields:

$$\tilde{Q}_t^{1-\theta} = \frac{\frac{N_{D,t}^*}{N_t^*}\left[TOL_t^{1-\alpha}\left(\frac{Z_t}{\tau_{G,t}Z_t^*}\right)^{\alpha}\frac{\tilde{z}_D}{\tilde{z}_D^*}\right]^{1-\theta} + \frac{N_{X,t}}{N_t^*}\left[\frac{\tau\tilde{z}_D}{\tilde{z}_{X,t}}\right]^{1-\theta}}{\frac{N_{D,t}}{N_t} + \frac{N_{X,t}^*}{N_t}\left[TOL_t^{1-\alpha}\left(\frac{Z_t}{\tau_{G,t}Z_t^*}\right)^{\alpha}\frac{\tau^*\tilde{z}_D}{\tilde{z}_{X,t}^*}\right]^{1-\theta}} \tag{20}$$

where $TOL_t \equiv Q_t(w_t^*/Z_t^*)/(w_t/Z_t)$. As in GM, this variable measures the relative cost of effective labor in the two countries. Interestingly, gas prices do not enter the real exchange rate expression directly. Factor prices enter the equation through cross-country ratios of variables. The ratio of Home to Foreign gas prices is such that $\rho_{G,t}/(Q_t\rho_{G,t}^*) = \tau_{G,t}$. Hence, only the iceberg cost paid by Home (the importer) appears in equation (20). In addition to the terms of labor and the iceberg cost of gas trade, the real exchange rate can change because of changes in the total number of products available to Home and Foreign consumers, in the numbers of producers serving the domestic or export market, and in average export productivities.

Consider a permanent decline in Home gas imports, a scenario that we study below as resulting from gas sanctions. In response to lower Home demand of Foreign gas, resources in the Foreign economy will be shifted toward production of consumption goods in order to sustain exports by increasing consumption-sector output. This translates into an increase in labor demand by Foreign consumption good producers, which puts upward pressure on consumption good sector wages. In turn, this leads to a depreciation (an increase) in TOL_t. We will show below that, to a first order, terms of labor depreciation is associated with depreciation of $\tilde{Q}_t$.

As for the gas price, we can build intuition on the determinants of the real exchange rate by considering the log-linear version of equation (20). Letting NUM_t denote the numerator of the expression in (20) and DEN_t the denominator, it is:

$$\tilde{\mathsf{Q}}_t = \frac{\overline{NUM} \cdot dDEN_t - \overline{DEN} \cdot dNUM_t}{(\theta - 1) \cdot \overline{NUM} \cdot \overline{DEN}} \tag{21}$$

where d is the differentiation operator. Hence, up to the constant $\frac{1}{(\theta-1)\cdot\overline{NUM}\cdot\overline{DEN}}$, the behavior of $\tilde{\mathsf{Q}}_t$ is determined by $\overline{NUM} \cdot dDEN_t - \overline{DEN} \cdot dNUM_t$. We show in Appendix A.4 that:

$$\begin{aligned}
&\overline{NUM} \cdot dDEN_t - \overline{DEN} \cdot dNUM_t \\
&= (\theta - 1)(\Phi_1 - \Phi_2)[(1-\alpha)\mathsf{TOL}_t + \alpha(\mathsf{Z}_t - \mathsf{Z}_t^* - \tau_{G,t})] \\
&\quad (\theta - 1)(\Phi_2 + \Phi_4)\tilde{z}^*_{X,t} - (\Phi_2 + \Phi_3)(\tilde{z}_{X,t} - \tau_t) \\
&\quad +\Phi_1[\mathsf{N_{D,t}} - \mathsf{N_t} - (\mathsf{N^*_{D,t}} - \mathsf{N^*_t})] \\
&\quad +\Phi_2[\mathsf{N^*_{X,t}} - \mathsf{N_t} - (\mathsf{N_{X,t}} - \mathsf{N^*_t})] \\
&\quad -\Phi_3[\mathsf{N_{X,t}} - \mathsf{N^*_t} - (\mathsf{N_{D,t}} - \mathsf{N_t})] \\
&\quad +\Phi_4[\mathsf{N^*_{X,t}} - \mathsf{N_t} - (\mathsf{N^*_{D,t}} - \mathsf{N^*_t})],
\end{aligned} \tag{22}$$

where

$$\Phi_1 \equiv \chi_1 \left(\frac{\bar{N}_D}{\bar{N}}\right)^2 \left(\overline{TOL}^{1-\alpha}\bar{\tau}_G^{-\alpha}\right)^{1-\theta} > 0,$$

$$\Phi_2 \equiv \gamma\chi_1 \left(\frac{\bar{N}^*_X}{\bar{N}}\right)^2 \left(\overline{TOL}^{1-\alpha}\bar{\tau}_G^{-\alpha}\chi_2\right)^{1-\theta} \left(\frac{\tau^*\tilde{z}_D}{\bar{\tilde{z}}^*_X}\right)^{2(1-\theta)} > 0,$$

$$\Phi_3 \equiv \gamma\chi_1 \frac{\bar{N}_D\bar{N}^*_X}{\bar{N}^2} \left(\chi_2\frac{\tau^*\tilde{z}_D}{\bar{\tilde{z}}^*_X}\right)^{1-\theta} > 0,$$

$$\Phi_4 \equiv \chi_1 \frac{\bar{N}_D\bar{N}^*_X}{\bar{N}^2} \left(\frac{\tau^*\tilde{z}_D}{\bar{\tilde{z}}^*_X}\right)^{1-\theta} \left(\overline{TOL}^{1-\alpha}\bar{\tau}_G^{-\alpha}\right)^{2(1-\theta)} > 0,$$

and we assumed $\bar{Z} = \bar{Z}^* = 1$. In the expressions above, the parameters χ_1, χ_2, and γ are defined implicitly by:

$$\frac{\bar{N}_D^*}{\bar{N}^*} = \chi_1 \frac{\bar{N}_D}{\bar{N}}, \quad \frac{\bar{N}_X}{\bar{N}^*} = \gamma\chi_1 \frac{\bar{N}_X^*}{\bar{N}}, \quad \text{and} \ \left(\frac{\tau \tilde{z}_D}{\bar{\tilde{z}}_X}\right)^{1-\theta} = \left(\chi_2 \frac{\tau^* \tilde{z}_D}{\bar{\tilde{z}}_X^*}\right)^{1-\theta}.$$

Equation (22) (or, more precisely, the equation that follows from combining equation (21) and equation (22)) is a more complicated version of the log-linear equation that is central to understanding real exchange rate dynamics in GM. Our version of the equation is more complicated because of the two-sector structure of production in each country and the fact that the steady state of the model is not symmetric. Nevertheless, it is still possible to obtain an equation that, to a first order, disentangles the different determinants of the real exchange rate that are at work in our model.

Consider the effect of TOL_t. We show in Appendix A.4 that $\Phi_1 - \Phi_2 > 0$ if and only if:

$$\left(\frac{\bar{N}_D}{\bar{N}_X^*}\right)^2 > \gamma\chi_2^{1-\theta} \left(\frac{\tau^* \tilde{z}_D}{\bar{\tilde{z}}_X^*}\right)^{2(1-\theta)}.$$

This condition is satisfied for all plausible calibrations of our model. It follows that, ceteris paribus, appreciation of the terms of labor (a downward movement in TOL_t) causes appreciation of the data-consistent real exchange rate (negative $\tilde{\mathsf{Q}}_t$) as in GM.

Higher average productivity of Foreign exporters (higher $\tilde{\mathsf{z}}_{X,t}^*$) causes $\tilde{\mathsf{Q}}_t$ to depreciate because it implies a lower domestic price index $\tilde{P}_t$, as more productive Foreign exporters charge lower prices.

The last four parts of equation (22) capture the effects of changes in the composition of consumption baskets in Home and Foreign. The first term measures the relative share of domestic goods in the total numbers of products available in Home and Foreign. The second term measures the relative share of imported goods in the total numbers of products available in Home and Foreign. If the share of imported goods in total Home variety rises relative to Foreign, the real exchange rate depreciates. An increase in Foreign exporter representation in the Home consumption basket relative to Home exporter representation in the Foreign consumption basket implies a lower price level $\tilde{P}_t$ in Home and a higher price level $\tilde{P}_t^*$ in Foreign because, on average, exporters charge lower prices. Hence, depreciation of $\tilde{\mathsf{Q}}_t$. The third and fourth terms measure the relative share of imported goods in total available variety versus domestic goods in total variety abroad in the two countries. If this share rises for Home, the real exchange rate depreciates; if it rises for Foreign, the real exchange rate appreciates. Consider, for example, the third term: If imported products representation in total variety available in Foreign rises relative to domestic products representation in total variety available in Home, $\tilde{P}_t^*$ falls and $\tilde{P}_t$ rise because, on average, exporters charge lower prices than non-exporters. Similarly, but with opposite effects on $\tilde{\mathsf{Q}}_t$ for the fourth term.

The results in the previous paragraphs help us understand the results of policy actions (including sanctions) that cause changes in the determinants of the real exchange rate. We use these results and those for the price of gas above to guide our interpretation of the numerical exercises in the next section.

4 Model Calibration and Simulations

In this section, we calibrate and solve our model numerically to provide illustrations in response to several exogenous changes in our model, including sanctions. We focus on transitional dynamics in the short, medium, and long term.[4] The intuition from the numerical results are following the mechanisms from our analytical results.

4.1 Calibration

We calibrate the model with parameter values that are widely used in the literature. This allows us to assess the implications of sanctions without the risk of our findings being the product of an unusual calibration. We set the discount factor and firm exit rates to $\beta = 0.99$ and $\delta = 0.025$, respectively. The disutility parameter from working is $\kappa = \kappa_G = 0.75$ to normalize the consumption good sector labor supply by one, approximately. The scale parameter for the costs of adjusting bond/share holdings, η, is 0.025, which is sufficient to induce stationarity. This value implies that this adjustment cost has a negligible impact on model dynamics, other than pinning down the non-stochastic steady state and ensuring mean reversion when shocks are transitory. Following Ghironi and Melitz (2005) again, we set the elasticity of substitution of varieties θ to 3.8. We assume that firm-level productivity z is drawn from a Pareto distribution with lower bound z_{min} and shape

[4] We solve the model as a nonlinear, forward-looking, deterministic system using Dynare's own nonlinear equation solver with line search. This method solves simultaneously all equations for each period, without relying on low-order, local approximations.

parameter k. We set k to 3.4 and normalize z_{min} and f_E to 1. This calibration ensures that z_{min} is smaller than the exporter cutoff, $z_{X,t}$. The fixed cost of exporting, $f_X = 0.085$, implies that in the initial steady state 17 and 24 percent of Home and Foreign firms export their good, respectively. We follow Kim et al. (2021) to set the share of gas in consumption good production and set α to 0.1. We set the iceberg costs for consumption good trade, τ and τ^*, to 1.3, and the iceberg cost for gas trade, τ_G, to 1.1.

An important dimension of our model is the asymmetry between two countries. We deviate from the symmetric two-country standard parameterization when calibrating cross-country productivities and natural gas endowments. First, we assume that Home is a gas importer and Foreign is a gas exporter country. It follows that Home's consumption good producing sector is more productive than Foreign's consumption good producing sector. On the other hand, Home is endowed with smaller natural gas resources than Foreign. Without loss of generality, we set $Z = 1.5$, $Z^* = 1$ and $G_N = 1$, $G^*_N = 1.5$. Our calibration indicates that Home GDP is about 52% larger than the Foreign GDP in the initial steady state, i.e., without sanctions.

On the financial front, Foreign has a positive initial NFA position. We set Home households' initial holdings of Home and Foreign bonds to $B_H = -5$ and $B^*_H = 3$, respectively. Our calibration implies that the value of Foreign households' initial holdings of Home bonds is 118% of Foreign GDP and Foreign NFA position is at 38% of Foreign GDP.

4.2 Effects of a Change in Aggregate Home Technology

First, we study how our model reacts to a permanent positive change in Home technology, before studying the sanctions imposed on Foreign economy. Our experiment with Home technology enables us to compare our model outcome with international trade and macroeconomic dynamics models in the literature that do not include an energy sector and/or assume symmetric countries. To make our comparison clearer, we also simulate our model with varying degrees of gas share in consumption good producer sector.

Figure 2 shows responses to a 1% permanent increase in Home consumption good sector aggregate productivity. Blue, green, and red lines indicate simulations when the share of gas in consumption good production is 20 percent, 10 percent, and 1 percent, respectively, i.e., $\alpha \in \{0.2, 0.1, 0.01\}$. The varying share of gas in consumption good production affects all model variables only quantitatively, except the short run behavior of number of Home exporters, Home exporter productivity cutoff, and data-consistent real exchange rate.

After the permanent change in productivity, Home becomes a more attractive business environment, which leads to larger Home entry in consumption good producer sector. Because of high productivity, marginal costs of production in the consumption good sector goes down in the long run. Consumption good producers demand more labor to expand production and wages for labor in consumption goods sector increase. There is a temporary increase in marginal costs because of higher factor demands after the realization of the shock. The cutoff productivity goes down for the least productive exporter in the short run because more producers are productive enough to cover the fixed export cost and the labor cost in the short run.

On the household side, appreciation in the wages for those employed in consumption good production generates an expansion of labor supply towards that sector but a reduction in the labor supply to gas producing sector. Therefore, the amount of Home produced gas diminishes. This is compensated by importing more Foreign gas in the consumption good production. The increase in gas demand by consumption good firms increase the Home gas price more than Foreign gas price, causing real exchange rate to depreciate (from Home perspective). It is important to note that the short run exchange rate depreciation is independent from variety effects, which is a result of the relative changes in labor supply to consumption good producing and gas producing sectors. In addition, Foreign becomes more concentrated in the gas sector than the consumption good sector as the share of gas gets larger in consumption good production.

4.3 The Sanctions

We assume that sanctions are imposed by Home, and we consider three types of sanctions: consumption good trade sanctions, financial sanctions, and gas trade sanctions.

We introduce consumption good trade sanctions by preventing trade for consumption good producers with productivity above a certain threshold. The idea is that sanctions imply a reduction in the trade of larger producers. Under financial market sanctions, a fraction of Foreign households is excluded from international bond trading and, in the limit, all Foreign households are excluded. To study the effects of gas trade sanctions, we conduct simulations with a permanent fall in gas imports from Foreign that takes place in the first period. The simulations describe the reaction to the shocks until the system returns to a new state of equilibrium.

4.3.1 Consumption Good Trade Sanctions

We introduce sanctions on consumption good trade through imposing another productivity cutoff, $z^S_{X,t}$, for Home producers exporting to Foreign. The sanction in consumption good trade is in two forms. First, Home consumption good producers with higher productivity level than the sanction cutoff level stop exporting to Foreign. Second, Home stops importing from the most productive Foreign producers. We set the sanction cutoff as a function of the number of consumption good exporters in the initial steady state. For example, in our simulations, we pin down the sanction cutoff productivity level by assuming that the top 10 percent most productive consumption good producers stop exporting.

Figure 3 presents transitional dynamics after the introduction of consumption good trade sanctions. Green lines indicate simulations after the introduction of export sanctions (EXS)–Home top 10% productive firms stop exporting to Foreign. Blue lines indicate simulations after the introduction of import sanctions (IMS)–Foreign top 10% productive firms cannot export to Home. Red lines indicate simulations when both import and export sanctions are in place simultaneously (TS).

Our first observation is that, under both export and import sanctions, consumption falls more in Foreign than in Home. Following export sanctions, the most productive producers in the Home export market drop from international trade, but Home economy still faces an external demand due to its comparative advantage of producing consumption goods. After the exclusion of most productive Home exporters, productivity cutoff level for the least Home exporter falls down, making less productive Home producers to join the export market. Therefore, the average price of Home exporters increase. The change in the average price of Home exports pushes exchange rate up, implying depreciation from Home perspective. Domestic demand in Foreign economy goes up because imports from Home became more expensive on average. Production in consumption good sector of the Foreign economy expands to accommodate the rise in domestic demand after the sanction. consumption good producers demand more inputs, namely gas and labor. The rise in factor demand translates into a rise in the prices of gas and labor in the Foreign economy.

In response to import sanctions, the most productive Foreign consumption producers drop from international trade. Less productive Foreign producers start to export, implying a fall in the productivity level of the least productive Foreign exporter. Number of exporters in Foreign increases and Foreign consumption exports become more expensive on average due to fall in Foreign exporter average productivity. The latter implies an appreciation of real exchange rate from Home perspective. Shrinking number of consumption good producers imply less demand for labor. In response, Foreign household labor supply to gas sector increases.

We observe that the effects of export sanctions dominate when both sanctions are introduced simultaneously. The reason is the asymmetricity between two regions. Namely, the relative advantage in producing consumption goods in Home makes export sanctions more effective than import sanctions.

4.3.2 Financial Sanctions

In this subsection, we, first, describe the changes in key relationships after Foreign agents are excluded from trading in international financial markets, and then, discuss the simulations under financial sanctions.

When Home imposes financial sanctions on Foreign, a fraction $\lambda > 0$ of Foreign households are excluded from participating in international financial markets. After the imposition of sanctions, these households can only trade Foreign bonds and shares with other Foreign households. When the entire Foreign economy is subject to financial sanctions with $\lambda = 1$, Foreign operates under financial autarky. In a two-country world, this means that also Home must live under financial autarky, a situation that we relax in a model extension below.

Once financial sanctions are imposed, Foreign population is divided into two groups of households: λ of them who are subject to the sanctions and $1 - \lambda$ who are not. The budget constraint of the representative sanctioned household becomes:

$$\begin{aligned} &C^*_{S,t} + B^*_{S,F,t+1} + \tfrac{\eta}{2}(B^*_{S,F,t+1} - B^*_F)^2 + \tfrac{\eta}{2}\tilde{v}^*_t N^*_t (x^*_{S,t+1} - 1)^2 + \tilde{v}^*_t N^*_t x^*_{S,t+1} \\ &\quad = (1 + r^*_t)\, B^*_{S,F,t} + w^*_{G,t} L^*_{S,G,t} + w^*_t L^*_{S,t} + \left(\tilde{d}^*_t + \tilde{v}^*_t\right) N^*_{D,t} x^*_{S,t} + T^{*f}_{S,t}. \end{aligned} \tag{23}$$

The subscript S denotes households that are subject to sanctions. The sanctioned household cannot trade Home bonds. They can still trade Foreign bonds, but its terminal steady state bond holding is zero, i.e., $B^*_{S,F} = 0$.

The budget constraint of the representative non-sanctioned household is:

$$\begin{aligned} C^*_{NS,t} + \tfrac{B_{F,t+1}}{Q_t} + \tfrac{\eta}{2Q_t}(B_{NS,F,t+1})^2 + B^*_{NS,F,t+1} + \tfrac{\eta}{2}(B^*_{NS,F,t+1})^2 \\ + \tfrac{\eta}{2}\tilde{v}^*_t N^*_t \left[x^*_{NS,t+1} - 1\right]^2 + \tilde{v}^*_t N^*_t x^*_{t+1} \\ = (1+r_t)\,\tfrac{B_{NS,F,t}}{Q_t} + (1+r^*_t)\,B^*_{NS,F,t} \\ + w^*_{G,t} L^*_{NS,G,t} + w^*_t L^*_{NS,t} + \left(\tilde{d}^*_t + \tilde{v}^*_t\right) N^*_{D,t} x^*_t + T^{*f}_{NS,t}. \end{aligned} \tag{24}$$

The subscript NS denotes non-sanctioned households. We omit it from Home and Foreign bond positions vis-a-vis Home households because only non-sanctioned Foreign households can trade bonds internationally. In the terminal steady state, non-sanctioned Foreign households' Home bond holdings are zero after financial sanctions, i.e., $B_{NS,F} = B^*_{NS,F} = 0$, but they can always trade Home bonds.

Asset market clearing conditions in every period t in the presence of financial market sanctions are as follows. Home and Foreign bonds:

$$B_{H,t+1} + (1-\lambda)\,B_{NS,F,t+1} = B_{H,t} + (1-\lambda)\,B_{NS,F,t} = 0;$$

$$B^*_{H,t+1} + (1-\lambda)\,B^*_{NS,F,t+1} + \lambda B^*_{S,F,t+1} = B^*_{H,t} + (1-\lambda)\,B^*_{NS,F,t} + \lambda B^*_{S,F,t} = 0.$$

Home and Foreign shares:

$$x_{t+1} = x_t = 1;$$

$$\lambda x^*_{S,t+1} + (1-\lambda)\,x^*_{NS,t+1} = \lambda x^*_{S,t} + (1-\lambda)\,x^*_{NS,t} = 1.$$

Figures 4 and 5 present transitional dynamics under financial sanctions. The figures plot transitional dynamics from the initial steady state in which Foreign has a positive NFA position to the terminal steady state in which Foreign has zero NFA position. The blue, green, and red lines show simulations for this transitional behavior when 99%, 90%, and 80% of Foreign households are excluded from international financial transactions, respectively (i.e., $\lambda \in \{0.8, 0.9, 0.99\}$).

The immediate observation is that financial sanctions generate a more pronounced drop in Foreign consumption, if a larger proportion of Foreign households is sanctioned. During the transition to zero NFA position, Foreign incentive to front-load entry increases and Foreign expands borrowing from abroad by non-sanctioned Foreign households. Proceedings of borrowing from Home is used for investment in new Foreign consumption good producers. The increase in entry in consumption good production translates into more labor demand and depreciation of terms of labor from Home perspective. Real exchange rate also depreciates in response to depreciation of terms of labor. Sanctioned households increase supply of labor to both consumption good production and gas production sector to compensate the fall in their financial income.

When greater share of households are financially sanctioned, international borrowing dampens and Foreign producer entry slows down. This is due to limitation of generating resources to facilitate entry. In this case, sanctioned households increase their labor supply further. Moreover, Foreign aggregate consumption increases if only a lower fraction of Foreign households are sanctioned. This is because non-sanctioned households increase their consumption while transitioning into a lower NFA steady state and reduce their savings. Sanctioned households cannot expand consumption during the transition. Therefore, if a larger share in Foreign is sanctioned, then Foreign trade balance improves and the number of exporters in Foreign increases (blue lines in Figures 4 and 5). Hence, more Foreign producers start to export and Foreign average export price increases, leading to an appreciation of the exchange rate (Home perspective)

4.3.3 Gas Sanctions

We study gas sanction by focusing on simulations after a permanent drop in Home imports of gas from Foreign in period 1. Figure 6 shows the behavior of several variables after gas sanctions are in place (blue lines). For comparison purposes, we also plot the transition dynamics under trade sanctions (green lines) and financial sanctions (red lines). Gas sanctions are not as effective as combined import and export sanctions of consumption good trade in terms of reducing Foreign consumption. However, in response to gas sanctions Foreign consumption drops more than Home consumption, even in the absence of ability of the Home economy to substitute toward gas imports from other regions/countries.

The fall in demand for Foreign gas induces a drop in gas production in Foreign and a subsequent jump in gas prices in Home. The gas price in Home economy rises because consumption good producing firms demand more domestic gas to compensate for the lost imported gas. With rising gas prices, marginal costs of production in consumption good sector increase. Rising costs are translated into less entrants in Home and the total number of producers decline. Home households increase labor supply to gas production and decrease labor supply to consumption good production.

Foreign economy rebalances itself in the opposite way. To compensate for the loss of gas exports, the economy shifts toward producing more in consumption good sector and import more of consumption goods. Consumption good producers increase demand for labor and wages rise. Whereas the fall in gas production means less labor is needed in the gas sector, and wages in gas sector decrease. The shift in the economy facilitates more entrants into the consumption good sector. Therefore the number of producers in Foreign consumption good sector increases. To compensate for the loss of energy exports, more firms in the consumption goods sector export and the cutoff productivity level for the least efficient exporter in Foreign goes down. This change translates into higher higher average export prices in Foreign, appreciating real exchange rate from Home perspective.

4.3.4 Combinations of Sanctions

In this subsection, we present the combined impact of several sanctions that are interoduced simultaneously. In particular, we consider three cases: (1) combination of financial sanctions and consumption good import and export sanctions (FS&TS), (2) combination of financial and gas sanctions (FS&GS) and, (3) combination of financial, consumption good trade, and gas sanctions, altogether (FS&TS&GS).

Figures 7 and 8 show transitional dynamics when several combination of sanctions are in place. The negative impact on Foreign consumption gets amplified if three sanctions are in place simultaneously. The impact is quantitatively similar when all sanctions applied simultaneously (FS&TS&GS, red solid lines) or when only consumption good trade and financial sanctions (FS&TS, green dashed lines with triangles) applied simultaneously. It is also important to note that all combinations of sanctions also generate a fall in Home consumption and GDP, although the fall is not as large as in Foreign. The effect of export sanction on consumption good trade dominates in terms of magnitudes. Therefore, Home exchange rate depreciates when trade sanctions are combined with any other sanction.

5 Conclusions

We studied sanctions in a model of international trade and macroeconomic dynamics. We examined how sanctions would work and their impact on international relative prices, balances, standard macroeconomic aggregates. Our analysis focuses on the impact of sanctions both in the country they are imposed and in the country which imposes them.

Product variety effects are central to transmission of sanctions. In response to sanctions by prohibiting consumption good exports of Home producers, average Home exporter price increases. Foreign households shift demand to domestically produces goods. Home exchange rate depreciates. In response to gas sanction that is introduced by prohibiting imported Foreign gas, Foreign economy rebalances itself by moving resources to consumption good sector. Number of exporters in Foreign increases, leading to higher average exporter price. Home exchange rate appreciates. Financial sanctions are generating drop in Foreign consumption, only if greater share of Foreign households are sanctioned.

These results are relevant for the ongoing discussions on geopolitical tensions affecting the global economy. Our analysis provides a roadmap on how several sanctions might work. A natural extension of our model would include financial imperfections. When these are included, interactions of sanctions with financial frictions can aggravate the impact on international transmission. We leave this extension for future work.

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Figures

Figure 1: Model Architecture

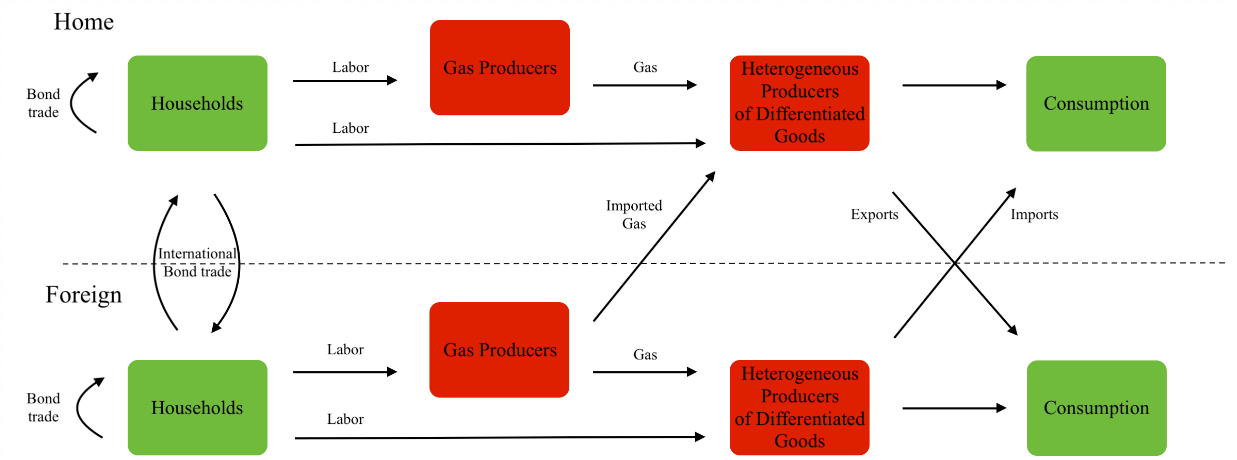


Figure 2: Responses to a 1% Permanent Increase in Home Productivity

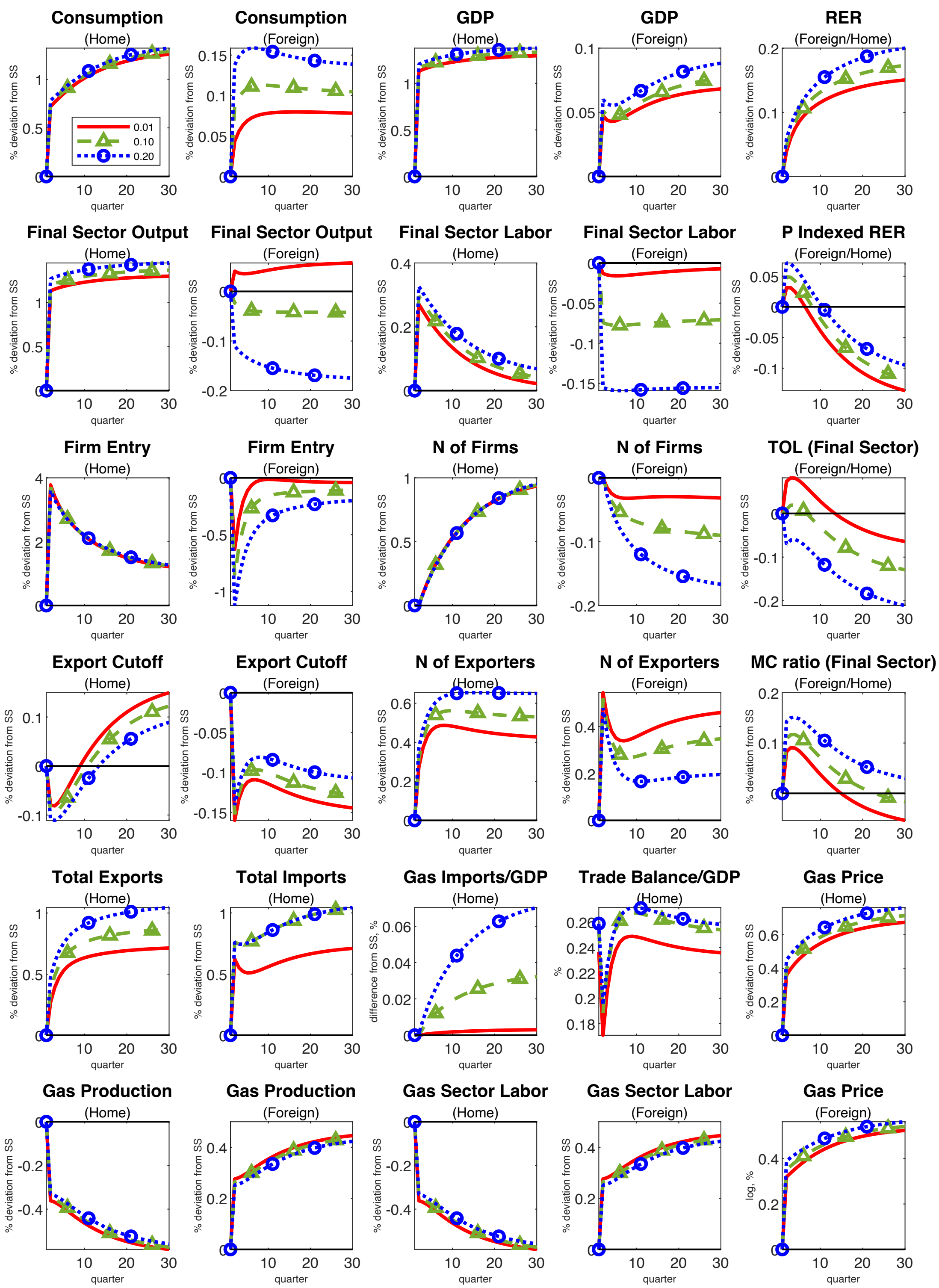


Figure 3: Transitional Dynamics after Consumption Good Export and Import Sanctions

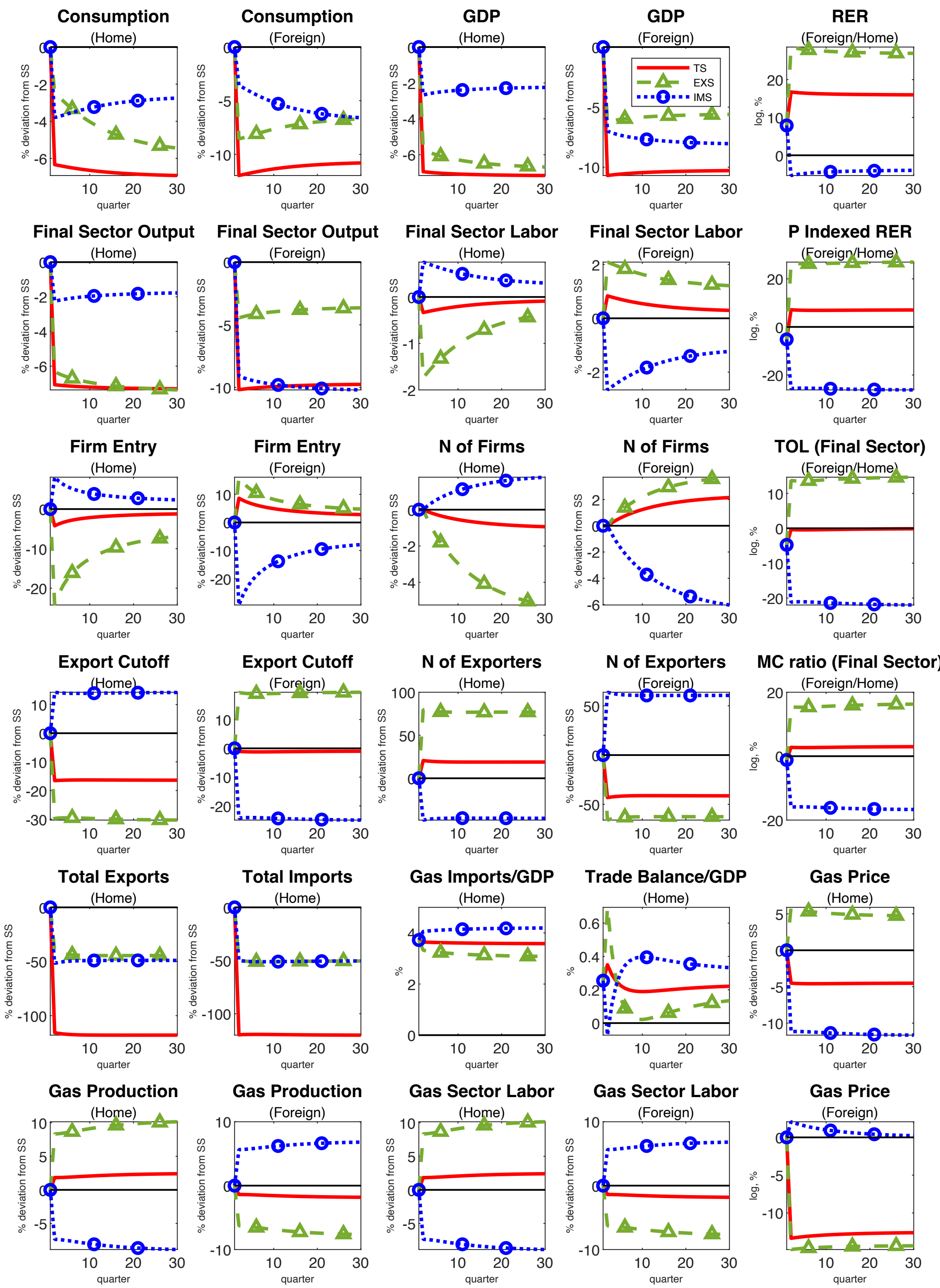


Figure 4: Transitional Dynamics after Financial Sanctions

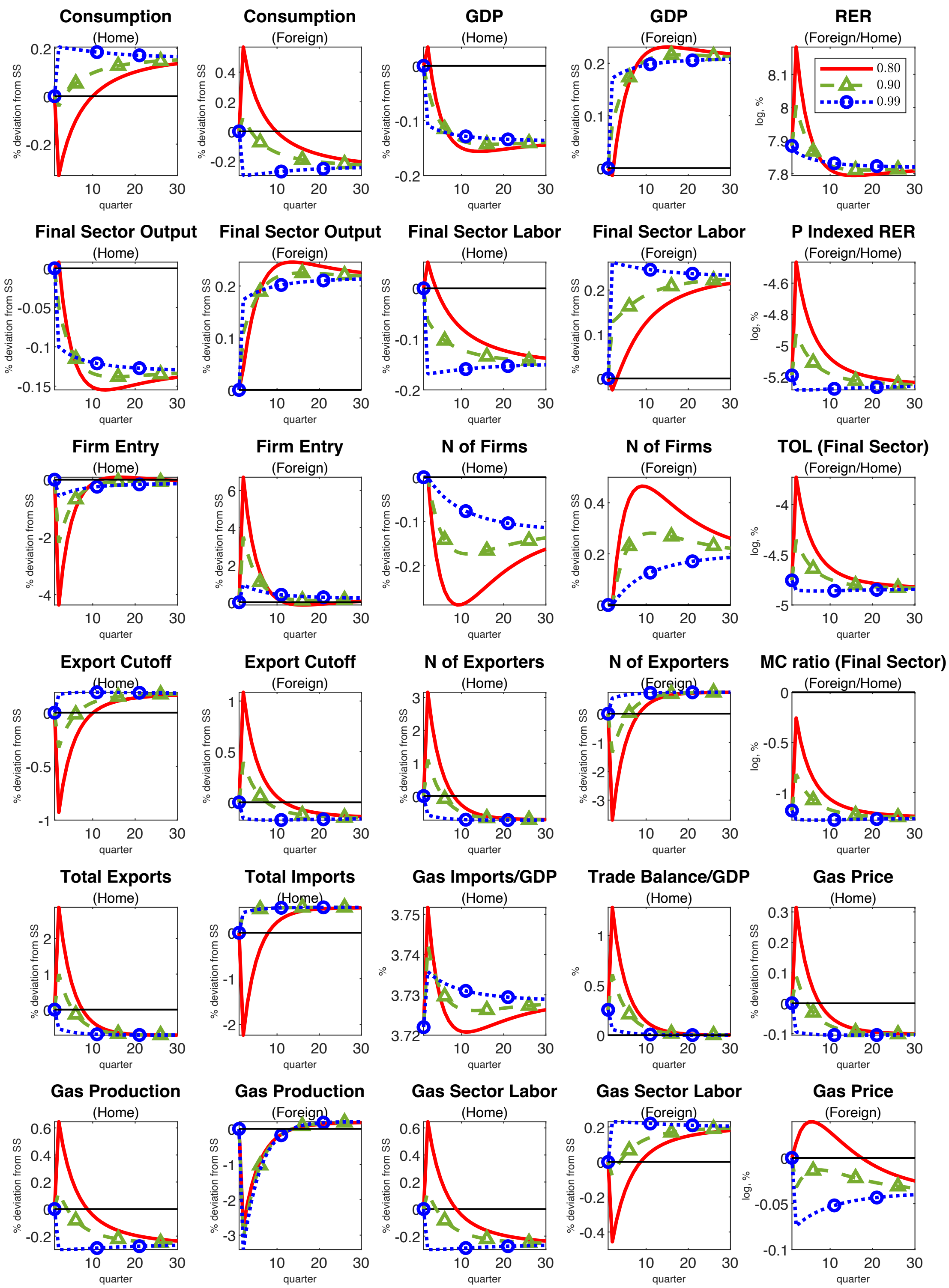


Figure 5: Foreign Household Consumption and Labor (Financial Sanctions)

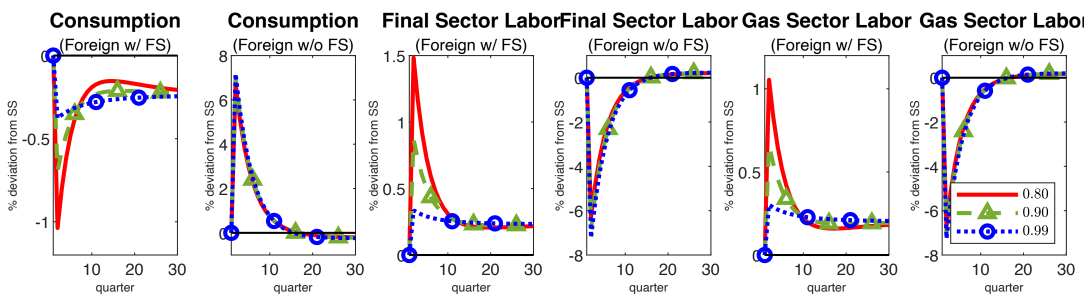


Figure 6: Transitional Dynamics after Trade, Financial and Gas Sanctions

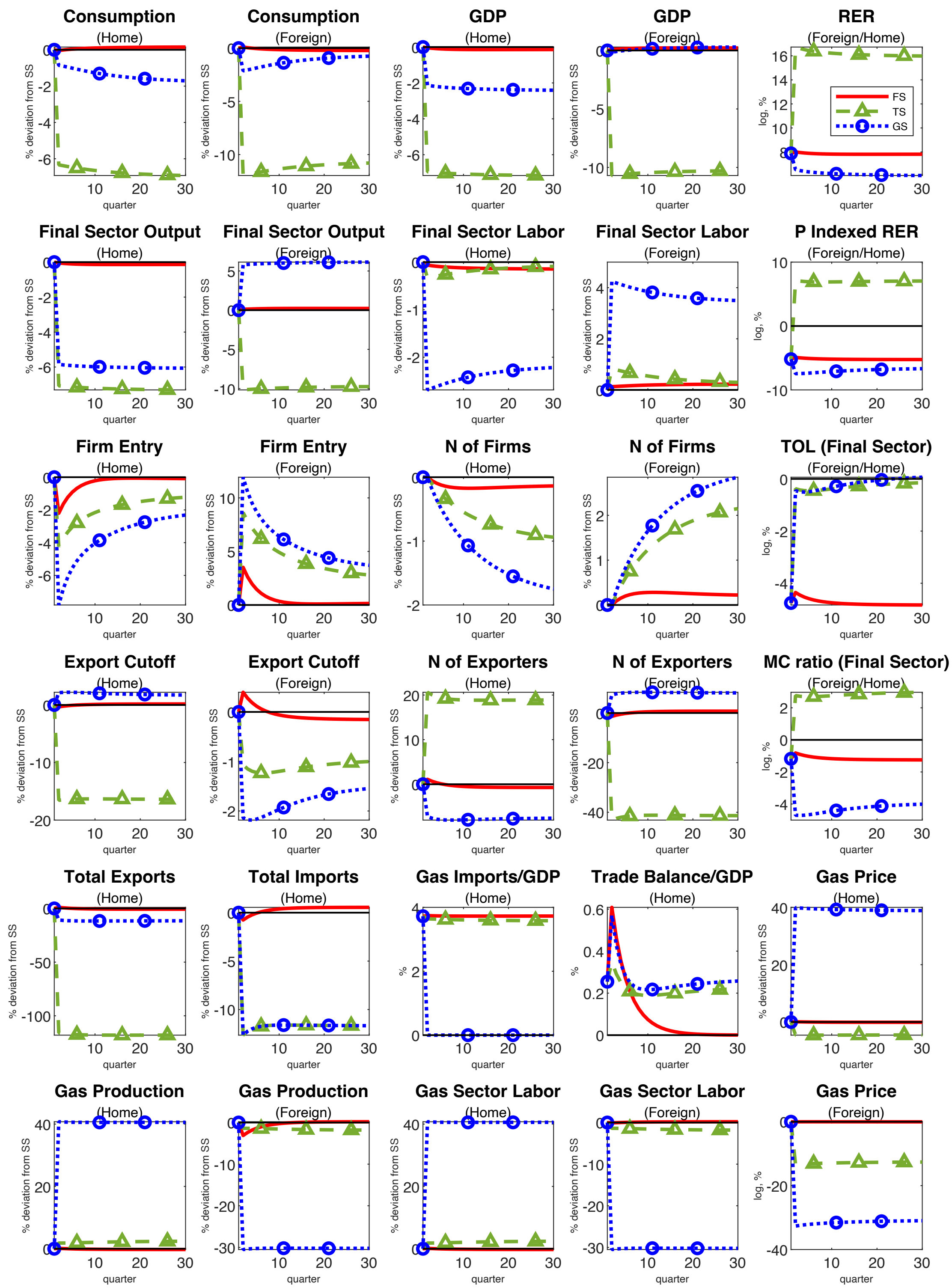


Figure 7: Transitional Dynamics after Multiple Sanctions Applied Simultaneously

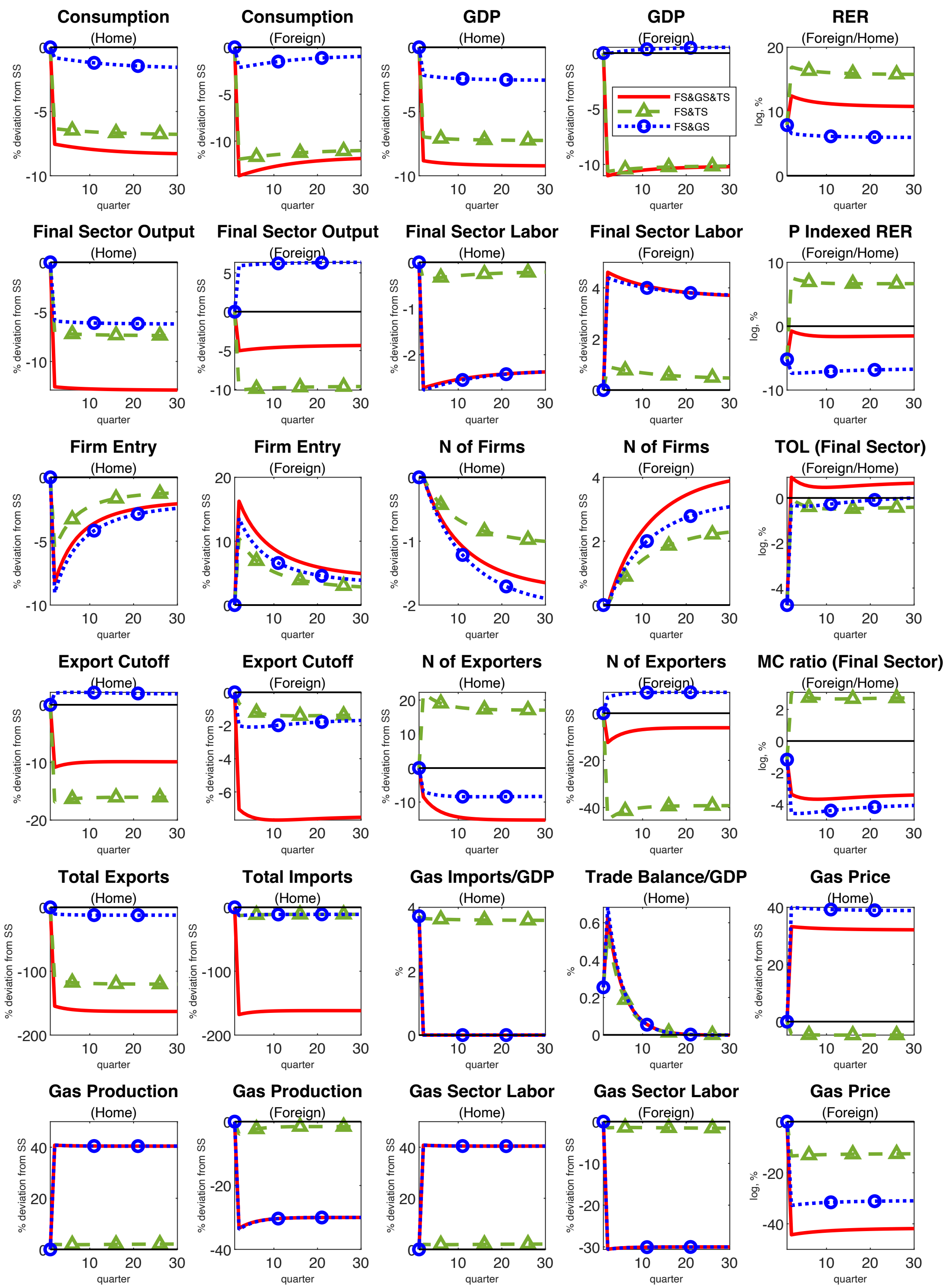
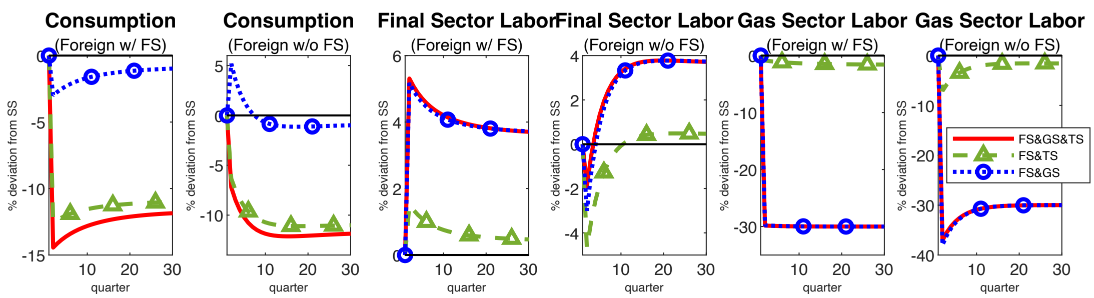


Figure 8: Foreign Household Consumption and Labor (under Different Combinations of Sanctions)



Appendix

A Mathematical Derivations

A.1 Gas Price Determination

Total demand for labor in the final Home sector can be written as $N_{D,t}L_t(\tilde{z}_D) + N_{X,t}L_t(\tilde{z}_{X,t})$. Since our assumptions are such that Home is the gas importer, total demand of gas by Home has to be equal to Home gas production ($G_N L_{G,t}$) plus Home imports of gas from Foreign ($\frac{G^*_{H,t}}{\tau_{G,t}}$). Optimal input demand conditions in Home then imply:

$$w_t\left(N_{D,t}L_t(\tilde{z}_D) + N_{X,t}L_t(\tilde{z}_{X,t})\right) = \left(\frac{1-\alpha}{\alpha}\right)\rho_{G,t}\left(G_N L_{G,t} + \frac{G^*_{H,t}}{\tau_{G,t}}\right).$$

Using final sector production functions, this equation can be rewritten as:

$$\begin{aligned} & w_t\left[N_{D,t}\left(\frac{y_t(\tilde{z}_D)}{Z_t\tilde{z}_D}\right)^{\frac{1}{1-\alpha}} G_t(\tilde{z}_D)^{-\frac{\alpha}{1-\alpha}} + N_{X,t}\left(\frac{y_t(\tilde{z}_{X,t})}{Z_t\tilde{z}_{X,t}}\right)^{\frac{1}{1-\alpha}} G_t(\tilde{z}_{X,t})^{-\frac{\alpha}{1-\alpha}}\right] \\ = & \left(\frac{1-\alpha}{\alpha}\right)\rho_{G,t}\left(G_N L_{G,t} + \frac{G^*_{H,t}}{\tau_{G,t}}\right). \end{aligned} \tag{25}$$

Next, note that optimal gas demand by a firm with productivity $\tilde{z}_D$ and market clearing for its output are such that $G_t(\tilde{z}_D) = \left(\frac{\alpha}{1-\alpha}\frac{w_t}{\rho_{G,t}}\right)^{1-\alpha}\frac{y_t(\tilde{z}_D)}{Z_t\tilde{z}_D}$ and $y_t(\tilde{z}_D) = \rho_{H,t}(\tilde{z}_D)^{-\theta}C_t$. Similarly, optimal gas demand by a firm with productivity $\tilde{z}_D$ and market clearing for its output satisfy $G_t(\tilde{z}_{X,t}) = \left(\frac{\alpha}{1-\alpha}\frac{w_t}{\rho_{G,t}}\right)^{1-\alpha}\frac{y_t(\tilde{z}_{X,t})}{Z_t\tilde{z}_{X,t}}$ and $y_t(\tilde{z}_{X,t}) = \rho_{H,t}(\tilde{z}_{X,t})^{-\theta}C_t + \tau_t\rho_{X,t}(\tilde{z}_{X,t})^{-\theta}C^*_t$. Substituting these equations into (25) and rearranging yields:

$$\begin{aligned} & \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\rho^{\alpha}_{G,t}w_t^{1-\alpha}\left[N_{D,t}\frac{\rho_{H,t}(\tilde{z}_D)^{-\theta}C_t}{Z_t\tilde{z}_D} + N_{X,t}\frac{\rho_{H,t}(\tilde{z}_{X,t})^{-\theta}C_t + \tau_t\rho_{X,t}(\tilde{z}_{X,t})^{-\theta}C^*_t}{Z_t\tilde{z}_{X,t}}\right] \\ = & \left(\frac{1-\alpha}{\alpha}\right)\rho_{G,t}\left(G_N L_{G,t} + \frac{G^*_{H,t}}{\tau_{G,t}}\right). \end{aligned} \tag{26}$$

Optimal price setting by Home final sector firms and the expression for final sector marginal cost imply:

$$\rho_{H,t}(\tilde{z}_D) = \left(\frac{\theta}{\theta-1}\right)\frac{\rho_{G,t}^{\alpha}w_t^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}\tilde{z}_D Z_t}, \tag{27}$$

$$\rho_{F,t}(\tilde{z}_{X,t}) = \frac{\tau_t}{Q_t}\frac{\tilde{z}_D}{\tilde{z}_{X,t}}\rho_{H,t}(\tilde{z}_D), \tag{28}$$

$$\rho_{H,t}(\tilde{z}_{X,t}) = \frac{\tilde{z}_D}{\tilde{z}_{X,t}}\rho_{H,t}(\tilde{z}_D), \tag{29}$$

Substituting equations (27)-(29) into equation (26) and rearranging yields:

$$\alpha\left(\frac{\theta-1}{\theta}\right)^{\theta} Z_t^{\theta-1}\left\{N_{D,t}\tilde{z}_D^{\theta-1}C_t + N_{X,t}\tilde{z}_{X,t}^{\theta-1}\left[C_t + \left(\frac{\tau_t}{Q_t}\right)^{1-\theta}Q_t C_t^*\right]\right\}\left[\frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}{\rho_{G,t}^{\alpha}w_t^{1-\alpha}}\right]^{\theta-1}$$
$$= \left(\frac{1-\alpha}{\alpha}\right)\rho_{G,t}\left(G_N L_{G,t} + \frac{G_{H,t}^*}{\tau_{G,t}}\right). \tag{30}$$

This equation can be solved for w_t as:

$$w_t = \left(\frac{\alpha}{\rho_{G,t}}\right)^{\frac{1+\alpha(\theta-1)}{(1-\alpha)(\theta-1)}}\left\{\frac{Z_t^{\theta-1}\left\{N_{D,t}\tilde{z}_D^{\theta-1}C_t + N_{X,t}\tilde{z}_{X,t}^{\theta-1}\left[C_t + \left(\frac{\tau_t}{Q_t}\right)^{1-\theta}Q_t C_t^*\right]\right\}}{\left(\frac{\theta}{\theta-1}\right)^{\theta}\left(G_N L_{G,t} + \frac{G_{H,t}^*}{\tau_{G,t}}\right)}\right\}^{\frac{1}{(1-\alpha)(\theta-1)}}. \tag{31}$$

Working in a similar way for the Foreign economy yields:

$$w_t^* = \left(\frac{\alpha\tau_{G,t}Q_t}{\rho_{G,t}}\right)^{\frac{1+\alpha(\theta-1)}{(1-\alpha)(\theta-1)}}\left\{\frac{Z_t^{*\theta-1}\left\{N_{D,t}^*\tilde{z}_D^{\theta-1}C_t^* + N_{X,t}^*\tilde{z}_{X,t}^{*\theta-1}\left[C_t^* + (\tau_t^*Q_t)^{1-\theta}\frac{C_t}{Q_t}\right]\right\}}{\left(\frac{\theta}{\theta-1}\right)^{\theta}G_N^*L_{G,t}^*}\right\}^{\frac{1}{(1-\alpha)(\theta-1)}}. \tag{32}$$

To economize on notation in the following steps, rewrite the last two equations as:

$$w_t = f(\rho_{G,t}) \quad \text{and} \quad w_t^* = f^*(\rho_{G,t}). \tag{33}$$

Equilibrium in the world market for gas requires total supply to be equal to demand. Hence, using production functions and optimal demand conditions:

$$
\begin{aligned}
G_N L_{G,t} + G_N^* L_{G,t}^* &= N_{D,t}\left(\frac{1-\alpha}{\alpha}\frac{w_t}{\rho_{G,t}}\right)^{1-\alpha}\frac{y_t\left(\tilde{z}_D\right)}{\tilde{z}_D Z_t} + N_{X,t}\left(\frac{1-\alpha}{\alpha}\frac{w_t}{\rho_{G,t}}\right)^{1-\alpha}\frac{\tau_t y_t\left(\tilde{z}_{X,t}\right)}{\tilde{z}_{X,t} Z_t} \\
&+ N_{D,t}^*\left(\frac{1-\alpha}{\alpha}\frac{w_t^*}{\rho_{G,t}^*}\right)^{1-\alpha}\frac{y_t^*\left(\tilde{z}_D\right)}{\tilde{z}_D Z_t^*} + N_{X,t}^*\left(\frac{1-\alpha}{\alpha}\frac{w_t^*}{\rho_{G,t}^*}\right)^{1-\alpha}\frac{\tau_t^* y_t^*\left(\tilde{z}_{X,t}^*\right)}{\tilde{z}_{X,t}^* Z_t^*} \qquad (34)
\end{aligned}
$$

$$(35)$$

Optimal labor supply for Home and Foreign gas production is given by, respectively:

$$L_{G,t} = \frac{w_{G,t}}{\kappa_G C_t} \quad \text{and} \quad L_{G,t}^* = \frac{w_{G,t}^*}{\kappa_G^* C_t^*}.$$

Therefore, it is:

$$L_{G,t} = \frac{\rho_{G,t} G_N}{\kappa_G C_t}, \tag{36}$$

and:

$$L_{G,t}^* = \frac{\rho_{G,t} G_N^*}{\kappa_G^* \tau_{G,t} Q_t C_t^*}. \tag{37}$$

Substituting $w_t = f\left(\rho_{G,t}\right)$, $w_t^* = f^*\left(\rho_{G,t}\right)$, and equations (36) and (37) into equation (34), using market clearing conditions for Home and Foreign final sector products, and rearranging yields:

$$
\begin{aligned}
& \rho_{G,t}\left(\frac{G_N^2}{\kappa_G C_t} + \frac{G_N^{*2}}{\tau_{G,t}\kappa_G^* Q_t C_t^*}\right) \\
= \; & N_{D,t}\left(\frac{\alpha}{1-\alpha}\frac{f\left(\rho_{G,t}\right)}{\rho_{G,t}}\right)^{1-\alpha}\left(\tilde{z}_D Z_t \rho_{H,t}\left(\tilde{z}_D\right)\right)^{-\theta}\left(\tilde{z}_D Z_t\right)^{\theta-1} C_t + \\
& N_{X,t}\left(\frac{\alpha}{1-\alpha}\frac{f\left(\rho_{G,t}\right)}{\rho_{G,t}}\right)^{1-\alpha}\tau_t\left(\tilde{z}_D Z_t \rho_{H,t}\left(\tilde{z}_D\right)\right)^{-\theta}\left(\tilde{z}_{X,t} Z_t\right)^{\theta-1}\left[C_t + \left(\frac{\tau_t}{Q_t}\right)^{1-\theta} Q_t C_t^*\right] + \\
& N_{D,t}^*\left(\frac{\alpha}{1-\alpha}\frac{f\left(\rho_{G,t}\right)/X_t}{\rho_{G,t}/\left(\tau_{G,t} Q_t\right)}\right)^{1-\alpha}\left(\tilde{z}_D Z_t^* \rho_{F,t}^*\left(\tilde{z}_D\right)\right)^{-\theta}\left(\tilde{z}_D Z_t^*\right)^{\theta-1} C_t^* + \\
& N_{X,t}^*\left(\frac{\alpha}{1-\alpha}\frac{f\left(\rho_{G,t}\right)/X_t}{\rho_{G,t}/\left(\tau_{G,t} Q_t\right)}\right)^{1-\alpha}\tau_t^*\left(\tilde{z}_D Z_t^* \rho_{F,t}^*\left(\tilde{z}_D\right)\right)^{-\theta}\left(\tilde{z}_{X,t}^* Z_t^*\right)^{\theta-1}\left[\begin{array}{c} C_t^* + \\ \left(\tau_t^* Q_t\right)^{1-\theta}\frac{C_t}{Q_t}\end{array}\right], \qquad (38)
\end{aligned}
$$

In this equation, we used the fact that $f^*(\rho_{G,t}) = f(\rho_{G,t})/X_t$, with:

$$X_t \equiv (\tau_{G,t}Q_t)^{-\frac{1+\alpha(\theta-1)}{(1-\alpha)(\theta-1)}} \left\{ \frac{\frac{Z_t^{\theta-1}\left\{N_{D,t}\tilde{z}_D^{\theta-1}C_t+N_{X,t}\tilde{z}_{X,t}^{\theta-1}\left[C_t+\left(\frac{\tau_t}{Q_t}\right)^{1-\theta}Q_tC_t^*\right]\right\}}{\left(\frac{\theta}{\theta-1}\right)^{\theta}\left(G_NL_{G,t}+\frac{G_{H,t}^*}{\tau_{G,t}}\right)}}{\frac{Z_t^{*\theta-1}\left\{N_{D,t}^*\tilde{z}_D^{\theta-1}C_t^*+N_{X,t}^*\tilde{z}_{X,t}^{*\theta-1}\left[C_t^*+(\tau_t^*Q_t)^{1-\theta}\frac{C_t}{Q_t}\right]\right\}}{\left(\frac{\theta}{\theta-1}\right)^{\theta}G_N^*L_{G,t}^*}} \right\}^{\frac{1}{(1-\alpha)(\theta-1)}}.$$

Notice that X_t does not depend directly on $\rho_{G,t}$.

The expressions for optimal $\rho_{H,t}(\tilde{z}_D)$ and $\rho_{F,t}^*(\tilde{z}_D)$ and tedious manipulation then make it possible to rewrite (38) as:

$$\begin{aligned}
&\rho_{G,t}\left[\frac{G_N}{\kappa_G C_t}+\left(\frac{G_N^*}{G_N}\right)\frac{G_N^*}{\tau_{G,t}\kappa_G^*Q_tC_t^*}\right] \\
= &\frac{\alpha}{G_N\rho_{G,t}}\left(\frac{\theta}{\theta-1}\right)^{-\theta}\left[\frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}{\rho_{G,t}^{\alpha}f(\rho_{G,t})^{1-\alpha}}\right]^{\theta-1}\times \\
&\left\{\begin{array}{l} N_{D,t}(\tilde{z}_DZ_t)^{\theta-1}C_t+N_{X,t}\tau_t(\tilde{z}_{X,t}Z_t)^{\theta-1}\left[C_t+\left(\frac{\tau_t}{Q_t}\right)^{1-\theta}Q_tC_t^*\right]+ \\ \tau_{G,t}Q_t\left[(\tau_{G,t}Q_t)^{\alpha}X_t^{1-\alpha}\right]^{\theta-1}\left[\begin{array}{l} N_{D,t}^*(\tilde{z}_DZ_t^*)^{\theta-1}C_t^*+ \\ N_{X,t}^*\tau_t^*\left(\tilde{z}_{X,t}^*Z_t^*\right)^{\theta-1}\left[C_t^*+(\tau_t^*Q_t)^{1-\theta}\frac{C_t}{Q_t}\right]\end{array}\right]\end{array}\right\}. \quad (39)
\end{aligned}$$

Equation (31) implies:

$$f(\rho_{G,t})^{-(1-\alpha)(\theta-1)} = \left(\frac{\alpha}{\rho_{G,t}}\right)^{-[1+\alpha(\theta-1)]}\left(\frac{\theta}{\theta-1}\right)^{\theta}\frac{G_NL_{G,t}+\frac{G_{H,t}^*}{\tau_{G,t}}}{N_{D,t}Z_t^{\theta-1}A_t}. \quad (40)$$

where:

$$A_t \equiv \tilde{z}_D^{\theta-1}C_t+\tau_t(\nu z_{\min})^k\,\tilde{z}_{X,t}^{-[k-(\theta-1)]}\left[C_t+\left(\frac{\tau_t}{Q_t}\right)^{1-\theta}Q_tC_t^*\right]. \quad (41)$$

In this expression, we used the relation between $N_{X,t}$ and $N_{D,t}$ implied by the assumption of a Pareto distribution of firm-specific productivity draws: $N_{X,t} = \left(\frac{\nu z_{\min}}{\tilde{z}_{X,t}}\right)^k N_{D,t}$.

It is also possible to verify that:

$$X_t^{(1-\alpha)(\theta-1)} = (\tau_{G,t}Q_t)^{-[1+\alpha(\theta-1)]}\left(\frac{N_{D,t}Z_t^{\theta-1}A_t}{G_NL_{G,t}+\frac{G_{H,t}^*}{\tau_{G,t}}}\right)\left(\frac{G_N^*L_{G,t}^*}{N_{D,t}^*Z_t^{*\theta-1}B_t}\right), \quad (42)$$

where:

$$B_t \equiv \tilde{z}_D^{\theta-1} C_t^* + \tau_t^* \left(\nu z_{\min}\right)^k \tilde{z}_{X,t}^{*-[k-(\theta-1)]} \left[C_t^* + \left(\tau_t^* Q_t\right)^{1-\theta} \frac{C_t}{Q_t}\right]. \tag{43}$$

Equation (39) can be rewritten as:

$$\begin{aligned} & \rho_{G,t}\left[\frac{G_N}{\kappa_G C_t} + \left(\frac{G_N^*}{G_N}\right)\frac{G_N^*}{\tau_{G,t}\kappa_G^* Q_t C_t^*}\right] \\ = \; & \frac{\alpha}{G_N \rho_{G,t}}\left(\frac{\theta}{\theta-1}\right)^{-\theta}\left[\frac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{\rho_{G,t}^\alpha}\right]^{\theta-1} f\left(\rho_{G,t}\right)^{-(1-\alpha)(\theta-1)} \\ & \times \left[N_{D,t} Z_t^{\theta-1} A_t + \left(\tau_{G,t} Q_t\right)^{1+\alpha(\theta-1)} X_t^{(1-\alpha)(\theta-1)} N_{D,t}^* Z_t^{*\theta-1} B_t\right]. \end{aligned} \tag{44}$$

Then, substituting equations (40)-(43) into equation (44) yields:

$$\rho_{G,t}\left[\frac{G_N}{\kappa_G C_t} + \left(\frac{G_N^*}{G_N}\right)\frac{G_N^*}{\tau_{G,t}\kappa_G^* Q_t C_t^*}\right] = \frac{(1-\alpha)^{(1-\alpha)(\theta-1)}}{G_N}\left(1 + \frac{G_N^* L_{G,t}^*}{G_N L_{G,t} + \frac{G_{H,t}^*}{\tau_{G,t}}}\right). \tag{45}$$

Finally, using $L_{G,t} = \frac{G_N \rho_{G,t}}{\kappa_G C_t}$ and $L_{G,t}^* = \frac{G_N^* \rho_{G,t}}{\tau_{G,t}\kappa_G^* Q_t C_t^*}$ and rearranging gives us:

$$\begin{aligned} & \rho_{G,t}\frac{G_N}{\kappa_G C_t}\left[1 + \left(\frac{G_N^*}{G_N}\right)^2\left(\frac{\kappa_G}{\kappa_G^* \tau_{G,t}}\right)\left(\frac{C_t}{Q_t C_t^*}\right)\right] \\ = \; & \frac{(1-\alpha)^{(1-\alpha)(\theta-1)}}{G_N}\left[1 + \left(\frac{G_N^*}{G_N}\right)^2\left(\frac{\kappa_G}{\kappa_G^* \tau_{G,t}}\right)\left(\frac{C_t}{Q_t C_t^*}\right)\left(\frac{\rho_{G,t}}{\rho_{G,t} + \kappa_G \tau_{G,t}^{-1} G_N^{-2} C_t G_{H,t}^*}\right)\right]. \end{aligned} \tag{46}$$

Home imports of Foreign gas are given by:

$$G_{H,t}^* = G_N^* L_{G,t}^* - G_{F,t}^* = \frac{\rho_{G,t} G_N^{*2}}{\kappa_G^* \tau_{G,t} Q_t C_t^*} - G_{F,t}^*, \tag{47}$$

where the second equality follows from using equation (37).

Optimal input demands by Foreign final sector firms and the relation $N_{X,t}^* = \left(\frac{\nu z_{\min}}{\tilde{z}_{X,t}^*}\right)^k N_{D,t}^*$ imply:

$$G_{F,t}^* = N_{D,t}^* \left(\frac{\alpha}{1-\alpha}\frac{w_t^*}{\rho_{G,t}^*}\right)^{1-\alpha}\left[\frac{y_t^*\left(\tilde{z}_D\right)}{\tilde{z}_D Z_t^*} + \left(\frac{\nu z_{\min}}{\tilde{z}_{X,t}^*}\right)^k \tau_t^* \frac{y_t^*\left(\tilde{z}_{X,t}^*\right)}{\tilde{z}_{X,t}^* Z_t^*}\right]. \tag{48}$$

Substituting market clearing conditions for Foreign final sector products and optimal price setting by Foreign firms into equation (48) yields:

$$G^*_{F,t} = \left(\frac{\theta-1}{\theta}\right)^{\theta} N^*_{D,t} \left(\frac{\alpha Q_t \tau_{G,t}}{\rho_{G,t}}\right)^{1+\alpha(\theta-1)} \left(\frac{1-\alpha}{w^*_t}\right)^{(1-\alpha)(\theta-1)} Z^{*\theta-1}_t \cdot$$
$$\tilde{z}^{\theta-1}_D C^*_t + \tau^*_t \left(\nu z_{\min}\right)^k \tilde{z}^{*-[k-(\theta-1)]}_{X,t} \left[C^*_t + \left(\tau^*_t Q_t\right)^{1-\theta} \frac{C_t}{Q_t}\right]. \quad (49)$$

Finally, substituting $N^*_{X,t} = \left(\frac{\nu z_{\min}}{\tilde{z}^*_{X,t}}\right)^k N^*_{D,t}$ and equation (37) into equation (32), and plugging the resulting expression for w^*_t into equation (49) makes it possible to obtain:

$$G^*_{F,t} = (1-\alpha)^{(1-\alpha)(\theta-1)} \frac{G^{*2}_N \rho_{G,t}}{\tau_{G,t}\kappa^*_G Q_t C^*_t}. \quad (50)$$

Equations (47) and (50) then imply:

$$G^*_{H,t} = \left[1-(1-\alpha)^{(1-\alpha)(\theta-1)}\right] \frac{G^{*2}_N \rho_{G,t}}{\tau_{G,t}\kappa^*_G Q_t C^*_t}. \quad (51)$$

This expression can be substituted into equation (46). Then, defining $\xi_t \equiv \left(\frac{G^*_N}{G_N}\right)^2 \left(\frac{\kappa_G}{\kappa^*_G \tau_{G,t}}\right) \left(\frac{C_t}{Q_t C^*_t}\right)$ and rearranging the resulting equation, we have:

$$\rho_{G,t} = \frac{(1-\alpha)^{(1-\alpha)(\theta-1)} \kappa_G C_t}{G^2_N} \left\{ \frac{1+\xi_t+\tau^{-1}_{G,t}\left[1-(1-\alpha)^{(1-\alpha)(\theta-1)}\right]\xi_t}{\left[1+\tau^{-1}_{G,t}\left[1-(1-\alpha)^{(1-\alpha)(\theta-1)}\right]\xi_t\right](1+\xi_t)} \right\}. \quad (52)$$

A.2 Gas Price and Gas Share

The gas price equation in the special case of complete markets, $\tau_{G,t} = 1$, $G_N = G^*_N$, and $\kappa_G = \kappa^*_G$ is reproduced below for your convenience:

$$\rho_{G,t} = \frac{(1-\alpha)^{(1-\alpha)(\theta-1)} \kappa_G C_t}{G^2_N} \left\{ \frac{3-(1-\alpha)^{(1-\alpha)(\theta-1)}}{2\left[2-(1-\alpha)^{(1-\alpha)(\theta-1)}\right]} \right\}. \quad (53)$$

Let $\psi \equiv (1-\alpha)^{(1-\alpha)(\theta-1)}$. The derivative of ψ with respect to α is given by:

$$\psi_\alpha = -(\theta-1)(1-\alpha)^{(1-\alpha)(\theta-1)}[1+ln(1-\alpha)].$$

Now let $\Lambda_t \equiv \frac{\kappa_G C_t}{G_N^2}$. Then, equation (53) can be rewritten as:

$$\rho_{G,t} = \psi \Lambda_t \left[\frac{3-\psi}{2(2-\psi)} \right]. \tag{54}$$

Our interest is in determining how $\psi \left[\frac{3-\psi}{2(2-\psi)} \right]$ varies with α. Taking the derivative and rearranging yields:

$$\frac{\partial \psi \left[\frac{3-\psi}{2(2-\psi)} \right]}{\partial \alpha} = \frac{\psi_\alpha \left[2(3-2\psi) + \psi^2 \right]}{2(2-\psi)^2}.$$

The definition of ψ, $0 \leq \alpha \leq 1$, and $\theta > 1$ imply $3 > 2\psi$. Thus, the sign of the derivative we are interested in is determined by the sign of ψ_α. Since $\theta > 1$, the sign of ψ_α depends on the sign of $1 + \ln(1-\alpha)$. This expression is a monotonically decreasing function of α. It is positive if α is smaller than (approximately) 0.63. It is negative if α is higher than this number. It follows that $\psi \left[\frac{3-\psi}{2(2-\psi)} \right]$ is a monotonically decreasing function of α if $0 \leq \alpha \leq 0.63$, and it increases with α if $0.63 < \alpha \leq 1$. Since $\psi \left[\frac{3-\psi}{2(2-\psi)} \right] = 1$ when $\alpha = 0$ and $\alpha = 1$, the relation between the price of gas and its share in production of final goods when markets are complete, countries are fully symmetric, and there is no iceberg cost of gas trade is U-shaped.

A.3 The Log-Linear Gas Price Equation

The non-linear equation for the gas price $\rho_{G,t}$ is reproduced below for your convenience:

$$\rho_{G,t} = \frac{(1-\alpha)^{(1-\alpha)(\theta-1)} \kappa_G C_t}{G_N^2} \left\{ \frac{1 + \xi_t + \tau_{G,t}^{-1} \left[1 - (1-\alpha)^{(1-\alpha)(\theta-1)} \right] \xi_t}{\left[1 + \tau_{G,t}^{-1} \left[1 - (1-\alpha)^{(1-\alpha)(\theta-1)} \right] \xi_t \right] (1+\xi_t)} \right\} \tag{55}$$

where $\xi_t \equiv \left(\frac{G_N^*}{G_N} \right)^2 \left(\frac{\kappa_G}{\kappa_G^* \tau_{G,t}} \right) \left(\frac{C_t}{Q_t C_t^*} \right)$.

Let $NUM_{\rho,t}$ denote the numerator of the expression inside curly brackets in this equation and $DEN_{\rho,t}$ the denominator. Then, the log-linear version of equation (55) can be written as:

$$\rho_{G,t} = \mathsf{C}_t + \frac{dNUM_{\rho,t}}{\overline{NUM}_\rho} - \frac{dDEN_{\rho,t}}{\overline{DEN}_\rho}, \tag{56}$$

where $\rho_{G,t}$ is the percentage deviation of $\rho_{G,t}$ from the steady state: $\rho_{G,t} \equiv \frac{d\rho_{G,t}}{\bar{\rho}_G}$, C_t is the percentage deviation of C_t from the steady state: $\mathsf{C}_t \equiv \frac{dC_t}{\bar{C}}$, d is the differentiation operator, and Sans Serif variables in equations below are defined similarly.

Differentiating $NUM_{\rho,t}$ and using the definitions of log-linearized variables yields:

$$dNUM_{\rho,t} = d\xi_t + \eta\bar{\xi}\bar{\tau}_G^{-1}\left(\xi_t - \tau_{G,t}\right), \tag{57}$$

where $\eta \equiv 1 - (1-\alpha)^{(1-\alpha)(\theta-1)}$. Equation (57) can be rewritten as:

$$dNUM_{\rho,t} = \bar{\xi}\left[\left(1 + \eta\bar{\tau}_G^{-1}\right)\xi_t - \eta\bar{\tau}_G^{-1}\tau_{G,t}\right]. \tag{58}$$

Proceeding similarly with $DEN_{\rho,t}$ yields:

$$dDEN_{\rho,t} = \bar{\xi}[1 + \eta\bar{\tau}_G^{-1}(1 + 2\bar{\xi})]\xi_t - (1 + \bar{\xi})\eta\bar{\xi}\bar{\tau}_G^{-1}\tau_{G,t} \tag{59}$$

The definition of ξ_t implies:

$$\xi_t = \mathsf{C}_t - \mathsf{Q}_t - \mathsf{C}_t^* - \tau_{G,t}. \tag{60}$$

Substituting equation (60) into equations (58) and (59), we have:%

$$dNUM_{\rho,t} = \bar{\xi}[(1 + \eta\bar{\tau}_G^{-1})(\mathsf{C}_t - \mathsf{Q}_t - \mathsf{C}_t^* - \tau_{G,t}) - \eta\bar{\tau}_G^{-1}\tau_{G,t}] \tag{61}$$

and:

$$DEN_{\rho,t} = \bar{\xi}[1 + \eta\bar{\tau}_G^{-1}(1 + 2\bar{\xi})](\mathsf{C}_t - \mathsf{Q}_t - \mathsf{C}_t^* - \tau_{G,t}) - (1 + \bar{\xi})\eta\bar{\xi}\bar{\tau}_G^{-1}\tau_{G,t} \tag{62}$$

Finally, substituting equations (61), (62), and the expressions for $\overline{NUM}_\rho$ and $\overline{DEN}_\rho$ into equation (56), and rearranging, we obtain:

$$\rho_{G,t} = (1 - \Gamma_1)\,\mathsf{C}_t - \mathsf{Q}_t - \mathsf{C}_t^* - (\Gamma_1 - \Gamma_2)\,\tau_{G,t} \tag{63}$$

with:

$$\Gamma_1 \equiv \frac{\eta\bar{\xi}\bar{\tau}_G^{-1}(1 - \eta\bar{\tau}_G^{-1})[1 + 2\bar{\xi}(1 + \bar{\xi})]}{(1 + \bar{\xi})(1 + \eta\bar{\xi}\bar{\tau}_G^{-1})[1 + \bar{\xi}(1 + \eta\bar{\tau}_G^{-1})]} > 0$$

and:

$$\Gamma_2 \equiv \frac{\eta\bar{\xi}^2\bar{\tau}_G^{-1}}{(1 + \eta\bar{\xi}\bar{\tau}_G^{-1})[1 + \bar{\xi}(1 + \eta\bar{\tau}_G^{-1})]} > 0$$

A.3.1 On Γ_1 and Γ_2

The parameter Γ_1 is strictly smaller than 1 if and only if:

$$\eta\bar{\xi}\bar{\tau}_G^{-1}\left(1-\eta\bar{\tau}_G^{-1}\right)\left[1+2\bar{\xi}\left(1+\bar{\xi}\right)\right]<\left(1+\bar{\xi}\right)\left(1+\bar{\xi}\eta\bar{\tau}_G^{-1}\right)\left[1+\bar{\xi}\left(1+\eta\bar{\tau}_G^{-1}\right)\right]$$

Tedious algebra shows that this inequality is equivalent to:

$$-\eta\bar{\xi}\bar{\tau}_G^{-1}\left[1+\eta\bar{\tau}_G^{-1}+\bar{\xi}(1-\bar{\xi})+3\eta\bar{\xi}\bar{\tau}_G^{-1}(1+\bar{\xi})\right]<\left(1+\bar{\xi}\right)^2$$

Hence, $\bar{\xi}<1$ is sufficient (but not necessary) to ensure $\Gamma_1<1$.

The parameter Γ_1 is strictly larger than Γ_2 if and only if:

$$\frac{\eta\bar{\xi}\bar{\tau}_G^{-1}(1-\eta\bar{\tau}_G^{-1})[1+2\bar{\xi}(1+\bar{\xi})]}{(1+\bar{\xi})(1+\eta\bar{\xi}\bar{\tau}_G^{-1})[1+\bar{\xi}(1+\eta\bar{\tau}_G^{-1})]}>\frac{\eta\bar{\xi}^2\bar{\tau}_G^{-1}}{(1+\eta\bar{\xi}\bar{\tau}_G^{-1})[1+\bar{\xi}(1+\eta\bar{\tau}_G^{-1})]},$$

or:

$$\left(1-\eta\bar{\tau}_G^{-1}\right)\left[1+2\bar{\xi}(1+\bar{\xi})\right]>\bar{\xi}(1+\bar{\xi}).$$

This inequality can be rewritten as:

$$1-\eta\bar{\tau}_G^{-1}+\bar{\xi}\left(1+\bar{\xi}\right)\left(1-2\eta\bar{\tau}_G^{-1}\right)>0.$$

Since $0\leq\eta\leq 1$ and $\bar{\tau}_G\geq 1$, $2\eta\bar{\tau}_G<1$ is sufficient (but not necessary) to ensure $\Gamma_1>\Gamma_2$.

A.4 The Log-Linear Real Exchange Rate Equation

The non-linear equation for the data-consistent real exchange rate $\tilde{Q}_t$ is reproduced below for your convenience:

$$\tilde{Q}_t^{\,1-\theta}=\frac{\frac{N^*_{D,t}}{N^*_t}\left[TOL_t^{1-\alpha}\left(\frac{Z_t}{\tau_{G,t}Z^*_t}\right)^{\alpha}\frac{\tilde{z}_D}{\tilde{z}^*_D}\right]^{1-\theta}+\frac{N_{X,t}}{N^*_t}\left[\frac{\tau\tilde{z}_D}{\tilde{z}_{X,t}}\right]^{1-\theta}}{\frac{N_{D,t}}{N_t}+\frac{N^*_{X,t}}{N_t}\left[TOL_t^{1-\alpha}\left(\frac{Z_t}{\tau_{G,t}Z^*_t}\right)^{\alpha}\frac{\tau^*\tilde{z}_D}{\tilde{z}^*_{X,t}}\right]^{1-\theta}}. \tag{64}$$

Let NUM_t denote the numerator of this equation and DEN_t the denominator. Then, the log-linear version of equation (64) can be written as:

$$\tilde{\mathsf{Q}}_t = \frac{\overline{NUM} \cdot dDEN_t - \overline{DEN} \cdot dNUM_t}{(\theta - 1)\overline{NUMDEN}} \tag{65}$$

Assume $\bar{Z} = \bar{Z}^* = 1$. Differentiating NUM_t and using the definitions of log-linearized variables yields:

$$\begin{aligned} dNUM_t = \; & \frac{\bar{N}_D^*}{\bar{N}^*}\left(\mathsf{N}^*_{\mathsf{D,t}} - \mathsf{N}^*_{\mathsf{t}}\right)\left(\overline{TOL}^{1-\alpha}\overline{\tau}_G^{-\alpha}\right)^{1-\theta} \\ & +(1-\theta)\frac{\bar{N}_D^*}{\bar{N}^*}\left[\mathsf{TOL_t}\overline{TOL}^{1-\alpha}\overline{\tau}_G^{-\alpha} + \alpha\overline{TOL}^{1-\alpha}\overline{\tau}_G^{-\alpha}(\mathsf{Z}_t - \mathsf{Z}_t^* - \tau_{G,t})\right](\overline{TOL}^{1-\alpha}\overline{\tau}_G^{-\alpha})^{-\theta} \\ & +\frac{\bar{N}_X}{\bar{N}^*}\left(\mathsf{N_{X,t}} - \mathsf{N}^*_{\mathsf{t}}\right)\left(\frac{\overline{\tau}\tilde{z}_D}{\bar{\tilde{z}}_X}\right)^{1-\theta} + \frac{\bar{N}_X}{\bar{N}^*}\frac{\overline{\tau}\tilde{z}_D}{\bar{\tilde{z}}_X}(\tau_t - \bar{\tilde{z}}_{X,t})\left(\frac{\overline{\tau}\tilde{z}_D}{\bar{\tilde{z}}_X}\right)^{-\theta}, \end{aligned}$$

or, after rearranging:

$$\begin{aligned} dNUM_t = \; & \frac{\bar{N}_D^*}{\bar{N}^*}\left(\overline{TOL}^{1-\alpha}\overline{\tau}_G^{-\alpha}\right)^{1-\theta}\left\{\mathsf{N}^*_{\mathsf{D,t}} - \mathsf{N}^*_{\mathsf{t}} + (1-\theta)[(1-\alpha)\mathsf{TOL_t} + \alpha(\mathsf{Z}_t - \mathsf{Z}_t^* - \tau_{G,t})]\right\} \\ & +\frac{\bar{N}_X}{\bar{N}^*}\left(\frac{\overline{\tau}\tilde{z}_D}{\bar{\tilde{z}}_X}\right)^{1-\theta}\left(\mathsf{N_{X,t}} - \mathsf{N}^*_{\mathsf{t}} + \tau_t - \bar{\tilde{z}}_{X,t}\right). \end{aligned} \tag{66}$$

Proceeding similarly with DEN_t yields:

$$\begin{aligned} dDEN_t = \; & \frac{\bar{N}_D}{\bar{N}}\left(\mathsf{N_{D,t}} - \mathsf{N_t}\right) \\ & +\frac{\bar{N}_X^*}{\bar{N}}\left(\overline{TOL}^{1-\alpha}\overline{\tau}_G^{-\alpha}\frac{\overline{\tau}^*\tilde{z}_D}{\bar{\tilde{z}}^*_X}\right)^{1-\theta}\left\{\mathsf{N}^*_{\mathsf{X,t}} - \mathsf{N_t} + (1-\theta)\begin{bmatrix} -\bar{\tilde{z}}^*_{X,t} + (1-\alpha)\mathsf{TOL_t} \\ +\alpha(\mathsf{Z}_t - \mathsf{Z}_t^* - \tau_{G,t}) \end{bmatrix}\right\}. \end{aligned} \tag{67}$$

Let the parameters, χ_1, χ_2, and γ be defined implicitly by:
$\frac{\bar{N}_D^*}{\bar{N}^*} = \chi_1\frac{\bar{N}_D}{\bar{N}}$, $\frac{\bar{N}_X}{\bar{N}^*} = \gamma\chi_1\frac{\bar{N}_X^*}{\bar{N}}$, and $\left(\frac{\overline{\tau}\tilde{z}_D}{\bar{\tilde{z}}_X}\right)^{1-\theta} = \left(\chi_2\frac{\overline{\tau}^*\tilde{z}_D}{\bar{\tilde{z}}^*_X}\right)^{1-\theta}$. Then, equation (66) can be written as:

$$\begin{aligned} dNUM_t = \; & \chi_1\frac{\bar{N}_D}{\bar{N}}\left(\overline{TOL}^{1-\alpha}\overline{\tau}_G^{-\alpha}\right)^{1-\theta}\left\{\mathsf{N}^*_{\mathsf{D,t}} - \mathsf{N}^*_{\mathsf{t}} + (1-\theta)[(1-\alpha)\mathsf{TOL_t} + \alpha(\mathsf{Z}_t - \mathsf{Z}_t^* - \tau_{G,t})]\right\} \\ & +\gamma\chi_1\frac{\bar{N}_X^*}{\bar{N}}\left(\chi_2\frac{\overline{\tau}^*\tilde{z}_D}{\bar{\tilde{z}}^*_X}\right)^{1-\theta}\left(\mathsf{N_{X,t}} - \mathsf{N}^*_{\mathsf{t}} + \tau_t - \bar{\tilde{z}}_{X,t}\right). \end{aligned} \tag{68}$$

Substituting equations (67) and (68) and the expressions for $\overline{NUM}$ and $\overline{DEN}$ into $\overline{NUM} \cdot dDEN_t - \overline{DEN} \cdot dNUM_t$ (the numerator of the expression for $\tilde{\mathsf{Q}}_t$ in equation 65), and rearranging yields:

$$
\begin{aligned}
&\overline{NUM} \cdot dDEN_t - \overline{DEN} \cdot dNUM_t \\
&= (\theta - 1)(\Phi_1 - \Phi_2)[(1-\alpha)\mathsf{TOL}_t + \alpha(\mathsf{Z}_t - \mathsf{Z}_t^* - \tau_{G,t})] \\
&\qquad +(\theta-1)(\Phi_2 + \Phi_4)\tilde{z}^*_{X,t} - (\Phi_2 + \Phi_3)(\tilde{z}_{X,t} - \tau_t) \\
&\qquad +\Phi_1[\mathsf{N_{D,t}} - \mathsf{N_t} - (\mathsf{N^*_{D,t}} - \mathsf{N^*_t})] + \Phi_2[\mathsf{N^*_{X,t}} - \mathsf{N_t} - (\mathsf{N_{X,t}} - \mathsf{N^*_t})] \\
&\qquad -\Phi_3[\mathsf{N_{X,t}} - \mathsf{N^*_t} - (\mathsf{N_{D,t}} - \mathsf{N_t})] + \Phi_4[\mathsf{N^*_{X,t}} - \mathsf{N_t} - (\mathsf{N^*_{D,t}} - \mathsf{N^*_t})
\end{aligned}
\tag{69}
$$

where

$$
\begin{aligned}
\Phi_1 &\equiv \chi_1 \left(\frac{\bar{N}_D}{\bar{N}}\right)^2 \left(\overline{TOL}^{1-\alpha}\bar{\tau}_G^{-\alpha}\right)^{1-\theta} > 0, \\
\Phi_2 &\equiv \gamma\chi_1 \left(\frac{\bar{N}^*_X}{\bar{N}}\right)^2 \left(\overline{TOL}^{1-\alpha}\bar{\tau}_G^{-\alpha}\chi_2\right)^{1-\theta} \left(\frac{\tau^*\tilde{z}_D}{\bar{\tilde{z}}^*_X}\right)^{2(1-\theta)} > 0, \\
\Phi_3 &\equiv \gamma\chi_1 \frac{\bar{N}_D\bar{N}^*_X}{\bar{N}^2}\left(\chi_2\frac{\tau^*\tilde{z}_D}{\bar{\tilde{z}}^*_X}\right)^{1-\theta} > 0, \\
\Phi_4 &\equiv \chi_1 \frac{\bar{N}_D\bar{N}^*_X}{\bar{N}^2}\left(\frac{\tau^*\tilde{z}_D}{\bar{\tilde{z}}^*_X}\right)^{1-\theta} \left(\overline{TOL}^{1-\alpha}\bar{\tau}_G^{-\alpha}\right)^{2(1-\theta)} > 0.
\end{aligned}
$$